Article 15:

Formulating the theories of gravity/intrinsic gravity and motion/intrinsic motion and their union at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field. Part II.

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An analytical approach to the theory of gravitational relativity (TGR) and combined theory of gravitational relativity and special theory of relativity (TGR+SR) on the flat spacetime of TGR in a gravitational field of arbitrary strength, is developed to complement the graphical approach developed in the first part of this paper. The analytical approach to TGR bears an interesting analogy to the analytical approach to SR developed by Albert Einstein. Relations for (or transformations of) mass, force, energy, gravitational potential, gravitational field (or acceleration), gravitational velocity, frequency and other parameters, are derived on flat spacetime in the context of TGR. These are relations that incorporate the effect of gravitational relativity into the classical and special-relativistic values of parameters at every point on flat spacetime in a gravitational field of arbitrary strength. Local Lorentz invariance (LLI) is validated on the flat spacetime of TGR. The weak equivalence principle (WEP) is shown to be valid in the context of TGR as long as it is valid in classical gravitation. The modified (or gravitational-relativistic) form of the Newtonian gravitational force law on flat spacetime in the context of TGR is derived. The non-trivial relationships among the various mass concepts in physics namely, the inertial mass, the passive gravitational mass, the active gravitational mass and the rest mass, which the derived mass relation in the context of TGR implies, are highlighted and the elusive rest mass and classical radius of a gravitational field source are calculated from the observed (or inertial) mass and the observed radius of the field source. The exterior Schwarzschild line element in the general theory of relativity is shown to pertain to the measurable sub-space of TGR (which is a fictitious curved spacetime with sub-Riemannian metric tensor), whereas the total space of TGR is the observed flat spacetime with constant Lorentzian metric tensor.

1 Theory of gravitational relativity by analytical approach

The global spacetime/intrinsic spacetime geometry of Fig. 11 of [1] in the absence of relative gravity at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field does not exist in nature, while the global spacetime/intrinsic spacetime geometry of combined first and

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second stages of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field of Fig. 7 and its complementary geometry of Fig. 8, along with their inverses Figs. 9 and 10 of [2], made fuller by incorporating the constantly flat absolute-absolute intrinsic-intrinsic spacetime ($\phi\phi\hat{\hat{\rho}}, \phi\phi\hat{\hat{c}}\phi\phi\hat{\hat{t}}$) in Fig.9 of [3] and reproduced as Fig. 1 and its complementary diagram of Fig. 2 of [4], are the global spacetime/intrinsic spacetime geometries that exist in every gravitational field.

Never the less it is theoretically valid that the flat four-dimensional proper spacetime (Σ', ct') containing proper parameters and its underlying flat two-dimensional proper intrinsic spacetime $(\phi\rho', \phi c\phi t')$ containing proper intrinsic parameters in Fig. 11 of [1], in the absence of relative gravity at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field, effectively transform into the flat four-dimensional relativistic spacetime (Σ, ct) containing gravitational-relativistic parameters and its underlying flat two-dimensional relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ containing gravitational-relativistic intrinsic parameters in Figs. 1 and 2 of [4], in a relative gravitational field, at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field.

Now let the center of the assumed spherical rest mass M_0 of a gravitational field source be located at a point O and let the rest mass m_0 of a test particle be located along a radial direction at radial distance r' from the center of M_0 in the proper Euclidean 3-space Σ' of the flat four-dimensional proper spacetime (Σ', ct'), which evolves at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field. This situation cannot endure, since the first and second stages of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters commence simultaneously and propagate together at the speed of light away from the location of the gravitational field source, as explained in sub-section 1.1 of [2] and section 3 of [3].

The rest mass M_0 of the gravitational field source in the proper Euclidean 3-space Σ' at position O and the rest mass m_0 of the test particle in Σ' at radial distance r' from the center of M_0 will, as soon as they are formed at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters, transform into the gravitational-relativistic mass M of the gravitational field source at position O in the relativistic Euclidean 3-space Σ and the gravitational-relativistic mass m of the test particle at radial distance r from the center of M in Σ respectively in the context of TGR at the second stage.

Every other proper (or classical) parameter (with prime label), such as proper gravitational potential $\Phi'(r')$, proper gravitational field $\vec{g}'(r')$, proper energy E', proper inertial and gravitational forces $\vec{F}_i'$ and $\vec{F}_g'$, proper frequency ν_0, etc, that evolved in the proper Euclidean 3-space Σ' at the neighborhood of the rest mass M_0 of the gravitational field source at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters, will likewise, as soon as they are formed, transform into the respective gravitational-relativistic parameters namely, gravitational-relativistic gravitational potential $\Phi(r)$, gravitational-relativistic gravitational field $\vec{g}(r)$, gravitational-relativistic energy E, gravitational-relativistic inertial and gravitational forces $\vec{F}_i$ and $\vec{F}_g$, gravitational-relativistic frequency ν, etc, in the relativistic Euclidean 3-space Σ in the context of TGR at the second stage.

It is the gravitational-relativistic masses m and M of test particles and gravitational field sources and gravitational-relativistic physical parameters and physical constants in the relativistic Euclidean 3-space Σ of TGR that are observed in every gravitational field, assuming the absence of special relativity. In a situation where special relativity is also present, then the effect of special relativity must be incorporated into the gravitational-relativistic parameters giving gravitational-relativistic cum special-relativistic parameters in the context of combined TGR and SR.

It then follows that the entire universe is a flat four-dimensional relativistic spacetime domain (Σ, ct) of TGR, containing gravitational-relativistic masses m and M in the context of TGR and gravitational-relativistic cum special-relativistic masses $\overline{m}$ and $\overline{M}$ in the context of TGR+SR, of particles and bodies and other gravitational-relativistic and gravitational cum special-relativistic parameters and its underlying flat two-dimensional relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ of ϕTGR, containing gravitational-relativistic intrinsic masses ϕm and ϕM of particles and bodies and gravitational-relativistic cum special-relativistic intrinsic masses $\phi\overline{m}$ and $\phi\overline{M}$ of particles and bodies and other gravitational-relativistic and gravitational-relativistic cum special-relativistic intrinsic parameters. The flatness of the universal four-dimensional spacetime and consequently of the universal 3-space are *a priori* in the present theory.

Although spacetime is flat within a gravitational field, gravitational velocity $\vec{V}g(r)$ and gravitational potential $\Phi(r)$ vary with radial distance r from the center of the mass M of the gravitational field source in the Euclidean 3-space Σ. It is therefore mandatory to restrict the formulations of the gravitational-relativistic non-gravitational laws and gravitational-relativistic classical theory of gravity (CG) to the interiors of local Lorentz frames on flat spacetime (Σ, ct) in every gravitational field.

Certainly there are transformations of elementary coordinate intervals dx'^0, dx'^1, dx'^2 and dx'^3 of the flat proper metric spacetime (Σ', ct') into the elementary coordinate intervals dx^0, dx^1, dx^2 and dx^3 of the flat relativistic metric spacetime (Σ, ct) at every point in spacetime at the neighborhood of every gravitational field source, in addition to transformations of proper parameters in (Σ', ct') into gravitational-relativistic parameters in (Σ, ct) in the context of TGR. The accomplishment of an analytical approach TGR must therefore start with the derivations of elementary metric spacetime coordinate interval transformation. Although this has been accomplished by the graphical approach in the first part of this paper [4], it shall be re-done analytically in this second part for completeness.

1.1 Gravitational local Lorentz transformation and gravitational local Lorentz invariance in the context of TGR

Let us consider a proper (or primed) local Lorentz frame on the flat proper spacetime (Σ', ct') (within which gravitational velocity $\vec{V}_g'(r')$ is constant), which is located at radial distance r' from the center of the assumed spherical rest mass M_0 of a gravitational field source in the proper Euclidean 3-space Σ'. (We shall restrict to spherically symmetric gravitational fields until the Maxwellian theory of gravity (MTG) is developed in later papers and non-spherically symmetric gravitational field sources shall be considered.)

Let us prescribe elementary metric spacetime coordinate intervals $cdt', dr', r'd\theta'$ and $r' \sin\theta' d\varphi'$ within the primed local Lorentz frame. Whether a particle is located within this proper local Lorentz frame or not, the primed metric coordinate intervals will, by virtue of the presence of gravitational speed $V_g'(r')$ within the local Lorentz frame, transform into relativistic (or unprimed) metric coordinate intervals cdt , dr , $rd\theta$ and $r\sin\theta d\varphi$, within the unprimed local Lorentz frame on the flat relativistic spacetime (Σ, ct), at the corresponding radial distance r from the center of the gravitational-relativistic mass M of the gravitational field source in the relativistic Euclidean 3-space Σ.

Thus there is a transformation of the proper (or primed) elementary metric coordinate intervals cdt' , dr' , $r'd\theta'$ and $r' \sin\theta' d\varphi'$) into relativistic (or unprimed) metric coordinate intervals cdt , dr , $rd\theta$ and $r\sin\theta d\varphi$) of the form $\mathbf{dx}' \to \mathbf{dx} = \mathbf{L}_g \mathbf{dx}$, in the context of the theory of gravitational relativity (TGR) at every point within every local Lorentz frame on flat spacetime in every gravitational field. We shall now determine the matrix $\mathbf{L}_g$ of this transformation.

It is appropriate to recall the further discussion on the concept of gravitational velocity $\vec{V}_g'(r')$ in sub-section 1.2 of the first part of this paper [4]. In a nutshell,

1. The gravitational (or static) velocity $\vec{V}_g'(r')$ serves the role in TGR that dynamical velocity $\vec{v}$ serves in SR. Thus $\vec{V}_g'(r')$ effects TGR on the flat relativistic specetime (Σ, ct) (that evolves in the context od TGR), just as $\vec{v}$ effects SR on (Σ, ct) within local Lorentz frames in every gravitational field;
2. The gravitational velocity is invariant, that is, $\vec{V}_g(r) = \vec{V}_g'(r')$ in the context of TGR, just as the dynamical velocity $\vec{v}$ is invariant, that is $\vec{v} = \vec{v}$ (between particle's and observer's frames) in the context of SR;
3. The gravitational velocity $\vec{V}_g'(r')$ points radially towards the centroid of every gravitational field source, spherically-symmetric or not, corresponding to the fact that the dynamical velocity $\vec{v}$ of relative motion of every pair of frames and their coordinates along which relative motion occurs, to be taken as $\tilde{x}'$ and $\tilde{x}$ or $\tilde{x}$ and $\tilde{\tilde{x}}$) always by convention, are naturally collinear (i.e. $\vec{v}$, $\tilde{x}'$ and $\tilde{x}$ are naturally collinear) and not by assumption or prescription;
4. The maximum over all gravitational velocities that can be established at a point in space by a gravitational field source or a combination of gravitational field sources, or which can be acquired in space by particles and bodies, including massless gravitons, is $c_g = 3 \times 10^8$ m/s, corresponding to the maximum over all dynamical velocities of 3×10^8 m/s, which particles and bodies, including massless photons, can attain in motion.

Now just as the speed c_γ (of electromagnetic waves) is a constant in all local Lorentz frames on the flat spacetime (Σ, ct) in a gravitational field, so is the speed c_g (of gravitational waves) a constant in all local Lorentz frames on the flat spacetime (Σ, ct) in a gravitational field. We shall make this subject more humorous and the formal analogy between TGR and SR more striking by starting the analytical approach to TGR with the following statements of two principles of the theory of gravitational relativity

1. Natural laws (gravitational and non-gravitational) are invariant with local Lorentz frame.
2. The speed of gravitational waves is a constant, $c_g = 3 \times 10^8\,\mathrm{m\,s^{-1}}$, in all local Lorentz frames.

These are the counterparts in TGR of the two principles of the special theory of relativity, which Albert Einstein started with in deriving the Lorentz transformation

(LT) and its inverse in his 1905 special relativity paper [5, 1923]. The validity of the first statement shall be demonstrated in a paper shortly in this volume.

One important aspect of the close analogy between SR and TGR, which follows from the two principles of TGR above, is that there exists a Lorentz-like transformation of elementary coordinate intervals cdt', dr', $r'd\theta'$ and $r'\sin\theta' d\varphi'$ within a proper (or primed) local Lorentz frame on the flat proper spacetime (Σ', ct') into the elementary unprimed coordinate intervals cdt, dr, $rd\theta$ and $r\sin\theta d\varphi$ within the corresponding relativistic (or unprimed) local Lorentz frame on the flat relativistic spacetime (Σ, ct) and its inverse, in terms of gravitational speed $V'_g(r')$, at every point in spacetime within every local Lorentz frame in a gravitational field of arbitrary strength, in the context of TGR. This corresponds to Lorentz transformation and its inverse in terms of dynamical speed v in the context of SR.

Thus the desired transformation and its inverse in the context of TGR, at an arbitrary radial distance r' from the center of the assumed spherical rest mass M_0 of the gravitational field source in the proper Euclidean 3-space Σ', which correspond to radial distance r from the center of the gravitational-relativistic mass M of the gravitational field source in the relativistic Euclidean 3-space Σ of TGR, should be linear in the elementary coordinate intervals, like the the Lorentz transformation and its inverse in SR. In other words, the elementary coordinate interval transformation at every point in spacetime within every local Lorentz frame in the context of TGR should take on the following form,

$$\left.\begin{array}{rcl} dt' &=& Adt - Bdr;\;\; dr' = Cdr - Ddt; \\ r'd\theta' &=& rd\theta \;\text{ and }\; r'\sin\theta' d\varphi' = r\sin\theta d\varphi \end{array}\right\} \tag{1}$$

and its inverse,

$$\left.\begin{array}{rcl} dt &=& Adt' + Bdr';\, dr = Cdr' + Ddt'; \\ rd\theta &=& r'd\theta' \;\text{ and }\; r\sin\theta d\varphi = r'\sin\theta' d\varphi' \end{array}\right\} \tag{2}$$

where A, B, C and D are functions of the gravitational speed $V'_g(r')$.

By using the general forms of coordinate interval transformation (1) and its inverse (2) along with the analogy between the roles of the gravitational speed in TGR and dynamical speed in SR, we must simply replace the dynamical speed v by gravitational speed $V'_g(r')$ and the speed of light $c \equiv c_\gamma$ by the speed of gravitational waves c_g in the Lorentz transformation (the Lorentz boost) in SR to obtain the explicit forms of systems (1) and (2) respectively as follows

$$\left.\begin{aligned} dt' &= \gamma_g(r')\left(dt - \frac{V'_g(r')}{c_g^2}dr\right); \\ dr' &= \gamma_g(r')(dr - V'_g(r')dt); \\ &r'd\theta' = rd\theta;\;\; r'\sin\theta' d\varphi' = r\sin\theta d\varphi \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} dt &= \gamma_g(r')\left(dt' + \frac{V'_g(r')}{c_g^2}dr'\right); \\ dr &= \gamma_g(r')(dr' + V'_g(r')dt'); \\ &rd\theta = r'd\theta';\;\; r\sin\theta d\varphi = r'\sin\theta' d\varphi' \end{aligned}\right\} \tag{4}$$

where

$$\gamma_g(r') = (1 - V'_g(r')^2/c_g^2)^{-1/2} \tag{5}$$

Also by using the established relation, $V'_g(r')^2 = 2GM_{0\mathrm{a}}/r'$, known since [2], systems (3) and (4) can be put in the following alternative forms

$$\left.\begin{aligned} dt' &= \gamma_g(r')\left(dt - \sqrt{\frac{2GM_{0\mathrm{a}}}{r'c_g^2}dr}\right); \\ dr' &= \gamma_g(r')\left(dr - \sqrt{\frac{2GM_{0\mathrm{a}}}{r'}dt}\right); \\ &r'd\theta' = r\sin\theta d\varphi;\;\; r'\sin\theta' d\varphi = r\sin\theta d\varphi \end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned} dt &= \gamma_g(r')\left(dt' + \sqrt{\frac{2GM_{0\mathrm{a}}}{r'c_g^2}dr'}\right); \\ dr &= \gamma_g(r')\left(dr' + \sqrt{\frac{2GM_{0\mathrm{a}}}{r'}dt'}\right); \\ &rd\theta = r'\sin\theta' d\varphi';\;\; r\sin\theta d\varphi = r'\sin\theta' d\varphi' \end{aligned}\right\} \tag{7}$$

where

$$\gamma_g(r') = \left(1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2}\right)^{-1/2} \tag{8}$$

Although this second part of this article shall be made independent of the first part on graphical approach as much as possible, it should be recalled that the coordinate transformation (3) and its inverse (4) or their explicit forms (6) and (7) in the context of TGR, have been derived graphically *via* the graphical approach to the intrinsic theory of gravitational relativity (ϕTGR) in part one of this paper [4], and referred to as gravitational local Lorentz transformation (GLLT) and its inverse in that paper, as shall also be done in this second part.

System (3) or (4) or the alternative form (6) or (7) leads to gravitational local Lorentz invariance (GLLI)

$$c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) = c^2dt'^2 - dr'^2 - r'^2(d\theta'^2 + \sin^2\theta' d\varphi'^2) \qquad (9)$$

The elementary metric coordinate intervals cdt' , dr' , $r'd\theta'$ and $r'\sin\theta'd\varphi'$ can be taken about every point in spacetime within the proper (or primed) local Lorentz frame on the flat proper spacetime (Σ', ct') and the elementary metric coordinate intervals cdt , dr , $rd\theta$ and $r\sin\theta d\varphi$ can be taken about every point in spacetime within the corresponding relativistic (or unprimed) local Lorentz frame on the flat relativistic spacetime (Σ, ct) in the GLLT of system (3) and its inverse (4) or their alternative forms (6) and (7). Consequently the GLLI (9) is valid at every point in spacetime within every local Lorentz frame in a gravitational field of arbitrary strength. This guarantees that the metric four-dimensional relativistic spacetime (Σ, ct) that evolves in the context of the gravitational theory of relativity (TGR), is flat (with constant Lorentzian metric tensor) in all finite neighborhood of every gravitational field source and by extension, in the entire universe.

The analytical approach to the derivation of the GLLT and its inverse in TGR done in this sub-section, corresponds to the analytical approach to the derivation of the Lorentz transformation (LT) and its inverse in SR, developed by Albert Einstein in 1905 [5, ibid.], as mentioned earlier. There is also the graphical approach to the derivation of the GLLT and its inverse *via* the graphical approach to the derivation of ϕGLLT and its inverse in the context of ϕTGR, which corresponds to the graphical approach to the derivation of local Lorentz invariance transformation (LLT) and its inverse in a gravitational field in SR *via* the graphical approach to the derivation of ϕLLT and its inverse in the context of ϕSR, both of which have been presented in the first part of this paper [4].

A comparison of the graphical and the analytical approaches to the derivations of the GLLT and its inverse in TGR and the LLT and its inverse in SR, shows that the graphical approaches are greatly superior to and by far more complete that the analytical approaches. This is so because the common four-world backgrounds of GLLT and its inverse and LLT and its inverse revealed by the graphical approaches are completely obliterated in the analytical approaches. The entire ϕTGR and ϕSR on flat two-dimensional intrinsic spacetime underlying the flat four-dimensional spacetime in a gravitational field, encompassed by the graphical approaches, are likewise obliterated in the analytical approaches.

Also the concept of 3-observers in the Euclidean 3-space and 1-observers in the time dimension, who jointly derive the GLLT and its inverse in TGR and the LLT and its inverse in SR, revealed by the graphical approaches, are unknown in the analytical approaches. Evidently the knowledge of physics afforded by the analytical approaches is superficial.

It may be recalled that only the general theory of relativity (GR) on a prescribed curved four-dimensional spacetime in a gravitational field and the special theory of relativity (SR) on flat four-dimensional spacetime, both developed by Albert Einstein [5, 1923], existed until now. The analytical approaches to these existing theories have no bearing with the four-world picture and the existence of two-dimensional intrinsic spacetime $(\phi\rho, \phi c\phi t)$ that underlies the four-dimensional spacetime and the the intrinsic theories of relativity on the intrinsic spacetime, now discovered in the present theory. The graphical approaches to SR by H. Minkowski namely, the Minkowski diagrams within the existing one-world picture, which has been critiqued and concluded to be not suitable for relativity in [6], have no bearing with these newly discovered items either. On the other hand, there is no graphical approach to GR as far as I can find.

1.1.1 Gravitational length contraction and gravitational time dilation formulae implied by GLLT and its inverse

It is nature that establishes gravitational local Lorentz transformation (GLLT) (3) and its inverse (4) or their alternative forms (6) and (7) and the gravitational local Lorentz invariance (GLLI) (9) at every point in spacetime in a gravitational field. Some of the terms of the GLLT and its inverse will not appear in the coordinate transformation and its inverse that man can establish through measurement by laboratory rod and clock. First of all, we must collect the transformations derived by 3-observers in the relativistic Euclidean 3-space Σ (courtesy the graphical approach in [4]) in systems (3) and (4) to have as follows

$$dr' = \gamma_g(r')\left(dr - V_g'(r')dt\right);\ r'd\theta' = rd\theta;\ r'\sin\theta' d\varphi' = r\sin\theta d\varphi; \quad (10a)$$

$$dt = \gamma_g(r')\left(dt' + (V_g'(r')/c_g^2\right)dr'); \quad (10b)$$

(w.r.t. 3-observer in Σ).

Now when the observer in Σ picks his laboratory rod to measure the interval of space traversed by the event, he will be able to measure the term $\gamma_g(r')dr$ at the right-hand side of the first equation of system (10a), while the term $-\gamma_g(r')V_g'(r')dt$ will be non-measurable with his rod. He will be able to measure the terms at the right-hand sides of the second and third equations of system (10a) with his laboratory rod and protractor. Likewise when a 3-observer in Σ' picks his laboratory clock to measure the time interval of the event, he will be able to measure the term $\gamma_g(r')dt'$ at the right-hand side of Eq. (10b), while the term $\gamma_g(r')(V_g'(r')/c_g^2)dr'$ will be non-measurable by his clock.

Thus from the point of view of what can be measured with laboratory rod and clock by man, system (10a) and Eq. (10b) reduce as follows

$$dr = \gamma_g(r')dr'\,;\; rd\theta = r'd\theta'\,;\; r\sin\theta d\varphi = r'\sin\theta' d\varphi' \tag{11a}$$

$$dt = \gamma_g(r')dt' \tag{11b}$$

or

$$dr = (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{1/2}dr'\,;\; rd\theta = r'd\theta'\,; r\sin\theta d\varphi = r'\sin\theta' d\varphi' \tag{12a}$$

$$dt = (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{-1/2}dt' \tag{12b}$$

1.1.2 The exterior Schwarzschild line element in the general theory of relativity pertains to the measurable sub-space of the space of the theory of gravitational relativity

If the gravitational time dilation and gravitational length contraction formulae expressed by systems (12a) and Eq. (12b), which man can discover through measurements of the intervals of space and times of non-special-relativistic events in gravitational fields by atomic clock and laboratory rod, is all the information that is available to man, then man would have established the following from the invariance of geodesic

$$\begin{aligned} ds'^2 &= c^2dt'^2 - dr'^2 - r'^2(d\theta'^2 + \sin\theta'^2 d\varphi'^2) \\ &= (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})c^2dt^2 - (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ &= ds^2 \end{aligned}$$

Hence man would have written the following line element in empty space at the exterior of a spherically symmetric gravitational field source

$$ds^2 = (1 - \frac{2GM_{0\mathrm{a}}}{r' c_g^2})c^2 dt^2 - (1 - \frac{2GM_{0\mathrm{a}}}{r' c_g^2})^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (13a)$$

However the active gravitational mass $M_{0\mathrm{a}}$ should be replaced by the rest mass M_0 in the context of GR from the known equivalence of active gravitational mass and rest mass in GR [7,8]. The gravitational speed c_g of gravitational waves should also be replaced by the only known speed of light c in GR, thereby converting the line element (13b) to the following equivalent form in terms of the familiar rest mass and the familiar speed of signal c in GR

$$ds^2 = (1 - \frac{2GM_0}{r' c^2})c^2 dt^2 - (1 - \frac{2GM_0}{r' c^2})^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (13b)$$

Finally the invariance of gravitational potential, $-GM_0/r' = -GM/r$, of mass, $M = M_0$, and of other parameters in the context of GR, known to be implied by the principle of equivalence, would have been used to put the line element (13b) in its following final equivalent form in terms of inertial mass of the gravitational field source

$$ds^2 = (1 - \frac{2GM}{rc^2})c^2 dt^2 - (1 - \frac{2GM}{rc^2})^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (14)$$

Indeed the gravitational local Lorentz transformation (GLLT) (6) and its inverse (7) and gravitational Lorentz invariance (GLLI) (9) they imply have been unknown in physics until now. System (12a) and Eq. (12b) that have been discovered through measurements of the intervals of space and times of non-special-relativistic events in spherically symmetric gravitational fields by man has indeed been deemed to imply the metric tensor of Eq. (14) in a gravitational field in the general theory of relativity [9]. However this is apart from the fact that the Schwarzschild's solution to Einstein's free space field equations at the exterior of a spherical gravitational field source yields the line element (14) directly.

What can be concluded from the foregoing is that the exterior Schwarzschild line element in GR pertains to the measurable sub-space (a curved four-dimensional spacetime with sub-Riemannian metric tensor), which supports gravitational length contraction and gravitational time dilation, of the space of the theory of gravitational relativity (TGR) namely, the flat four-dimensional spacetime with constant Lorentzian metric tensor. Despite this conclusion however, there is nothing so far in the present theory to rule out the general theory of relativity (GR) as a possible geometrical model of gravity on an hypothetical curved spacetime in a gravitational field. Our focus in the present theory shall not be on debasing or repudiating the GR, but on formulating a fundamental (or natural) and complete theory of gravity.

As a final remark in this sub-section, the gravitational local Lorentz transformation (GLLT) of system (3) and its inverse of system (4) (or systems (6) and (7)) have important application consequences in TGR, quite apart from establishing gravitational local Lorentz invariance (GLLI) from them above. They are as valuable in TGR as Lorentz transformation (LT) and its inverse in SR. For instance, gravitational length contraction and gravitational time dilation formulae have been deduced from them above.

The transformations of physical parameters and physical constants in the context of TGR shall likewise be derived with the aid of the GLLT and its inverse in the next sub-section. The GLLT and its inverse are the relevant transformations for deriving the superposition at a point in space of the resultant gravitational field of several gravitational field sources that are scattered in space about that point, as shall be done elsewhere with further development.The GLLT and its inverse shall find application in the transformations of the classical and special-relativistic non-gravitational laws, as well as classical gravitational law, in the context of TGR in a paper shortly in this volume.

1.2 Transformations of physical parameters in the context of the gravitational theory of relativity

The transformations of physical parameters in the context of TGR shall be derived with the aid of the gravitational local Lorentz transformation (GLLT) within local Lorentz frames in a gravitational field of system (3) or (6) and its inverse of system (4) or (7), in addition to the derivation of gravitational time dilation and gravitational length contraction formulae from them earlier. The results of TGR shall be derived with the aid of the GLLT and its inverse, in analogy to the derivation of the results of SR with the aid of the LT and its inverse in SR.

1.2.1 *Transformations of mass, linear momentum, kinetic energy, force, angular momentum and torque in the context of TGR*

Let us divide the second equation into the first equation of the inverse GLLT of system (4) to have as follows

$$\frac{dr}{dt} = \frac{\gamma_g(r')(dr' + V_g'(r')dt')}{\gamma_g(r')(dt' + \frac{V_g'(r')}{c_g^2}dr')}$$

or

$$\frac{dr}{dt} = \frac{\frac{dr'}{dt'} + V_g'(r')}{1 + \frac{V_g'(r')}{c_g^2}\frac{dr'}{dt'}} \tag{15}$$

We shall consider a test particle in motion within the proper (or primed) local Lorentz frame on the flat proper spacetime (Σ', ct') and take dr'/dt' as the dynamical speed v_r' of the rest mass m_0 of the particle within the proper local Lorentz frame, while dr/dt is the resultant of the dynamical speed v_r' and gravitational speed $V_g'(r')$ in the proper local Lorentz frame with respect to observers in the relativistic (or unprimed) local Lorentz frame on the flat relativistic spacetime (Σ, ct) of TGR. Then Eq. (15) becomes the following

$$v_r = \frac{v_r' + V_g'(r')}{1 + \frac{V_g'(r')v_r'}{c_g^2}} \tag{16}$$

Now the gravitational speed $V_g'(r')$ is an absolute speed from the point of view of dynamics, since it is not made manifest in actual translation of a particle that acquires it and since it is the same relative to all observers or frames of reference. The dynamical speed v_r' must likewise be an absolute speed for the composition of $V_g'(r')$ and v_r' to be possible. The speed v_r' must be the non-detectable absolute speed of the rest mass m_0 of the particle relative to its frame.

Let us put the discussion in the foregoing paragraph in perspective. Let us temporarily write the intrinsic form in two-dimensional intrinsic spacetime of Eq. (16) as follows

$$\phi v_r = \frac{\phi v_r' + \phi V_g'(\phi r')}{1 + \frac{\phi V_g'(\phi r')\phi v_r'}{\phi c_g^2}} \tag{17}$$

Again the intrinsic dynamical speed $\phi v_r'$ must be an absolute intrinsic speed for the composition of $\phi V_g'(\phi r')$ and $\phi v_r'$ to be possible.

Graphically, let us consider the lower half of the first quadrant in Fig. 1 of [4]. The intrinsic rest mass ϕm_0 of the test particle lying along the curved proper intrinsic space $\phi\rho'$, acquires the proper intrinsic gravitational speed $\phi V_g'(\phi r')$ established at its location by the intrinsic rest mass ϕM_0 of the gravitational field source at the origin of the curved $\phi\rho'$. It (i.e. ϕm_0) also acquires the absolute intrinsic gravitational speed $\phi\hat{V}_g(\phi\hat{r})$ that is invariantly projected into its location along the curved $\phi\rho'$ by $\phi\hat{V}_g(\phi\hat{r})$ along the curved absolute intrinsic space $\phi\hat{\rho}$ and possesses the absolute intrinsic dynamical speed $\phi\hat{V}_d$ that is also invariantly projected into its location along the curved $\phi\rho'$ by the absolute intrinsic rest mass $\phi\hat{m}_0$ of the test particle in absolute intrinsic motion along the curved absolute intrinsic space $\phi\hat{\rho}$. The three intrinsic speeds $\phi V_g'(\phi r')$, $\phi\hat{V}_g(\phi\hat{r})$ and $\phi\hat{V}_d$ at the location of ϕm_0 along the curved $\phi\rho'$ (which ϕm_0 acquires), are indicated in Fig. 1 of [4].

Let us re-denote the absolute intrinsic dynamical speed $\phi\hat{V}_d$ along the curved $\phi\rho'$ at the location of ϕm_0 within the proper (or primed) intrinsic local Lorentz frame on the curved proper intrinsic spacetime $(\phi\rho', \phi c\phi t')$ in Fig. 1 of [3] by $\phi V_d'$. It is the proper intrinsic gravitational speed $\phi V_g'(\phi r')$ and the absolute intrinsic dynamical speed $\phi V_d'$ acquired by the intrinsic rest mass ϕm_0 at its location along the curved $\phi\rho'$ that can be composed according to the formula of Eq. (17), yielding the resultant absolute intrinsic speed $\phi V_d' \equiv \phi\hat{V}_d$ relative to the relativistic (or unprimed) intrinsic local Lorentz frame on the flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ in Fig. 1 of [4]. In other words, we must replace $\phi v_r'$ by $\phi V_d'$ and ϕv_r by ϕV_d in Eq. (17) to have

$$\phi\hat{V}_d = \frac{\phi V_d' + \phi V_g'(\phi r')}{1 + \dfrac{\phi V_g'(\phi r')\phi V_d'}{\phi c_g^2}} \tag{18}$$

Since Eq. (18) contains only absolute intrinsic speeds from the point of view of intrinsic dynamics (or from the point of view of intrinsic special theory of relativity), it is a valid expression. The outward manifestation in spacetime of Eq. (18), which must be obtained by simply removing the symbol ϕ, is the following

$$V_d = \frac{V_d' + V_g'(r')}{1 + \dfrac{V_g'(r')V_d'}{c_g^2}} \tag{19}$$

This equation is valid for all values of $V_d' \leq c_\gamma$ and all values of $V_g'(r') \leq c_g$.

Since relative motion in the context of primed special theory of relativity (SR′) on flat proper spacetime (Σ', ct'), on which Eq. (19) has effectively been written, is absent, let us multiply through Eq. (19) by the rest mass m_0 of the test particle to have

$$m_0 V_d = \frac{m_0(V'_d + V'_g(r'))}{1 + \frac{V'_g(r')V'_d}{c_g^2}} \tag{20}$$

Let us now consider the rest mass of the test particle to be in accelerated absolute motion from an initial absolute speed V'_{di} to a final absolute speed V'_{df} within its local Lorentz frame within which gravitational speed is constant at $V'_g(r')$, which is located at radial distance r' from the center of the rest mass M_0 of the gravitational field source in the proper Euclidean 3-space Σ'. This corresponds to allowing the absolute intrinsic speed $\phi\hat{V}_d$ of the absolute intrinsic rest mass $\phi\hat{m}_0$ of the particle along the curved absolute intrinsic space $\phi\hat{\rho}$ to vary from $\phi\hat{V}_{di}$ to $\phi\hat{V}_{df}$ in Fig. 1 of [4]. Then Eq. (20) can be written separately for V'_{di} and V'_{df} and the following expression for relative momentum of the test particle obtained from the difference of the resulting equations

$$\begin{aligned} p &= m_0(V_{df} - V_{di}) \\ &= m_0\left(\frac{V'_{df} + V'_g(r')}{1 + \frac{V'_g(r')V'_{df}}{c_g^2}} - \frac{V'_{di} + V'_g(r')}{1 + \frac{V'_g(r')V'_{di}}{c_g^2}}\right) \end{aligned}$$

or

$$p = \frac{m_0\left(1 - \frac{V'_g(r')^2}{c_g^2}\right)(V'_{df} - V'_{di})}{1 + \frac{V'_g(r')}{c_g}\frac{V'_{df}}{c_g} + \frac{V'_g(r')}{c_g}\frac{V'_{di}}{c_g} + \frac{V'_g(r')^2}{c_g^2}\frac{V'_{df}}{c_g}\frac{V'_{di}}{c_g}} \tag{21}$$

Now the difference of two non-detectable absolute dynamical speeds of the rest mass m_0 in the proper Euclidean 3-space Σ' is a detectable relative dynamical speed. Thus let replace $V'_{df} - V'_{di}$ by a relative dynamical speed v in Eq. (19) to have

$$p = \frac{m_0\left(1 - \frac{V'_g(r')^2}{c_g^2}\right)v}{1 + \frac{V'_g(r')}{c_g}\frac{V'_{df}}{c_g} + \frac{V'_g(r')}{c_g}\frac{V'_{di}}{c_g} + \frac{V'_g(r')^2}{c_g^2}\frac{V'_{df}}{c_g}\frac{V'_{di}}{c_g}} \tag{22}$$

The speed v is a speed of relative motion, hence the momentum p is a momentum of relative motion in Eq. (22). Thus Eq. (22) is an expression in the context of relative motion. Then the absolute dynamical speeds V'_{df} and V_{di} of whatever magnitudes in the denominator in Eq. (22) must be allowed to vanish, which is so since any magnitude of each of these absolute dynamical speeds is equivalent to zero magnitude of relative dynamical speed of relative motion. The gravitational speed $V'_g(r')$ in the denominator in Eq. (22) must also be allowed to vanish since $V'_g(r')$ is an absolute speed in the context of relative motion, since any magnitude of $V'_g(r')$ is equivalent to zero magnitude of relative dynamical speed.

On the other hand, the factor $(1 - V'_g(r')^2/c_g^2)$ in the numerator in Eq. (20) must be retained. This is so because $1 - V'_g(r')^2/c_g^2 = (1/c_g^2)(c_g^2 - V'_g(r')^2)$ is a measurable or observable quantity in space, since $c_g^2 - V'_g(r')^2$ is a measurable difference of gravitational potential, as follows from the relation of gravitational speed to gravitational potential $(\Phi'(r') = -\frac{1}{2}V'_g(r')^2)$ in sub-section 2.1 of [2]. Therefore $-\frac{1}{2}c_g^2$ is also a gravitational potential Φ_0 (corresponding to a field g_0), which is localized at a point of zero extension at the center of the gravitational field source.

Equation (20) simplifies as follows by virtue of the foregoing two paragraphs

$$\vec{p} = m_0 \left(1 - \frac{V'_g(r')^2}{c_g^2} \right) \vec{v} \tag{23}$$

The vector sign has been introduced on $\vec{v}$ and $\vec{p}$ since there are 3-vectors in the Euclidean 3-space. The primed (or Newtonian) momentum $\vec{p}\,' = m_0\vec{v}$ is measured in the proper (or primed) local Lorentz frame on flat proper spacetime (Σ', ct') (in Fig. 11 of [1]), while the gravitational-relativistic (or unprimed) momentum $\vec{p}$, given by the right-hand side of Eq. (23), is measured in the relativistic (or unprimed) local Lorentz frame on flat relativistic spacetime (Σ, ct) in Fig. 1 of [4]. In other words, $\vec{p}$ is the gravitational-relativistic momentum on the flat relativistic spacetime in the context of TGR, while $\vec{p}\,' = m_0\vec{v}$ is the momentum of primed classical mechanics (CM′) (assuming $\vec{v}$ is a low velocity) on the flat proper spacetime (Σ', ct').

In order to further elucidate the foregoing discussion, let us write the intrinsic form in two-dimensional intrinsic spacetime of Eq. (13) as follows

$$\phi p = \phi m_0 \left(1 - \frac{\phi V'_g(\phi r')^2}{\phi c_g^2} \right) \phi v \tag{24}$$

where $\phi m_0 \phi v = \phi p'$ is the intrinsic momentum within the proper intrinsic local Lorentz frame on curved proper intrinsic spacetime $(\phi\rho', \phi c\phi t')$ and ϕp given by

Eq. (24) is the gravitational-relativistic intrinsic momentum projected into the corresponding relativistic (or unprimed) intrinsic local Lorentz frame on the flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ in Fig. 1 of [4].

The expression (23) for momentum in the context of TGR looks different from the usual Newtonian form because of the appearance of the factor $(1 - V_g'(r')^2/c_g^3)$. However according to the second principle of the theory of gravitational relativity stated earlier, non-gravitational laws and expressions must retain their usual classical and special-relativistic forms in the context of TGR, except that the proper (or classical) and special-relativistic physical parameters that appear in the usual forms of the laws and expressions must be replaced by their gravitational-relativistic forms in the context of TGR.

Let us then convert the expression (23) to its usual form in classical mechanics by introducing a new mass m of the particle in the context of TGR through the following relation,

$$m = \gamma_g(r')^{-2} m_0 = m_0(1 - \frac{V_g'(r')^2}{c_g^2}) \tag{25}$$

or

$$m = \gamma_g(r')^{-2} m_0 = m_0(1 - \frac{2GM_{0\mathrm{a}}}{r' c_g^2}) \tag{26}$$

Eq. (24) then takes its usual Newtonian form in terms of the new mass m as follows

$$\vec{p} = m\vec{v} \tag{27}$$

While the rest mass m_0 of the test particle is observed in the proper Euclidean 3-space Σ' of the flat proper spacetime (Σ', ct') in Fig. 11 of [1] at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field, although that diagram does not exist in reality, the mass m given by Eq. (25) or (26) is observed in the relativistic Euclidean 3-space Σ of the flat relativistic spacetime (Σ, ct) of Fig. 1 of [4], which evolves at combined first and second stages of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field. Eqs. (25) or (26) has been derived by the graphical approach in the first part of this paper [4] and the mass m has been referred to as the gravitational-relativistic mass.

The corresponding intrinsic mass relation $\phi m_0 = \phi\hat{m}_0 \times (1 - 2G\phi\hat{M}_{0\mathrm{a}}/\phi\hat{r}\phi\hat{c}_g^2)$ in the context of the theory of absolute intrinsic gravity (ϕAG), on the curved '2-dimensional' absolute intrinsic metric spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$, with absolute intrinsic sub-Riemannian metric tensor $\phi\hat{g}_{ik}$, in Fig. 3 of [1], which corresponds to the mass relation $m_0 = \hat{m}_0(1 - 2G\hat{M}_{0\mathrm{a}}/\hat{r}\hat{c}_g^2)$ in the context of the theory of absolute gravity (AG) on the flat proper spacetime (Σ', ct') in Fig. 3 of [1], has also been derived from the absolute intrinsic geodesic line on the curved $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$, by an absolute intrinsic tensorial approach in sub-sub-section 2.1.2 of [1]. We have therefore derived the same mass relation by three different approaches so far.

The transformation of the classical linear momentum in the context of TGR that follows from Eq. (23) is

$$\begin{aligned} \vec{p} &= (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})m_0\vec{v} \\ &= (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})\vec{p}\,' \end{aligned} \tag{28}$$

The transformation of classical kinetic energy that follows from Eq. (28) is

$$\begin{aligned} E_K &= (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})(\frac{1}{2}m_0v^2) \\ &= (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})E'_K \end{aligned} \tag{29}$$

Let us differentiate both sides of Eq. (28) with respect to unprimed time t of the flat relativistic spacetime (Σ, ct) of TGR. Knowing that the factor $(1 - 2GM_{0\mathrm{a}}/r'c_g^2)$ is time-independent we have

$$\frac{d\vec{p}}{dt} = (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})\frac{d\vec{p}\,'}{dt} \tag{30}$$

Eq. (30) is not in its final form. In order to obtain its final form, we must obtain the transformation of the differential operator d/dt into d/dt' in the context of TGR in $d\vec{p}\,'/dt$ at the right-hand side, so that primed momentum $\vec{p}\,'$ and primed time coordinate interval dt' in the proper (or primmed) local Lorentz frame on flat proper spacetime (Σ', ct') appear at the right-hand side.

As established under the Appendix, the following trivial transformations of elementary coordinate interval and coordinates within local Lorentz frames in a gravitational field and the implied trivial transformations of differential operators must be used in deriving the transformations of non-gravitational laws in the context of TGR.

$$dt = dt'; \;\; dx = dx'; \;\; dy = dy'; \;\; dz = dz' \tag{31}$$

or

$$t = t'; \quad x = x'; \quad y = y'; \quad z = z' \tag{32}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}; \quad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t'^2}; \quad \vec{\nabla} = \vec{\nabla}'; \quad \nabla^2 = \nabla'^2; \text{ etc} \tag{33}$$

It must be remembered that the elementary coordinate intervals and coordinates in systems (31) and (32) are limited in extensions to the interior of local Lorentz frames in an external gravitational field.

The final form of Eq. (30) must be obtained by replacing d/dt by d/dt' at the right-hand side of that equation by virtue of the operator transformations of system (33) to have

$$\frac{d\vec{p}}{dt} = (1 - \frac{2GM_{0a}}{r'c_g^2})\frac{d\vec{p}\,'}{dt'}$$

or

$$\vec{F} = (1 - \frac{2GM_{0a}}{r'c_g^2})\vec{F}' \tag{34}$$

This is the transformation of inertial force in the context of TGR.

Let us obtain the vector product of the unprimed coordinate 3-vector $\vec{q} = x\hat{i} + y\hat{j} + z\hat{k}$ within the unprimed local Lorentz frame on flat relativistic spacetime (Σ, ct) and the momenta $\vec{p}$ and $\vec{p}\,'$ in Eq. (28) as follows

$$\vec{q} \times \vec{p} = (1 - \frac{2GM_{0a}}{r'c_g^2})\vec{q} \times \vec{p}\,'$$

or

$$\vec{L} = (1 - \frac{2GM_{0a}}{r'c_g^2})\vec{q} \times \vec{p}\,' \tag{35}$$

In order to obtain the final form in Eq. (35), we must obtain the transformation of $\vec{q}$ into $\vec{q}\,'$ at the right-hand side, which from system (32), is given trivially as $\vec{q} = \vec{q}\,'$. Hence the final form of Eq. (35) must be obtained by replacing $\vec{q}$ by $\vec{q}\,'$ as follows

$$\vec{L} = (1 - \frac{2GM_{0a}}{r'c_g^2})\vec{q}\,' \times \vec{p}\,'$$

or

$$\vec{L} = (1 - \frac{2GM_{0a}}{r'c_g^2})\vec{L}' \tag{36}$$

This is the transformation of angular momentum in the context of TGR.

Let us differentiate Eq. (36) with respect to the unprimed time t of the unprimed local Lorentz frame on the flat four-dimensional relativistic spacetime (Σ, ct) as follows

$$\frac{d\vec{L}}{dt} = (1 - \frac{2GM_{0\mathrm{a}}}{r' c_g^2})\frac{d\vec{L}'}{dt} \tag{37}$$

We must then replace d/dt by d/dt' at the right-hand side of (37) by virtue of system (33) to have

$$\frac{d\vec{L}}{dt} = (1 - \frac{2GM_{0\mathrm{a}}}{r' c_g^2})\frac{d\vec{L}'}{dt'}$$

or

$$\vec{\tau} = (1 - \frac{2GM_{0\mathrm{a}}}{r' c_g^2})\vec{\tau}\,' \tag{38}$$

This is the transformation of torque in the context of TGR.

1.2.2 Invariance of active gravitational mass (or gravitational charge) in the context of TGR

The immaterial negative active gravitational mass (or gravitational charge) $-M_{0\mathrm{a}}$ of a body, which is imperceptibly contained within the rest mass of a body, is the source of gravitational velocity, Newtonian gravitational potential and Newtonian gravitational field of the body in the context of the present theory, as has been introduced since [1]. However the origin and the negative sign of the gravitational charge, as well as the model of how it is contained within the rest mass and the mechanism by which it establishes gravitational speed, gravitational potential and gravitational field at every point in space in all finite neighborhood of the body are yet to be explained.

The immaterial gravitational charge is absolute-absolute. It is invariant in the context of TGR. This is expressed as follows

$$-M_{\mathrm{a}} = -M_{0\mathrm{a}} \tag{39}$$

where $-M_{0\mathrm{a}}$ is the proper (or classical) gravitational charge within the rest mass M_0 in the proper Euclidean 3-space Σ' and $-M_{\mathrm{a}}$ is the gravitational-relativistic gravitational charge within the gravitational-relativistic mass M in the relativistic Euclidean 3-space Σ of TGR. The invariance (39) is another property of the gravitational charge to be explained elsewhere with further development. Recall that the invariance (39) has been discussed more fully in sub-section 3.1 of [3] and summarized in Table I of that paper.

The absolute-absolute active gravitational mass (or gravitational charge) is also invariant in the context of SR. Its correspondence in electromagnetism namely, the electric charge, is likewise absolute-absolute and is invariant in the context of TGR and SR.

1.2.3 Transformations of gravitational and inertial accelerations and dynamical and gravitational velocity in the context of TGR

The force $\vec{F}'$ in Eq. (34) can be any non-gravitational force acting on the rest mass of the particle in the proper local Lorentz frame on flat proper spacetime (Σ', ct'). Also if we let $\vec{F}' = m_0\vec{a}'$ and apply the equivalence of inertial acceleration and gravitational acceleration, $\vec{a}' = \vec{g}'$, then $\vec{F}'$ can be replaced by gravitational force $\vec{F}' = m_0\vec{g}'$ in Eq. (34).

It also follows from the end of the last paragraph that $\vec{F}'$ in Eq. (34) can be a test gravitational force $\vec{F}'_{g\mathrm{t}}$ acting on the rest mass m_0 of the test particle from another test particle or object of rest mass $m_{0\mathrm{t}}$ nearby in the external gravitational field of the source with rest mass M_0. The force $\vec{F}'$ in Eq. (34) can also be the gravitational force $\vec{F}'_g$ on the rest mass m_0 of the test particle from the external gravitational field source of rest mass M_0. Thus Eq. (34) can be written as transformations of gravitational forces as follows

$$\vec{F}_g = (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})\vec{F}'_g \tag{40}$$

and

$$\vec{F}_{g\mathrm{t}} = (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})\vec{F}'_{g\mathrm{t}} \tag{41}$$

or

$$m\vec{g} = (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})m_0\vec{g}' \tag{42}$$

and

$$m\vec{g}_\mathrm{t} = (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})m_0\vec{g}'_\mathrm{t} \tag{43}$$

Then by using the mass relation (26) in Eqs. (42) and (43) we have

$$\vec{g} = \vec{g}' \tag{44}$$

and

$$\vec{g}_\mathrm{t} = \vec{g}'_\mathrm{t} \tag{45}$$

These are the transformations in the context of TGR of the gravitational field (or acceleration) due to the source of the external gravitational field and due to a test gravitational field source located in the external gravitational field respectively.

The transformations of linear momentum and inertial force in the context of TGR derived earlier are the following

$$\vec{p} = m\vec{v} = (1 - 2GM_{0\mathrm{a}}/r'c_g^2)m_0\vec{v}$$

and

$$\vec{F} = m\vec{a} = (1 - 2GM_{0\mathrm{a}}/r'c_g^2)m_0\vec{a}$$

By using the mass relation (26) in these expressions we obtain the transformations of dynamical velocity and inertial acceleration in the context of TGR as follows

$$\vec{v} = \vec{v}' \text{ and } \vec{a} = \vec{a}' \tag{46}$$

Interestingly there is also the invariance of gravitational velocity in the context of TGR, as has been claimed without proof since [2]. We are still not at the point of providing the proof of the invariance of the gravitational velocity in the context of TGR at this point either, but shall simply re-write it as follows

$$\vec{V}_g(r) = \vec{V}_g'(r') \tag{47}$$

1.2.4 Transformations of gravitational potential in the context of TGR

Since the gravitational potential is time-independent and a function of the radial coordinate only in static spherically-symmetric gravitational fields being considered at present, let us express the derivative of the relativistic gravitational potential $\Phi(r)$ in the relativistic Euclidean 3-space Σ with respect to the radial coordinate r as follows

$$\frac{d\Phi(r)}{dr} = \frac{d\Phi(r)}{dr'}\frac{dr'}{dr} \tag{48}$$

Since this is an expression in the Newtonian law of gravity, its simplification must be done with the aid of the gravitational local Lorentz transformation (GLLT) and its inverse (and not with the aid of the trivial coordinate transformation of system (31), which will be appropriate if Eq. (48) is an expression in a non-gravitational law).

From the GLLT (3) we have

$$\frac{dr'}{dr} = \gamma_g(r') \tag{49}$$

By using this in Eq. (48) we have

$$\frac{d\Phi(r)}{dr} = \gamma_g(r')\frac{d\Phi(r)}{dr'} \tag{50}$$

Let us re-write the invariance of gravitational acceleration (or field) in the context of TGR of Eq. (44) as follows

$$-\frac{d\Phi(r)}{dr} = -\frac{d\Phi'(r')}{dr'} \tag{51}$$

This equation is valid since the vector equation (44) implies the scalar equation $g = g'$. The following ensues from Eqs. (50) and (51)

$$\gamma_g(r')\frac{d\Phi(r)}{dr'} = \frac{d\Phi'(r')}{dr'} \tag{52}$$

or

$$d\Phi(r) = \gamma_g(r')^{-1}d\Phi'(r')$$

Hence

$$\begin{aligned}\Phi(r') &= \gamma_g(r')^{-1}\Phi'(r') \\ &= \Phi'(r')(1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{1/2}\end{aligned} \tag{53}$$

or

$$\Phi(r') = -\frac{GM_{0\mathrm{a}}}{r'}(1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{1/2} \tag{54}$$

Eq. (53) expresses the transformation of the Newtonian gravitational potential in the context of TGR (or Eq. (54) gives the Newtonian gravitational potential expression in the context of TGR). The relativistic (or unprimed) gravitational potential function in the relativistic Euclidean 3-space Σ turns out to be a function of the proper (or primed) radial coordinate r' of the proper Euclidean 3-space Σ' in the final expression (53) or (54). Hence the gravitational-relativistic (or unprimed) gravitational potential function in Σ has been written as $\Phi(r')$ and not $\Phi(r)$. The classical (or primed) gravitational potential function in the proper Euclidean 3-space Σ' being denoted by $\Phi'(r')$.

The forms of Eq. (53) or (54) on two-dimensional intrinsic spacetime in the context of ϕTGR is the following

$$\phi\Phi(\phi r') = \phi\Phi'(\phi r')(1 - \frac{2G\phi M_{0\mathrm{a}}}{\phi r'\phi c_g^2})^{1/2} \tag{55}$$

or

$$\phi\Phi(\phi r') = -\frac{G\phi M_{0\mathrm{a}}}{\phi r'}(1 - \frac{2G\phi M_{0\mathrm{a}}}{\phi r'\phi c_g^2})^{1/2} \tag{56}$$

The proper (or primed) intrinsic gravitational potential $\phi\Phi'(\phi r')$ established along the curved proper intrinsic space $\phi\rho'$ by the intrinsic rest mass ϕM_0 of the gravitational field source at the origin of the curved $\phi\rho'$, at 'distance' $\phi r'$ along the curved $\phi\rho'$ from the base of ϕM_0 in Fig. 1 of [4], 'projects' the relativistic (or unprimed) intrinsic gravitational potential $\phi\Phi(\phi r')$ into the relativistic intrinsic space $\phi\rho$ along the horizontal, which appears to originate from the base of the gravitational-relativistic intrinsic mass ϕM in $\phi\rho$ of the gravitational field source, at 'distance' ϕr along $\phi\rho$ from the base of ϕM in the figure.

The intrinsic gravitational-relativistic mass ϕM of the gravitational field source in the straight line relativistic intrinsic space $\phi\rho$ along the horizontal in Fig. 1 of [4] is intrinsic inertial mass or intrinsic passive gravitational mass, as shall be shown shortly in his paper. It is not a source of intrinsic gravitational potential or field, which means that $\phi\Phi(r) = -G\phi M_{\mathrm{a}}/\phi r$ and $\phi g(\phi r) = -G\phi M_{\mathrm{a}}/\phi r^2$ that originate from ϕM do not exist along $\phi\rho$ along the horizontal in Fig. 1 of [4]. This has been well discussed in sub-section 3.1 of [4].

On the other hand, the intrinsic rest mass ϕM_0 of the gravitational field source on the curved proper intrinsic space $\phi\rho'$ is a source of intrinsic gravitational potential $\phi\Phi'(\phi r') = -G\phi M_{0\mathrm{a}}/\phi r'$ and intrinsic gravitational field $\phi g'(\phi r') = -G\phi M_{0\mathrm{a}}/\phi r'^2$ along the curved $\phi\rho'$ in Fig. 1 of [4]. Consequently only the non-uniform relativistic intrinsic gravitational potential $\phi\Phi(\phi r')$ given by Eq. (56), projected into $\phi\rho$ along the horizontal by the non-uniform $\phi\Phi'(\phi r')$ along the curved $\phi\rho'$, exist along $\phi\rho$ in Fig. 1 of [4].

The gravitational-relativistic mass M in the relativistic Euclidean 3-space Σ of the gravitational field source in Fig. 1 of [4], is an inertial mass or a passive gravitational mass, as shall be shown shortly in this paper. It is not a gravitational field source, which means it does not establish non-uniform gravitational potential $\Phi(r) = -GM_{\mathrm{a}}/r$ and non-uniform gravitational field $\vec{g} = -GM_{\mathrm{a}}\vec{r}/r^3$ in the relativistic Euclidean 3-space Σ. Consequently it is only the outward manifestation of the projective relativistic intrinsic gravitational potential $\phi\Phi(\phi r')$ of Eq. (56) namely, the gravitational-relativistic gravitational potential $\Phi(r')$ of Eq. (54), along with the invariant gravitational field of Eq. (44), which appear to originate from M, that exists in Σ in Fig. 1 of [4].

For the test Newtonian gravitational potential from a spherical test particle of rest mass m_{0t} in the external gravitational field, the proper test gravitational potential function is the following

$$\Phi'_{\rm t} = -\frac{Gm_{0\rm at}}{q'} \tag{57}$$

where q' is the distance along a radial direction from the center of $m_{0\rm t}$ to the test particle's rest mass m_0, where both m_0 and $m_{0\rm t}$ are located at radial distance r' from the center of M_0, and q' does not have to be along a radial direction from the center of M_0. Thus the proper test gravitational field from $m_{0\rm t}$ is the following

$$\vec{g}_{\rm t}{}' = -\frac{d\Phi'_{\rm t}}{dq'}\hat{q}\,' = -\frac{Gm_{0\rm at}}{q'^2}\hat{q}\,' \tag{58}$$

where $\hat{q}\,'$ is the unit vector along the radial direction from the center of $m_{0\rm t}$ towards m_0.

The corresponding gravitational-relativistic test gravitational potential function and gravitational-relativistic test gravitational field in Σ in the context of TGR shall be written respectively as follows

$$\Phi_{\rm t} = -\frac{Gm_{\rm at}}{q} \tag{59}$$

and

$$\vec{g}_{\rm t} = -\frac{Gm_{\rm at}}{q^2}\hat{q} \tag{60}$$

Eq. (43) shall be re-written as follows

$$-m\frac{d\Phi_{\rm t}}{dq} = -m_0\frac{d\Phi'_{\rm t}}{dq'}(1 - \frac{2GM_{0\rm a}}{r'c_g^2})$$

or

$$-m_0(1 - \frac{2GM_{0\rm a}}{r'c_g^2})\frac{d\Phi_{\rm t}}{dq} = -m_0\frac{d\Phi'_{\rm t}}{dq'}(1 - \frac{2GM_{0\rm a}}{r'c_g^2})$$

Hence

$$-\frac{d\Phi_{\rm t}}{dq} = -\frac{d\Phi'_{\rm t}}{dq'} \tag{61}$$

This is Eq. (45) again.

If q' is along a radial direction from the center of M_0, then as follows from Eq. (50), $d\Phi_{\rm t}/dq = \gamma_g(r')d\Phi_{\rm t}/dq'$. By using this in Eq. (61) we have

$$-\frac{d\Phi_{\rm t}}{dq} = -\gamma_g(r')\frac{d\Phi_{\rm t}}{dq'} = -\frac{d\Phi'_{\rm t}}{dq'}$$

Hence

$$\Phi_{\rm t} = \gamma_g(r')^{-1/2}\Phi'_{\rm t} = \Phi'_{\rm t}(1 - \frac{2GM_{0\rm a}}{r'c_g^2})^{1/2} \tag{62}$$

or

$$\Phi_{\rm t} = -\frac{Gm_{0\rm a\rm t}}{q'}(1 - \frac{2GM_{0\rm a}}{r'c_g^2})^{1/2};\ \text{(for } q' \text{ along } r') \tag{63}$$

On the other hand, for q' not along the radial direction from the center of M_0, then $d/dq = d/dq'$ (as follows from the trivial transformations of $r'd\theta'$ and $r'\sin\theta' d\varphi'$ into $rd\theta$ and $r\sin\theta d\varphi$ in the GLLT and its inverse), and Eq. (61) becomes the following

$$-\frac{d\Phi_{\rm t}}{dq'} = -\frac{d\Phi'_{\rm t}}{dq'}$$

Hence

$$\Phi_{\rm t} = \Phi'_{\rm t} = -\frac{Gm_{0\rm a\rm t}}{q'};\ \text{(for } q' \text{ not along } r') \tag{64}$$

We have the following from Eq. (62):

$$-\frac{d\Phi_{\rm t}}{dq} = -\frac{d\Phi'_t}{dq}(1 - \frac{2GM_{0\rm a}}{r'c_g^2})^{1/2} \tag{65}$$

But,

$$d/dq = \gamma_(r')d/dq' = (1 - 2GM_{0\rm a}/r'c_g^2)^{-1/2}d/dq',$$

from the GLLT (3), for q' along r'. Hence

$$-\frac{d\Phi_{\rm t}}{dq} = -\frac{d\Phi'_{\rm t}}{dq'}\frac{(1 - 2GM_{0\rm a}/r'c_g^2)^{1/2}}{(1 - 2GM_{0\rm a}/r'c_g^2)^{1/2}} = -\frac{d\Phi'_{\rm t}}{dq'}$$

or

$$\vec{g}_{\rm t} = \vec{g}_{\rm t}{}';\ \text{(for } q' \text{ along } r') \tag{$*$}$$

We also have the following from Eq. (64),

$$-\frac{d\Phi_{\rm t}}{dq} = -\frac{d\Phi'_{\rm t}}{dq} = -\frac{d\Phi'_{\rm t}}{dq'}$$

for q' not along r'. Hence

$$\vec{g}_{\rm t} = \vec{g}_{\rm t}{}';\ \text{(for } q' \text{ not along } r') \tag{$**$}$$

If we multiply through ($*$) and ($**$) by m we have:

$$m\vec{g}_{\mathrm{t}} = m\vec{g}_{\mathrm{t}}{}' = m_0\vec{g}_{\mathrm{t}}'(1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})$$

or

$$\vec{F}_{g\mathrm{t}} = (1 - \frac{2GM_{0\mathrm{a}}}{r'c_g^2})\vec{F}'_{g\mathrm{t}};\ \text{(for } q' \text{ along } r' \text{ or not)} \qquad (***)$$

Thus the transformation (43) of a test Newtonian gravitational force on a test particle due to another test particle in an external gravitational field, and the transformation (45) for the corresponding test Newtonian gravitational acceleration on the test particle, are valid irrespective of the orientation of the test gravitational force or acceleration relative to a radial direction from the center of the source of the external gravitational field. With the relations (26), (29), (34), (36), (38), (39) (40), (41), (44), (45), (46), (47), (54), (63) and (64), achieved, we shall defer the transformations of other physical parameters in the context of TGR to another paper.

It is important to answer a question that may arise namely, can the non-trivial parameter transformations (26), (29), (34), (36), (38), (40), (41), (54) and (63), derived in the context of TGR above be measured? In order to answer this question, we must show whether the factors $\gamma_g(r')^{-2}$, $\gamma_g(r')^{-1}$ and $\gamma_g(r')$ that appear in the relations can be measured or not.

Now, $\gamma_g(r')^{-2} = 1 - 2GM_{0\mathrm{a}}/r'c_g^2$. Hence

$$c_g^2\gamma_g(r')^{-2} = c_g^2 - 2GM_{0\mathrm{a}}/r' \qquad (66)$$

But $c_g^2 = 2GM_{0\mathrm{a}}/r'_b$, where r'_b is the gravitational (or Schwarzschild) radius of the field source of rest mass M_0. Eq. (66) can therefore be re-written as follows

$$c_g^2\gamma_g(r')^{-2} = 2GM_{0\mathrm{a}}/r_b - 2GM_{0\mathrm{a}}/r' \qquad (67)$$

Hence

$$\gamma_g(r')^{-2} = \frac{2}{c_g^2}(-\frac{GM_{0\mathrm{a}}}{r'} - (-\frac{GM_{0\mathrm{a}}}{r_b})) \qquad (68)$$

The factor $\gamma_g(r')^{-2}$ can be measured since it is a constant times the difference of gravitational potential as expressed by Eq. (68). Consequently the factors $\gamma_g(r')^{-1}$ and $\gamma_g(r')$ can be measured. It then follows that the derived parameter transformations in terms of these factors in the context of TGR in this section and others to be derived elsewhere with further development can be measured.

If the gravitational (or static) time dimension $c_g t$ has not been introduced, and one has erroneously formulated TGR on the flat spacetime of electromagnetism and dynamics (or special relativity) $(\Sigma, c_\gamma t)$, where $c_\gamma t$ is usually denoted by ct in SR and GR, then c_g must be replaced by, $c \equiv c_\gamma$, in the derived parameter transformations in the context of TGR. Then Eq. (66) would become the following

$$c_\gamma^2 \gamma_g(r')^{-2} = c_\gamma^2 - 2GM_{0\mathrm{a}}/r' \tag{69}$$

Now the square of the speed of light, $c^2 \equiv c_\gamma^2$, cannot be expressed as gravitational potential, and the potential at a point $-GM_{0\mathrm{a}}/r'$ cannot be measured. Thus from the point of view of what can be measured, Eq. (69) reduces as follows

$$c_\gamma^2 \gamma_g(r')^{-2} = c_\gamma^2 \quad \text{or} \quad \gamma_g(r')^{-2} = 1 \tag{70}$$

Thus if TGR has been formulated on the flat spacetime of SR with the dynamical time dimension, $c_\gamma t \equiv ct$, then the factors, $\gamma_g(r')^{-2} = (1 - 2GM_{0\mathrm{a}}/r'c_\gamma^2)$, $\gamma_g(r')^{-1}$ and $\gamma_g(r')$ are all equal to 1, as expressed by Eq. (70). This means that the non-trivial parameter transformations (26), (29), (34), (36), (38), (40), (41), (54) and (63), derived on the flat spacetime of SR (or in the Minkowski's space, $(\Sigma, ct) \equiv (\Sigma, c_\gamma t)$, with the gravitational (or static) speed c_g of gravitational waves replaced by the dynamical speed c_γ of electromagnetic waves in those relations, do not exist or are hypothetical and cannot be measured or observed. This shows explicitly that a relativistic theory of gravity formulated on the flat spacetime of special relativity (or in the Minkowski's space), $(\Sigma, ct) \equiv (\Sigma, c_\gamma t)$, which is the only flat spacetime known until now, is impossible, or do not exist and cannot be observed. This fact had also been arrived at but not as robustly as done here at different points since [2].

The transformations of various quantities in the context of TGR derived analytically in this sub-section, some of which have been derived graphically in [4], shall be considered adequate for this sub-section. More parameter transformations in the context of TGR shall be derived in other papers with further development.

2 Incorporating the special theory of relativity into the theory of gravitational relativity

The gravitational-relativistic mass m of the test particle, given by Eq. (26), evolved on the flat four-dimensional relativistic spacetme (Σ, ct) (in Fig. 1 of [4]) in the context of TGR, at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field. It is the gravitational-relativistic masses of particles and objects that undergo relative motions within local

Lorentz frames on the flat relativistic spacetime of TGR (Σ, ct) in the context of SR in every gravitational field in the universe.

2.1 Validating local Lorentz invariance (of SR) on flat four-dimensional spacetime prescribed by TGR in a gravitational field

The special theory of relativity (SR) in gravitational fields in the universe at present involves the motion of the gravitational-relativistic masses of particles and objects relative to observers within local Lorentz frames on the flat relativistic spacetime (Σ, ct) of TGR, as mentioned above. There is local Lorentz transformation and its inverse of the affine coordinates $(c_\gamma \tilde{t}\,,\ \tilde{x}\,,\ \tilde{y}\,, \tilde{z})$ of the particle's frame into the affine coordinates $(c_\gamma \tilde{\tilde{t}}\,,\ \tilde{\tilde{x}}\,, \tilde{\tilde{y}}\,,\ \tilde{\tilde{z}})$ of the observer's frame within a local Lorentz frame on the flat relativistic spacetime $\Sigma, ct)$, which are referred to as local Lorentz transformation (LLT) and its inverse because of the restriction to the interiors of local Lorentz frames, which are given respectively as follows (as derived by Albert Einstein in [, ibid.]

einstein and as has been re-derived by the graphical approach in part one of this paper [4].

$$\tilde{t} = \gamma_d(v)(\tilde{\tilde{t}} - \frac{v}{c_\gamma^2}\tilde{\tilde{x}});\ \ \tilde{x} = \gamma_d(v)(\tilde{\tilde{x}} - v\tilde{\tilde{x}});\ \ \tilde{y} = \tilde{\tilde{y}}\ \text{ and }\ \tilde{z} = \tilde{\tilde{z}} \tag{71}$$

and

$$\tilde{\tilde{t}} = \gamma_d(v)(\tilde{t} + \frac{v}{c_\gamma^2}\tilde{x});\ \ \tilde{\tilde{x}} = \gamma_d(v)(\tilde{x} + v\tilde{x});\ \ \tilde{\tilde{y}} = \tilde{y}\ \text{ and }\ \tilde{\tilde{z}} = \tilde{z} \tag{72}$$

where

$$\gamma_d(v) = (1 - v^2/c_\gamma^2)^{-1/2} \tag{73}$$

and v is the dynamical speed of of the gravitational-relativistic mass m of the particle relative to the observer.

The relative motion of the particle, taken to be along the $\tilde{x}-$ and $\tilde{\tilde{x}}-$ axes of the particle's and observer's frames respectively by convention in systems (71) and (72), which are collinear by the mandate of nature and not by man's assumption or prescription, as discussed in sub-section 4.2 of [4], can be along any direction in the Euclidean 3-space Σ within the local Lorentz frame of the motion of the particle.

The transformations (71) and (72) are local Lorentz transformation (LLT) and its inverse within a local Lorentz frame at radial distance r from the center of the assumed spherical gravitational field source of gravitational-relativistic mass M in the relativistic Euclidean 3-space Σ of TGR, within which gravitational speed $V_g'(r')$

and gravitational potential $\Phi'(r')$ are constant and, hence, within which a test particle can move at a uniform dynamical velocity relative to the observer. System (72) or (73) yields the following invariance

$$c^2\tilde{\tilde{t}}^2 - \tilde{\tilde{x}}^2 - \tilde{\tilde{y}}^2 - \tilde{\tilde{z}}^2 = c^2\tilde{t}^2 - \tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 \tag{74}$$

This Lorentz invariance in SR, which is limited to the interior of a local Lorentz frame in an external gravitational field, is usually referred to as local Lorentz invariance (LLI). The LLI has remained a postulate albeit with abundant experimental justification in general relativity [10, 1985],

prestage.

The special-relativistic time dilation and special-relativistic length contraction formulae, which follow from systems (71) and (72) are the following

$$\tilde{\tilde{t}} = (1 - v^2/c_\gamma^2)^{-1/2}\tilde{t}\ ;\ \ \tilde{\tilde{x}} = (1 - v^2/c_\gamma^2)^{1/2}\tilde{x}\ ;\ \tilde{\tilde{y}} = \tilde{y}\ ;\ \text{and}\ \tilde{\tilde{z}} = \tilde{z} \tag{75}$$

But there are also gravitational time dilation and gravitational length contraction formulae of systems (12b) and (12a) in the context of TRG, which when incorporated into system (75) gives the following

$$\tilde{\tilde{t}} = (1 - 2GM_{0\mathrm{a}}/r'c_g^2)^{-1/2}(1 - v^2/c_\gamma^2)^{-1/2}\tilde{t}'; \tag{76a}$$
$$\tilde{\tilde{x}} = (1 - 2GM_{0\mathrm{a}}/r'c_g^2)^{1/2}(1 - v^2/c_\gamma^2)^{1/2}\tilde{x}';\ \tilde{\tilde{y}} = \tilde{y}';\ \text{and}\ \tilde{\tilde{z}} = \tilde{z}' \tag{76b}$$

These are time dilation and length contraction formulae in the context of combined special theory of relativity and theory of gravitational relativity (SR+TGR).

However while the time dilation formulae (76a) is valid for arbitrary orientation of the direction of motion of the particle relative to a radial direction from the center of the gravitational field source within the local Lorentz frame in which relative motion occurs, the length contraction formula of system (76b) is valid when the collinear $\tilde{\tilde{x}}-$ and $\tilde{x}-$ axes of the observer's frame and the particle's frame respectively, along which motion of the particle occurs, are along a radial direction from the center of the gravitational field source within the local Lorentz frame.

In a situation where the $\tilde{\tilde{x}}-$ and $\tilde{x}-$ axes of the observer's frame and the particle's frame, along which the motion of the particle occurs, lie perpendicular to a radial direction from the center of the gravitational field source, but the $\tilde{z}-$axis lies along the radial direction from the center of the field source within the local Lorentz frame of motion of the particle, then Eq. (76a) must be retained while system (76b) must modified, thereby converting (76a) and (76b) to the following

$$\tilde{\tilde{t}} = (1 - 2GM_{0\mathrm{a}}/r'c_g^2)^{-1/2}(1 - v^2/c_\gamma^2)^{-1/2}\tilde{t}' \tag{77a}$$
$$\tilde{\tilde{x}} = (1 - v^2/c_\gamma^2)^{1/2}\tilde{x}';\ \ \tilde{\tilde{y}} = \tilde{y}' \text{ and } \tilde{\tilde{z}} = (1 - 2GM_{0\mathrm{a}}/r'c_g^2)^{1/2}\tilde{z}' \tag{77b}$$

It follows from the validity of local Lorentz invariance (LLI), expressed by Eq. (74) on flat spacetime (Σ, ct) of TGR in every gravitational field and the invariance of differential operators in the context of TGR discussed earlier, that non-gravitational natural laws should take on their usual classical and special-relativistic instantaneous differential forms on the flat spacetime of TGR in every gravitational field, (and not their covariant tensor forms adopted in the covariant tensor approach on the prescribed curved spacetime in context of the general theory of relativity). However the gravitational-relativistic values of physical quantities and constants in the context of TGR must be substituted into the laws. The principle of equivalence requires that the effect of such substitutions, (or the effect of gravity), should cancel out in all laws, as shall be investigated in a paper later in this volume.

It is clear from the derivation of Eq. (74) that local Lorentz invariance (LLI) would be impossible in a gravitational field without the gravitational local Lorentz transformation (GLLT) and its inverse (3) and (4) (or (6) and (7)) and the gravitational local Lorentz invariance (GLLI) (9) they imply in every local Lorentz frame in a gravitational field in the context of TGR. That is, in the absence of the flat relativistic spacetime (Σ, ct) established in a gravitational field in the context of TGR on which SR operates in a gravitational field.

The actual physics underlying the GLLT and its inverse of equations (3) and (4) or (6) and (7) and the GLLI of Eq. (9) they imply are the diagrams in the two-world picture of Figs.5 and 6 and their inverses of Figs. 7 and 8 of the first part of this paper [4], in the context of intrinsic two-dimensional theory of gravitational relativity (ϕTGR). Intrinsic gravitational local Lorentz transformation (ϕGLLT and its inverse on the two-dimensional intrinsic spacetime $(\phi\rho, \phi c\phi t)$ were derived from those diagrams. Then the GLLT and its inverse on the flat four-dimensional spacetime in the context of TGR were written directly from ϕGLLT and its inverse in the context of ϕTGR, since TGR is mere outward manifestation on flat four-dimensional spacetime of ϕTGR on flat two-dimensional intrinsic spacetime. This is the actual physics.

The re-derivation of GLLT on flat four-dimensional relativistic spacetime (Σ, ct) of TGR analytically in this paper requires us to start by stating two unproven principles of the theory of gravitational relativity, just as Einstein started the derivation of

Lorentz transformation (LT) analytically by starting from two principles of special relativity. This shows that the analytical approaches to the derivation of GLLT in TGR and LT in SR do not get down to the foundation of physics underlying GLLT and LT. They are incomplete and non-fundamental. This is the general characteristic of the hypothetico-deductive theories.

Now the diagrams in the two-world picture of Figs 5 and 6 of ϕTGR in the first part of this paper [4], take identical forms as the diagrams of Figs. 9 and 10 of ϕSR in that paper. Then as explained in [12], the spacetime/intrinsic spacetime diagrams in the two-world picture of SR/ϕSR, from which the LT/ϕLT and its inverse were re-derived in [6], which should now be replaced by Figs. 9 and 10 of [4] and their inverses, are immutable for as long as there is perfect symmetry of state among the four universes encompassed by the two-world picture identified in [12] – [13]. It equally follows that the spacetime/intrinsic spacetime diagrams in the two-world picture of TGR/ϕTGR, from which the GLLT/ϕGLLT and its inverse were derived are immutable. Hence the GLLT and its inverse, and the GLLI they imply in TGR are immutable for as long as there is a perfect symmetry of state among the four universes encompassed by the two-world picture. It follows from this, (and the fact that the validity of LLI depends the validity of on GLLI), that LLI is immutable for as long as there is perfect symmetry of state among the four universes in the two-world picture.

2.2 Mass and energy relations in combined gravitational theory of relativity and special theory of relativity

It is the gravitational-relativistic mass m in the context of TGR of Eq. (26) on the flat relativistic spacetime (Σ, ct) of TGR that moves relative to the observer in special relativity in an external gravitational field. Thus we must simply replace the rest mass m_0 on the flat proper spacetime $(\Sigma' ct')$, which usually appears in the primed special theory of relativity on the flat $(\Sigma' ct')$ until now, by the gravitational-relativistic mass m of equation (26) in the expression for gravitational-relativistic cum special-relativistic mass in in an external gravitational field to have as follows

$$\overline{m} = \gamma_d(v)m = \frac{m}{\sqrt{1 - v^2/c_\gamma^2}} \tag{78}$$

Then along with equation (26), equation (78) becomes the following

$$\overline{m} = \frac{m_0(1 - 2GM_{0\mathrm{a}}/r'c_g^2)}{\sqrt{1 - v^2/c_\gamma^2}} \tag{79}$$

The gravitational-relativistic cum special-relativistic total energy $\overline{E}$ in a gravitational field in the context of combined TGR and SR is given at radial distance r in the relativistic Euclidean 3-space Σ of TGR from the center of the gravitational field source as follows

$$\begin{aligned}\overline{E} &= mc_\gamma^2\gamma_d(v)\\ &= m_0c_\gamma^2\gamma_g(r')^{-2}\gamma_d(v)\\ &= m_0c_\gamma^2(1-2GM_{0\mathrm{a}}/r'c_g^2)(1-v^2/c_\gamma^2)^{-1/2}\end{aligned}$$

or

$$\overline{E} = (1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})E' \tag{80}$$

where E' is the usual special-relativistic total energy, which is valid on flat proper spacetime (Σ', ct') in the absence of relative gravitational field at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in an external gravitational field.

The gravitational-relativistic cum special-relativistic kinetic energy $\overline{T}$ in a gravitational field in the context of combined TGR and SR is likewise given as follows

$$\begin{aligned}\overline{T} &= \gamma_d(v)mc_\gamma^2 - mc_\gamma^2\\ &= m_0c_\gamma^2(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})(\gamma_d(v)-1)\\ &= m_0c_\gamma^2(1-\frac{2GM_{0\mathrm{a}}}{r'c_\gamma^2})[(1-\frac{v^2}{c_\gamma^2})^{-1/2}-1] \qquad (81)\end{aligned}$$

$$\overline{T} = (1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})T' \tag{82}$$

where T' is the usual special-relativistic kinetic energy, which is valid on flat proper spacetime (Σ', ct') in the absence of relative gravitational field at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in an external gravitational field.

An author [14, 1982], derived the following mass-energy relation in a spherically symmetric gravitational field in the context of the general theory of relativity

$$E^* = m^*c^2 = mc^2(1-\frac{2GM}{rc^2}) \tag{83}$$

Interestingly, relation (83), with c^2 canceled, is of the form of the mass relation (26) derived in the context of TGR. The derivation of Eq. (83) on curved four-dimensional spacetime with the aid of the line element in GR, corresponds to the derivation of the intrinsic mass relation, $\phi m_0 = \phi\hat{m}_0(1 - 2G\phi\hat{M}_{0\mathrm{a}}/\phi\hat{r}\phi\hat{c}_g^2)$, on the curved 'two-dimensional' absolute intrinsic spacetime $(\phi\hat{\rho}, \phi\hat{t})$ with the aid of the absolute intrinsic line element in the context of the metric theory of absolute intrinsic gravity ϕMAG in sub-section 2.1 of [1].

The principle of equivalence of Albert Einstein requires that the mass m of the test particle is constant with position in a gravitational field and therefore precludes relation (83) in GR. No clear meaning could be given the relation (83) in GR. On the other hand, the validity of Einstein's principle of equivalence shall be established in the context of TGR in this and another paper, and this shall be despite the existence of the mass relation (26) and other parameter relations in the context of TGR derived in this paper and those to be derived in later papers. The hierarchy of spacetimes/intrinsic spacetimes containing hierarchy of masses/intrinsic masses and parameters/intrinsic parameters and the associated hierarchy of spacetime/intrinsic spacetime geometries in the present theory, give physical meaning to parameter/intrinsic parameter relations in the present theory.

3 Modification of Newton's gravitational force law in the context of the theory of gravitational relativity

As mentioned in the first paragraph of section 1 of this second part of this paper, the theory of relativistic gravity and the commonly used terminology "relativity and gravitation", in the absence of the special theory of relativity, refers to the theory of gravitational relativity (TGR) and the classical (or Newton's) theory of gravity with relativistic correction in the context of TGR, also referred to as the unprimed classical theory of gravity and given the acronym CG.

The usual classical (or Newton's) theory of gravity, also referred to as the primed classical theory of gravity and given the acronym CG′ in the present theory, in sub-section 3.3 of [3] and sub-section 1.1 of [4], operates on the flat proper spacetime (Σ', ct') of Fig. 3 of [1] or Fig. 1 of [3], which evolves at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field. Although Fig. 1 of [3] at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field does not exist in nature, as discussed in section 3 of [3], it is nevertheless valid theoretically that CG′ on flat (Σ', ct') in that geometry transforms into CG on the flat (Σ, ct) in the geometry of Fig. 9 of [3] or Fig. 1 of [4].

However what happens in reality, as well discussed in section 3 of [3] and sub-section 1.1 of [4], is that the primed intrinsic classical (or Newton's) theory of gravity (ϕCG′) within a proper (or primed) intrinsic local Lorentz frame on the curved proper intrinsic spacetime ($\phi\rho', \phi c\phi t'$), projects unprimed intrinsic classical (or Newton's) theory of gravity (ϕCG) into the corresponding relativistic (or unprimed) intrinsic local Lorentz frame on flat relativistic intrinsic spacetime ($\phi\rho, \phi c\phi t$) in the context of ϕTGR, which is then made manifest outwardly in unprimed classical (or Newton's) theory of gravity (CG) on flat four-dimensional relativistic spacetime (Σ, ct) in the context of TGR in Fig. 9 of [3] or Fig. 1 of [4].

Having derived the gravitational local Lorentz transformation (GLLT) and its inverse in the context of TGR by the graphical approach *via* graphical approach to the intrinsic gravitational local Lorentz transformation (ϕGLLT) and its inverse in the context of ϕTGR and having established the gravitational local Lorentz invariance (GLLI), along with gravitational time dilation and gravitational length contraction formulae in the context of TGR in the first part of this paper [4] and having rederived these analytically along with the transformations of several other physical parameters in the context of TGR in this second part of this paper, the TGR on flat relativistic spacetime (Σ, ct) and ϕTGR on flat relativistic intrinsic spacetime ($\phi\rho, \phi c\phi t$) underlying (Σ, ct), shall be deemed to have been accomplished to a good extent. Further development of the TGR/ϕTGR, which entails the transformations of other physical parameters/intrinsic parameters and physical constants/intrinsic constants in the context of TGR/ϕTGR, shall be done in other papers.

What is left for us to do now in order to accomplish the theory of relativistic gravity (or the theory of relativity and gravitation without reference to the special theory of relativity) in the context of the present evolving theory, is to develop the gravitational-relativistic (or unprimed) classical (or Newton's) theory of gravity (CG) on the flat relativistic spacetime (Σ, ct) of TGR and the gravitational (or unprimed) intrinsic classical (or Newton's) theory of gravity (ϕCG) on the flat relativistic intrinsic spacetime ($\phi\rho, \phi c\phi t$) underlying (Σ, ct) in a gravitational field of arbitrary strength. This is the topic of this section.

Now the gravitational-relativistic mass m of the test particle that evolves in the relativistic Euclidean 3-space Σ in the context of TGR, given by Eq. (26), interacts with the gravitational-relativistic gravitational potential $\Phi(r')$ that evolves in the relativistic Euclidean 3-space Σ in the context of TGR, given by Eq. (54), yielding the gravitational-relativistic gravitational potential energy $U(r')$ in the relativistic Euclidean 3-space Σ (or on the flat relativistic spacetime (Σ, ct)) in the context of TGR as follows

$$U(r') = m(r')\Phi(r')$$

$$
\begin{aligned}
&= m_0(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})(-\frac{GM_{0\mathrm{a}}}{r'})(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{1/2} \\
&= -\frac{GM_{0\mathrm{a}}m_0}{r'}(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{3/2} \qquad (84)
\end{aligned}
$$

Then by dividing through Eq. (84) by the rest mass m_0 of the test particle, the effective gravitational potential function 'seen' by the test particle in the relativistic Euclidean 3-space Σ in the context of TGR is the following

$$
\Phi_{\mathrm{eff}}(r') = -\frac{GM_{0\mathrm{a}}}{r'}(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{3/2} \qquad (85)
$$

The effective gravitational-relativistic gravitational potential $\Phi_{\mathrm{eff}}(r')$ in Σ in the context of TGR is a function of the primed radial coordinate r' directly. Hence the effective gravitational acceleration suffered by the test particle in the context of TGR is given from definition as follows

$$
\begin{aligned}
\vec{g}_{\mathrm{eff}}(r') &= -\frac{d\Phi_{\mathrm{eff}}(r')}{dr'}\hat{r} \\
&= -\frac{GM_{0\mathrm{a}}}{r'^2}(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{3/2}\hat{r} + \frac{3G^2M_{0\mathrm{a}}^2}{r'^3c_g^2}(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{1/2}\hat{r} \qquad (86)
\end{aligned}
$$

or

$$
\vec{g}_{\mathrm{eff}}(r') = \vec{g}\,'(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{3/2} - 3\vec{g}\,'\frac{|\vec{g}\,'|r'}{c_g^2}(1-\frac{2GM_{0\mathrm{a}}}{r'c_g^2})^{1/2} \qquad (87)
$$

where $\hat{r}$ in Eq. (86) is the unit vector radially away from the center of the assumed spherical gravitational-relativistic mass M of the gravitational field source in the relativistic Euclidean 3-space Σ in the context of TGR, since $\vec{g}_{\mathrm{eff}}(r')$ is a vector in Σ.

Equation (86) or (87) gives the unprimed (or gravitational-relativistic) gravitational acceleration in the relativistic Euclidean 3-space Σ in a gravitational field of arbitrary strength, which the test particle suffers towards the center of the gravitational-relativistic mass M of the gravitational field source in Σ in the context of TGR. Eq. (86) or (87) is valid for a spherically symmetric gravitational field source only. the modified form in a non-spherically-symmetric gravitational field shall be derived elsewhere with further development.

By multiplying through Eqs. (86) and (87) by the rest mass m_0 of the test particle (and not its gravitational-relativistic mass m), as follows from the division of the gravitational-relativistic gravitational potential energy of Eq. (84) by m_0 to obtain the effective gravitational potential function of Eq. (85)), we obtain the unprimed (or gravitational-relativistic) gravitational force suffered by the test particle towards the center of the gravitational field source in Σ in the context of TGR as follows

$$\vec{F}_{\text{eff}}(r') = -\frac{GM_{0\text{a}}m_0}{r'^2}(1-\frac{2GM_{0\text{a}}}{r'c_g^2})^{3/2}\hat{r} + \frac{3G^2M_{0\text{a}}^2m_0}{r'^3c_g^2}(1-\frac{2GM_{0\text{a}}}{r'c_g^2})^{1/2}\hat{r} \quad (88)$$

or

$$\vec{F}_{\text{eff}}(r') = m_0\vec{g}\,'(1-\frac{2GM_{0\text{a}}}{r'c_g^2})^{3/2} - 3m_0\vec{g}\,'\frac{|\vec{g}\,'|r'}{c_g^2}(1-\frac{2GM_{0\text{a}}}{r'c_g^2})^{1/2} \quad (89)$$

The equivalence of inertial acceleration and gravitational acceleration hold on the flat relativistic spacetime (Σ, ct) in the context of TGR as well, that is $d^2\vec{x}/dt^2 \equiv \vec{g}_{\text{eff}}(r')$. Thus the equation of motion within a local Lorentz frame of the test particle that suffers the effective gravitational acceleration (86) or (87) in the relativistic Euclidean 3-space Σ in the context of TGR is the following

$$\frac{d^2\vec{x}}{dt^2} = -\frac{GM_{0\text{a}}}{r'^2}(1-\frac{2GM_{0\text{a}}}{r'c_g^2})^{3/2}\hat{r} + \frac{3G^2M_{0\text{a}}^2}{r'^3c_g^2}(1-\frac{2GM_{0\text{a}}}{r'c_g^2})^{1/2}\hat{r} \quad (90)$$

or

$$\frac{d^2\vec{x}}{dt^2} = \vec{g}\,'(1-\frac{2GM_{0\text{a}}}{r'c_g^2})^{3/2} - 3\vec{g}\,'\frac{|\vec{g}\,'|r'}{c_g^2}(1-\frac{2GM_{0\text{a}}}{r'c_g^2})^{1/2} \quad (91)$$

This equation describes the motion of the observed gravitational-relativistic mass m of the test particle on the flat relativistic spacetime (Σ, ct) in a gravitational field, under the sole influence of gravitational attraction towards the center of the assumed spherical gravitational-relativistic mass M of the gravitational field source in Σ.

Eq. (85), Eq. (86) or (87), Eq. (88) or (89) and Eq. (90) or (91) are expressions in the context of the gravitational-relativistic (or unprimed) classical theory of gravity (CG) on the flat relativistic spacetime (Σ, ct) of TGR at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in every gravitational field. They are the modified (or gravitational-relativistic) forms on the flat relativistic spacetime (Σ, ct) in the context of TGR of the respective equations (or expressions) in the context of the primed classical theory of gravity (CG′) on the flat proper spacetime (Σ', ct') at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field.

3.1 The Newtonian limit and post-Newtonian approximation to the gravitational-relativistic form of the Newtonian gravitational force law

Eqs. (85) – (91) are valid on the flat relativistic spacetime (Σ, ct) of TGR in a gravitational field of arbitrary strength. They simplify respectively as follows by letting $2GM_{0a}/r'c_g^2 = V_g'(r')^2/c_g^2 = 0$,

$$\Phi_{\text{eff}}(r') = \Phi'(r') = -GM_{0a}/r' \tag{92}$$

$$\vec{g}_{\text{eff}}(r') = \vec{g}\,'(r') = -GM_{0a}\vec{r}\,'/r'^3 \tag{93}$$

$$\vec{F}_{\text{eff}}(r') = \vec{F}'(r') = -GM_{0a}m_0\vec{r}\,'/r'^3 \text{ and} \tag{94}$$

$$\frac{d^2\vec{x}\,'}{dt'^2} = \vec{g}\,' \tag{95}$$

Eqs. (92) – (95) are the essential equations of the primed classical (or Newtonian) theory of relative gravity (CG′), where relative gravity means that the gravitational potential $\Phi'(r')$ and gravitational acceleration $\vec{g}\,'(r')$ are relative gravitational parameters, that is, they vary in magnitude with radial distance r from the center of the gravitational-relativistic mass M of the gravitational field source in Σ (corresponding to radial distance r' from the center of the rest mass M_0 of the gravitational field source in the elusive proper Euclidean 3-space Σ'). Hence Eqs. (92) – (95) are the Newtonian limits to Eqs. (86) – (91) of CG. They can be applied without significant loss of accuracy in very weak gravitational fields, such as the gravitational fields of the planets.

Now $V_g'(r')/c_g = 0$ corresponds to $\phi V_g'(\phi r')/\phi c_g = \sin\phi\psi_g(\phi r') = 0$. Hence $\phi\psi_g(\phi r') = 0$. The intrinsic angle $\phi\psi_g(\phi r')$ is the inclination of the curved proper intrinsic space $\phi\rho'$ to its projective straight line relativistic intrinsic space $\phi\rho$ along the horizontal, at 'distance' $\phi r'$ along the the curved $\phi\rho'$ from the base of the intrinsic rest mass ϕM_0 of the gravitational field source at the origin of the curved $\phi\rho'$ in Figs. 1 and 2 of [4]. The intrinsic angle $\phi\psi_g(\phi r')$ is likewise the inclination of the curved proper intrinsic time dimension $\phi c\phi t'$ to its projective straight line relativistic intrinsic time dimension $\phi c\phi t$ along the vertical, at equal 'distance' $\phi r'$ along the curved $\phi c\phi t'$ from the base of the intrinsic rest mass $\phi E'/\phi c^2$ of the gravitational field source at the origin of the curved $\phi c\phi t'$ in Figs. 1 and 2 of [4].

The vanishing of $\phi\psi_g(\phi r')$ at every 'distance' $\phi r'$ along the curved $\phi\rho'$ and curved $\phi c\phi t'$ implies that the curved $\phi\rho'$ becomes a straight line intrinsic dimension along the horizontal and the curved $\phi c\phi t'$ becomes a straight line intrinsic dimension along the vertical. They thereby constitute two-dimensional flat proper intrinsic spacetime $(\phi\rho', \phi c\phi t')$ underlying four-dimensional flat proper spacetime

(Σ', ct'). In other words, the condition $\phi V_g(\phi r')/\phi c_g = \sin\phi\psi_g(\phi r') = 0$, converts the geometries of Figs. 1 and 2 of combined first and second stages of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field or arbitrary strength to the geometry of Fig. 5 of [2] in two-world or Fig. 3 of [1] in partial one-world representation, at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in every gravitational field.

It follows from the foregoing two paragraphs that Eqs. (92) – (95) of CG$'$, obtained with the condition $V_g(r')/c_g = 0$ on CG, corresponding to $\phi V_g(\phi r') = \sin\phi\psi_g(\phi r') = 0$ on ϕCG$'$, pertain to the flat proper spacetime (Σ', ct') in Fig. 3 of [1], at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in every gravitational field.

However the condition $2GM_{0\mathrm{a}}/r'c_g^2 = 0$, which reduces Eqs. (85) – (91) to Eqs. (92) – (95), implies $M_{0\mathrm{a}} = 0$, hence $M_0 = 0$. Thus this condition implies we must let $M_{0\mathrm{a}} = 0$ in Eqs. (92) – (95), from which it follows that those equations are impossible (or do not exist). In other words, CG$'$ is impossible (or does not exist) on the flat proper spacetime (Σ', ct') to which Eqs. (85) – (91) pertain. It must also be recalled that the flat proper spacetime (Σ', ct') that evolves at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in every gravitational field, is devoid of relative gravity. Hence CG$'$ (Eqs. (92) – (95)) does not exist on the flat proper spacetime (Σ', ct') indeed. Only the primed Newtonian theory of absolute gravity (NAG$'$) and combined primed Newtonian theory of absolute gravity and primed Newtonian theory of absolute motion (NAG$'$+NAM$'$), involving absolute gravitational parameters $\hat{\Phi}(\hat{r})$ and $\hat{g}(\hat{r})$ and absolute dynamical parameters $\hat{V}_d$ and $\hat{a}$, exist on the flat proper spacetime (Σ', ct') at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in every gravitational field.

Never the less Eqs. (92) – (95) of CG$'$ can be applied without significant loss of accuracy in very weak gravitational fields on the flat proper spacetime (Σ', ct'), as approximate theory to CG with the exact equations (85) – (91) on the flat relativistic spacetime (Σ, ct). In this wise, the flat proper spacetime (Σ', ct') shall be referred to as the space of the primed classical theory of relative gravity CG$'$, although CG$'$ does not exist in (Σ', ct') in a strict sense.

On the other hand, let us consider the weak gravitational field limit for which $2GM_{0\mathrm{a}}/r'c_g^2 = V_g'(r')^2/c_g^2 \approx 0$ or in the limit as $2GM_{0\mathrm{a}}/r'c_g^2 \to 0$. This condition implies $M_{0\mathrm{a}} \approx 0$, hence $M_0 \approx 0$, that is, gravitational field sources of small but non-zero rest masses. By applying the limit $2GM_{0a}/r'c_g^2 \to 0$ on Eqs. (85) – (91) we have the following

$$\Phi_{\text{eff}}(r') \approx -\frac{GM_{0a}}{r'}(1 - \frac{3GM_{0a}}{r'c_g^2}) \tag{96}$$

$$\vec{g}_{\text{eff}}(r') \approx -\frac{GM_{0a}}{r'^2}\hat{r} + \frac{6G^2M_{0a}^2}{r'^3c_g^2}\hat{r} \tag{97}$$

$$\vec{F}_{\text{eff}}(r') \approx -\frac{GM_{0a}m_0}{r'^2}\hat{r} + \frac{6G^2M_{0a}^2m_0}{r'^3c_g^2}\hat{r} \tag{98}$$

$$\frac{d^2\vec{x}}{dt^2} \approx -\frac{GM_{0a}}{r'^2}\hat{r} + \frac{6G^2M_{0a}^2}{r'^3c_g^2}\hat{r} \tag{99}$$

The weak field limit approximations (96) – (99) shall be referred to as the post-Newtonian approximations to the exact equations (86) – (91) of CG on the flat relativistic spacetime (Σ, ct). The post-Newtonian approximation (Eqs. (96) – (99)) is a more accurate approximation to CG than the Newtonian limit (or CG′) (with Eqs. (92 – (95)) in weak gravitational fields. Since $V_g'(r')/c_g$ is small but non-zero, corresponding $\phi V_g'(\phi r')/\phi c_g = \sin\phi\psi_g(\phi r')$ is small but non-zero, the post-Newtonian approximation pertains to the flat relativistic spacetime (Σ, ct) like CG.

The invariance of gravitational acceleration (or field) (44) in the context of TGR is valid in empty space at the exterior of a gravitational field source, when the interaction of a test particle with the external gravitational field is not in consideration. On the other hand, it is the effective gravitational acceleration (86) or (87) that a test particle suffers towards the center of the assumed spherical external gravitational field source in the context of TGR (and not the direct invariant gravitational field $\vec{g} = \vec{g}'$ originating from the gravitational field source).

The closest to deriving a modification of the Newtonian gravitational potential function in general relativity, as far as I can find, arises in the study of the trajectory on curved spacetime of a planet in orbit round the sun by the geodesic approach in GR, where it has been deduced by formal analogy with the classical orbit, see, for example, page 206 of [15], that the approximate effective Newtonian gravitational potential function responsible for the observed perturbation of the classical orbit is the following

$$f(r) \approx -\frac{GM}{r}(1 + \frac{v_l^2}{c^2}) = -\frac{GM}{r}(1 + \frac{2GM}{rc^2}) \tag{100}$$

where, $v_l = rd\varphi/dt = (2GM/r)^{1/2}$, is the so-called lateral velocity of the planet. The notation in GR has been preserved in (100). We shall not go into a discussion of the discrepancy between $\Phi_{\text{eff}}(r')$ in Eq. (85) and $f(r)$ in Eq. (100). It is however safe to say that Eq. (100) has been derived from an inexact calculation in the quoted reference, unlike the derivation of the exact relation (85) in this paper. It is just important to note that the need for the modification of the Newtonian gravitational potential has been known in general relativity, although no physical meaning could be given to $f(r)$ in GR, as stated on page 206 of [15]. There is no spacetime to contain this derived function, just as said for the derived mass-energy expression of Eq. (83) in GR.

As another aspect of the modification of the classical (or Newton's) law of gravity in the context of TGR, let us consider a particle of rest mass m_0, which is moving at a low (non-relativistic) velocity $\vec{v}$ through radial distance r from the center of the assumed spherical gravitational-relativistic mass M of a gravitational field source on the flat relativistic spacetime (Σ, ct) relative to an observer. The relativistic kinetic energy in the context of TGR of the particle has been related to its classical kinetic energy by Eq. (29), which shall be rewritten here as follows

$$E_K = \frac{1}{2}mv^2 = \frac{1}{2}m_0v^2(1 - \frac{2GM_{0\text{a}}}{r'c_g^2}) \tag{101}$$

Now let the gravitational-relativistic mass m of the test particle above propagate at non-relativistic velocity radially from radial distance r_1 to radial distance r from the center of the gravitational-relativistic mass M of the gravitational field source on the flat relativistic spacetime (Σ, ct) of TGR, which correspond to radial distances r_1' and r' respectively from the center of the rest mass M_0 of the field source on the flat proper spacetime (Σ', ct'). The difference in kinetic energy of the particle as it moves from r_1 to r in (Σ, ct) is given in the context of TGR as follows

$$\Delta E_K = \frac{1}{2}m_0v^2(1 - \frac{2GM_{0\text{a}}}{r'c_g^2}) - \frac{1}{2}m_0v_1^2(1 - \frac{2GM_{0\text{a}}}{r_1'c_g^2}) \tag{102}$$

where $\vec{v}_1$ is its velocity while passing through r_1. On the other hand, the change in gravitational potential energy (84) in the context of TGR, as the particle moves from r_1 to r is the following

$$\Delta U = -\frac{GM_{0\text{a}}m_0}{r'}(1 - \frac{2GM_{0\text{a}}}{r'c_g^2})^{3/2} + \frac{GM_{0\text{a}}m_0}{r_1'}(1 - \frac{2GM_{0\text{a}}}{r_1'c_g^2})^{3/2} \tag{103}$$

Conservation of energy allows us to write, $\Delta U = \Delta E_K$. Hence

$$\frac{1}{2}m_0v^2(1 - \frac{2GM_{0\text{a}}}{r'c_g^2}) - \frac{1}{2}m_0v_1^2(1 - \frac{2GM_{0\text{a}}}{r_1'c_g^2}) = -\frac{GM_{0\text{a}}m_0}{r'}(1 - \frac{2GM_{0\text{a}}}{r'c_g^2})^{3/2} +$$

$$+\frac{GM_{0a}m_0}{r'_1}(1-\frac{2GM_{0a}}{r'_1c_g^2})^{3/2}$$

Therefore,

$$\begin{aligned} v^2 &= v_1^2(1-\frac{2GM_{0a}}{r'_1c_g^2})(1-\frac{2GM_{0a}}{r'c_g^2})^{-1} \\ &= -\frac{2GM_{0a}}{r'}(1-\frac{2GM_{0a}}{r'c_g^2})^{1/2}+\frac{2GM_{0a}}{r'_1}(1-\frac{2GM_{0a}}{r'_1c_g^2})^{3/2}(1-\frac{2GM_{0a}}{r'c_g^2})^{-1} \end{aligned} \tag{104}$$

This expression is valid in a gravitational field of arbitrary strength. Its post-Newtonian approximation on flat relativistic spacetime (Σ, ct) given in the limit as $2GM_{0a}/r'c_g^2 \to 0$ can be written. In the pure Newtonian limit on the flat proper spacetime (Σ', ct'), the factor $(1-2GM_{0a}/r'c_g^2) = (1-2GM_{0a}/r'_1c_g^2) = 1$, Eq. (104) simplifies as the familiar relation in classical mechanics, $v^2 = v_1^2 - 2\Delta\Phi' = v_1^2 + 2gH$.

Finally let us consider a test particle of rest mass m_0, which undergoes gravitational fall from rest at infinity to speed v at radial distance r from the center of the assumed spherical gravitational-relativistic mass M in the relativistic Euclidean 3-space Σ of TGR, of the only source of external gravitational field. At infinity, ($r = \infty$), it possesses zero gravitational potential energy and zero kinetic energy, and upon falling to radial distance r from the center of M, it possesses gravitational-relativistic gravitational potential energy $m_0\Phi_{\text{eff}}(r')$ in the context of TGR, and gravitational-relativistic kinetic energy $\frac{1}{2}mv^2$ in the context of TGR, (assuming v is non-relativistic thereby precluding the presence of SR). Then from the fact that the decrease in gravitational potential energy is equal to the increase in kinetic energy of the test particle, we have the following

$$\frac{1}{2}mv^2 - 0 = 0 - m_0\Phi_{\text{eff}}(r')$$

or

$$\frac{1}{2}m_0v^2(1-\frac{2GM_{0a}}{r'c_g^2}) = \frac{GM_{0a}m_0}{r'}(1-\frac{2GM_{0a}}{r'c_g^2})^{3/2}$$

Hence

$$v^2 = \frac{2GM_{0a}}{r'}(1-\frac{2GM_{0b}}{r'c_g^2})^{1/2} \tag{105}$$

The dynamical velocity v given by Eq. (105) is the escape velocity of the test particle, usually denoted by v_{esc}. Thus the escape velocity with gravitational-relativistic correction (in the context of TGR) is the following

$$v_{\rm esc} = \sqrt{\frac{2GM_{0\rm a}}{r'}}(1 - \frac{2GM_{0\rm a}}{r'c_g^2})^{1/4} \tag{106}$$

Obviously the observed (or unprimed) classical theory of gravity CG on the flat relativistic spacetime (Σ, ct) of TGR is the the fictitious primed classical theory of gravity (CG′) on flat proper spacetime (Σ', ct') with relativistic correction in the context of TGR, just as the observed (or unprimed) classical dynamics is the primed classical dynamics with relativistic correction in the context of SR. The existence of CG′ on flat proper spacetime (Σ', ct') is described as fictitious because CG′ is impossible (or does not exist) on (Σ', ct') in reality, as explained above.

Local Lorentz invariance (LLI) obtains naturally on the flat relativistic spacetime (Σ, ct) of TGR in an external gravitational field, as demonstrated in sub-sub-section 3.3.1 earlier. This coupled with the fact that the gravitational velocity $\vec{V}_g(r')$ of any magnitude of TGR is equivalent to zero dynamical speed v of SR, the transformation of non-gravitational dynamical natural laws in the context of TGR, should retain the usual instantaneous differential classical and special-relativistic forms of the laws, (and not their covariant tensor forms), on the flat (Σ, ct) in a gravitational field of arbitrary strength. However the gravitational-relativistic values in the context of TGR of physical quantities and constants that appear in the natural laws, usually as differential coefficients, must be substituted. Whether the effect of the substitutions (or of gravity) will cancel in all natural laws so that the strong equivalence principle (SEP) is strictly valid or not shall be investigated in detail in a paper shortly in this volume.

Also since the effective gravitational acceleration $\vec{g}_{\rm eff}$ of Eq. (96) or (97) in the context of CG, which a test particle suffers towards the center of a gravitational field source, does not depend on the property of the test particle, the validity of the weak equivalence principle (WEP), usually considered to be confirmed by the Eötvös-Dicke experiment and others in the context of primed classical gravity (CG′), see, for example, page 251 of [16, 1972] and page 15 of [17, 1972], holds good in the context of the gravitational-relativistic (or unprimed) classical gravitation (CG), for as long as it is valid in the context of CG′. There is a high prospect for theoretical confirmation of the principle of equivalence of Albert Einstein (EEP) in the contexts of TRG. We only need to demonstrate the validity of SEP theoretically in the context of TGR in addition to the theoretical confirmations of LLI and WEP already shown.

4 Mass concepts in the context of the theory of gravitational relativity

Let us consider a spherically symmetric gravitational field source of rest mass M_0 and classical radius R_0 in the proper Euclidean 3-space Σ' of the flat proper spacetime (Σ', ct') in Fig. 3 or Fig. 11 of [1], at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters within the gravitational field of the body, which shall be assumed to be isolated from the gravitational field of every other source. The rest mass M_0 of this body will interact with its own gravitational field to become the gravitational-relativistic mass M in the relativistic Euclidean 3-space Σ of the flat relativistic spacetime (Σ, ct) in Fig. 1 of [4] at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters within its gravitational field, in the context of the the theory of gravitational relativity (TGR). Likewise the radius R_0 of M_0 will be identically contracted along all radial directions from the center of M_0 to become the gravitational-relativistic radius R of M in Σ. Thus the spherical symmetry of the field source will be preserved in Σ.

The mass relation (26) in the context of TGR must be written with the test particle's rest mass m_0 replaced by the rest mass M_0 of the isolated gravitational field source and the proper radial distance r' from the center of M_0 to the location of the rest m_0 in Σ' replaced by the classical radius R_0 of the field source in the present case to have as follows

$$M = M_0\left(1 - \frac{2GM_{0\mathrm{a}}}{R_0 c_g^2}\right) \tag{107}$$

This is the mass relation in the context of TGR for a gravitational field source of rest mass M_0 and classical radius R_0 (of M_0), which is isolated from the gravitational field of every other source.

If the body is then brought near another gravitational field source, it is the gravitational-relativistic mass M in the context of TGR, due to its own gravitational field that will interact with the external gravitational field. Hence by definition [8, 16, 17], the gravitational-relativistic mass M given by Eq. (107) is the passive gravitational mass. It shall therefore be re-denoted M_p in Eq. (107) to have

$$M_p = M_0\left(1 - \frac{2GM_{0\mathrm{a}}}{R_0 c_g^2}\right) \tag{108}$$

If a mechanical (or inertial) force is impressed on the body isolated from the gravitational field due to any other body, it is also its gravitational-relativistic mass M that will move in response. Hence the gravitational-relativistic mass is also the inertial mass to be denoted by M_i of the body. Again let us re-denote M by M_i in Eq. (107) to have

$$M_i = M_0(1 - \frac{2GM_{0\mathrm{a}}}{R_0 c_g^2}) \tag{109}$$

The equivalence of the passive gravitational mass and the inertial mass of a body in the context of TGR follows from Eqs. (108) and (109). That is,

$$M_p = M_i \tag{110}$$

Relation (110) is usually known as equality of gravitational mass and inertial mass and written as $m_g = m_i$ in classical mechanics. Thus the equivalence of gravitational mass and inertial mass in classical mechanics, becomes the equivalence of passive gravitational mass and inertial mass in the context of TGR.

On the other hand, Eqs. (108) and (109) imply non-equivalence of rest mass and passive gravitational mass and non-equivalence of rest mass and inertial mass in the context of TGR,

$$M_p = M_i \neq M_0 \tag{111}$$

The known equivalence of inertial mass and rest mass $m_i = m_0$ in classical mechanics and general theory of relativity is invalid in the context of TGR.

We have finally identified the gravitational-relativistic mass M or m of a body or particle formed in the relativistic Euclidean 3-space Σ in the context of TGR, at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in every gravitational field as the passive gravitational mass and inertial mass of the body or particle.

As first introduced in sub-sub-section 2.1.5 of [1], the entity $M_{0\mathrm{a}}$ in Eq. (102) is non-observable and non-detectable to observers in Σ. It is an absolute-absolute immaterial entity with unit of mass, which is imperceptibly hidden within the rest mass M_0 in the proper Euclidean 3-space Σ'. As suggested in sub-sub-section 2.1.5 of [1], $M_{0\mathrm{a}}$ is a negative entity (i.e. we must write $-M_{0\mathrm{a}}$), and it is the source of the attractive gravitational speed $V'_g(r') = -(2GM_{0\mathrm{a}}/r')^{1/2}$, attractive proper (or classical) gravitational potential $\Phi'(r') = -GM_{0\mathrm{a}}/r'$ and attractive proper (or classical) gravitational field $\vec{g}\,' = -GM_{0\mathrm{a}}\vec{r}\,'/r'^3$. The attractive nature of these gravitational parameters arise by virtue of the negative sign of the immaterial $-M_{0\mathrm{a}}$.

The gravitational speed $V'_g(r')$ due to the non-detectable immaterial gravitational charge $-M_{0\mathrm{a}}$ within the rest mass M_0 in the proper Euclidean 3-space Σ' then interacts with M_0 thereby converting the rest M_0 in Σ' to the passive gravitational mass M_p, which is also the inertial mass M_i of the body in the relativistic Euclidean 3-space Σ in the context of TGR, as expressed by Eq. (108) and Eq. (109).

The negative absolute-absolute immaterial entity $-M_{0\mathrm{a}}$ with the unit of mass, which is imperceptibly hidden within the rest mass M_0 in Σ', being the source of the classical (or Newtonian) gravitational potential and field observed to originate from M_0, is the active gravitational mass of the body. This is so since the source of the classical gravitational potential and field of a body is the active gravitational mass of the body by definition [8, 16, 17]. The active gravitational mass $-M_{0\mathrm{a}}$ has also been referred to as the gravitational charge since its incorporation into the present theory in sub-sub-section 2.1.5 of [1].

The equivalence of the rest mass and the active gravitational mass, $M_0 = M_{0\mathrm{a}}$ is known in classical mechanics [8]. Consequently the classical gravitational potential and field are usually written in terms of the rest mass as $\Phi'(r') = -GM_0/r'$ and $\vec{g}' = -GM_0\vec{r}'/r'^3$. The classical gravitational potential and field originating from the material rest mass M_0 are identical in magnitude to those originating from the immaterial active gravitational mass (or gravitational charge) — in a spherically symmetric gravitational field. That is,

$$\Phi'(r') = -GM_0/r' = -GM_{0\mathrm{a}}/r' \tag{112a}$$

$$\vec{g}'(r') = -GM_0\vec{r}'/r'^3 = -GM_{0\mathrm{a}}\vec{r}'/r'^3 \tag{112b}$$

Since the gravitational field is massless, its source must also be massless. This thereby makes the the immaterial active gravitational mass (or massless gravitational charge) $-M_{0\mathrm{a}}$ the source of gravitational potential and field. In analogy, the massless electric charge is the source of the massless electrostatic field in electromagnetism, and the sign of electric charge determines the sign (attractive or repulsive) of electrostatic field.

For the gravitational potential and gravitational field that originate from the rest mass to be identical to the gravitational potential and gravitational field that originate from the active gravitational mass (or gravitational charge) according to Eqs. (112a) and (112b), it must be that the immaterial negative active gravitational mass (or gravitational charge) $-M_{0\mathrm{a}}$ is equal in magnitude to the rest mass. In other words, there is equivalence (in magnitude but not in nature) of the positive material rest mass and negative immaterial active gravitational mass (or gravitational charge) in the context of TGR, expressed as follows

$$M_0 = |-M_{0\mathrm{a}}| \tag{113}$$

Further more the fact that the gravitational potential and gravitational field that originate from the rest mass to be identical to the gravitational potential and gravitational field that originate from the active gravitational mass (or gravitational charge) according to Eqs. (112a) and (112b), implies that the negative immaterial active gravitational mass in its own domain (yet to be identified) is effectively wholly embedded in the material rest mass (or wholly occupies the rest mass) M_0 in the proper Euclidean 3-space Σ'. That is, $-M_{0\mathrm{a}}$ has the same shape and same volume as the rest mass M_0 in Σ' as the rest mass M_0 (although $-M_{0\mathrm{a}}$ does not exist in Σ' but in a different space that is embedded in Σ' yet to be isolated). Consequently the density ϱ' of the rest mass M_0 and the density $-\varrho'_{\mathrm{a}}$ of the active gravitational mass (or gravitational charge) $-M_{0\mathrm{a}}$ that occupies the whole of M_0 are equal in magnitude. The model of the containment of $-M_{0\mathrm{a}}$ in M_0 that follows from this discussion is illustrated for a non-spherical gravitational field source as Fig. 1.

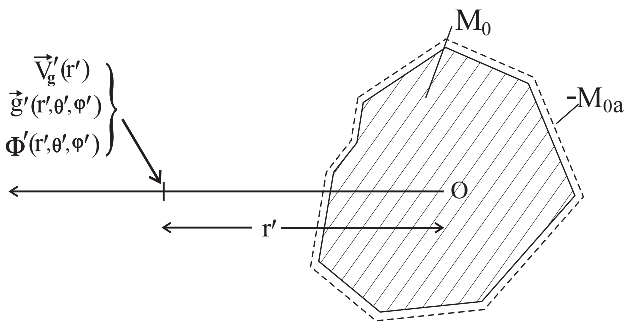

Figure 1: Model of the containment of the gravitational charge in the rest mass of a gravitational field source that is consistent with the identical shapes and densities of the gravitational charge in its space (to be identified) and the rest mass in the physical proper Euclidean 3-space.

Fig. 1a is consistent with the fact that the active gravitational mass (or gravitational charge) $-M_{0\mathrm{a}}$, in its space (to be identified), has the same shape and density as the rest mass M_0 in the proper physical Euclidean 3-space Σ'. It therefore gives rise to identical gravitational potential and field (spherically symmetric or not), as the rest mass would. On the other hand fact that the gravitational velocity $\vec{V}_g(r')$ that the gravitational charge also gives rise to is purely radial in all gravitational fields (see sub-section 2.1 of [2]), is not consistent with the model of the containment of $-M_{0\mathrm{a}}$ in M_0 Fig. 1. It requires further development of the present theory beyond the current level to explain.

The negative active gravitational mass (or negative gravitational charge) cannot annihilate the rest mass containing it in Fig. 1, because the active gravitational charge does not exist in the space Σ' of the rest mass, but in a different space of its own. Consequently M_0 and $-M_{0\mathrm{a}}$ do not touch. The non-spherically symmetric gravitational field $\vec{g}\,'(r', \theta', \varphi')$ originating from the non-spherical rest mass will be identical to the gravitational field originating from the active gravitational mass of the same magnitude and shape as the rest mass in Fig. 1. According to the present theory, the Newton's law of gravity in the differential form must be written in terms of the negative density $-\varrho'_{\mathrm{a}}$ of gravitational charge as

$$\vec{\nabla}' \cdot \vec{g}\,' = 4\pi G(-\varrho'_{\mathrm{a}}) = -4\pi G\varrho'_{\mathrm{a}} \tag{114}$$

Or in integral form as

$$\iint \vec{g}\,' \cdot d\vec{A}' = 4\pi G(-M_{0\mathrm{a}}) = -4\pi G M_{0\mathrm{a}} \tag{115}$$

where $\vec{\nabla}'$ is differential operator in the proper Euclidean 3-space Σ' in which Eqs. (114) and (115) are written.

The origin of the immaterial active gravitational mass (or gravitational charge) and its negative sign, as well as the correct model of its containment within the rest mass, considered to be as illustrated in Fig. 1 for now, and the mechanism by which $-M_{0\mathrm{a}}$ contained within M_0 establishes gravitational speed $V'_g(r')$, gravitational potential $\Phi'(r', \theta', \varphi')$ and gravitational field $\vec{g}\,'(r', \theta', \varphi')$ at every point in space in all finite neighborhood of M_0, shall be investigated elsewhere with further development.

Now a spherical test particle of rest mass m_0 and classical radius r_{0p}, which is isolated from the gravitational field of every other source possesses passive gravitational mass m_p, which is the same as its inertial mass m_i determined by its own gravitational field solely, which is given as follows

$$m_p = m_i = m_0(1 - \frac{2Gm_{0\mathrm{a}}}{r_0 c_g^2}) \tag{116}$$

If the test particle is now located at radial distance r from the center of the passive gravitational mass M_p, (or inertial mass M_i), of a body in the relativistic Euclidean 3-space Σ of TGR, corresponding to radial distance r' from the center of the rest mass M_0 in the proper Euclidean 3-space Σ' of the massive body, then Eq. (116) must be modified as follows

$$m = m_p(1 - \frac{2GM_{0a}}{r'c_g^2}) = m_0(1 - \frac{2Gm_{0a}}{r_0c_g^2})(1 - \frac{2GM_{0a}}{r'c_g^2}) \tag{117}$$

or

$$m = m_i(1 - \frac{2GM_{0a}}{r'c_g^2}) = m_0(1 - \frac{2Gm_{0a}}{r_0c_g^2})(1 - \frac{2GM_{0a}}{r'c_g^2}) \tag{118}$$

The mass m in Eq. (117) or (118) is the passive gravitational mass or the inertial mass of the test particle in the external gravitational field of the massive body of rest mass M_0, where the effect of the gravitational field of the test particle has been taken into consideration. It is inherently assumed that other gravitational field sources are absent in Eq. (117) or (118), otherwise their effect must be taken into account.

The inertial mass or passive gravitational mass of a particle or body determined by the gravitational field of the particle or body solely, shall be denoted by m_i (or M_i) and m_p (or M_p) as done in Eq. (116), while the inertial mass or passive gravitational mass of a particle or body in the external gravitational field of other gravitational field sources shall be denoted by m (or M) as done in Eq. (117) or (118) henceforth. In other words, we shall retain the notation m or M for the gravitational-relativistic mass in the relativistic Euclidean 3-space Σ of TGR, now confirmed to be the inertial mass or passive gravitational mass.

The passive gravitational mass m_p, which is the same as the inertial mass m_i, is the mass observed in the physical 3-space. On the other hand, the rest mass m_0 and the active gravitational mass $-m_{0a}$ hidden within the rest mass, of a particle or body cannot be observed. As Eq. (115) shows, the gravitational field of the particle or body, (no matter how weak), causes its rest mass to evolve into its inertial or passive gravitational mass, even in the absence of external gravitational field, in the context of TGR.

Finally, let us make connection to the so-called "dressed mass" and "undressed mass" of a particle. Let us expand Eq. (118) to have the following

$$m = m_0 - \frac{2Gm_{0a}m_0}{r_0c_g^2} - \frac{2GM_{0a}m_0}{r'^2c_g^2} + \frac{4G^2M_{0a}m_{0a}m_0}{r'r_0c_g^4} \tag{119}$$

The inertial mass m is the observed mass of the particle. It shall also be referred to (humorously) as the "dressed mass", while the rest mass m_0 at the right-hand side shall be referred to as the "undressed mass" (or "naked mass") and the other terms at the right-hand side as the "dressing". These are "dressed mass", "undresses mass" and "dressing" of a test particle at rest relative to the observer in the external gravitational field of a spherical body of rest mass M_0.

4.1 Calculating the rest mass and classical radius of a gravitational field source

The proper (or primed) parameters on the flat proper spacetime (Σ', ct') that evolved at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field, have transformed into the observable (or measurable) gravitational-relativistic parameters on the flat relativistic spacetime (Σ, ct) in the context of TGR, at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field. Consequently the proper (or primed) parameters do not exist to be observed or measured by 3-observers in the relativistic Euclidean 3-space Σ. They are elusive with respect to these observers.

One usefulness of the parameter relations in the context of TGR is that the proper durations of time dt' and proper intervals of space dr', $r'd\theta'$ and $r' \sin\theta' d\varphi'$ involved in events and proper physical parameters Q' on the flat proper spacetime (Σ', ct'), can be calculated from the corresponding measured relativistic (or unprimed) durations of time dt and relativistic (or unprimed) intervals of space dr, $rd\theta$ and $r\sin\theta d\varphi$ of the events and the gravitational-relativistic (or unprimed) physical parameters Q on the flat relativistic spacetime (Σ, ct) of TGR.

It is important to calculate the elusive proper (or classical) values of various physical parameters on the elusive flat proper spacetime (Σ', ct') from their observed and measurable gravitational-relativistic values on the flat relativistic spacetime (Σ, ct) of TGR. This is so because the classical values of physical parameters so calculated become useful data in physics. In order to do this, we must evaluate the factors $\gamma_g(r')^{-2} = (1 - 2GM_{o\text{a}}/r'c_g^2)$, $\gamma_g(r')^{-1}$ and $\gamma_g(r')$ that appear in coordinate interval and parameter transformations in the context of TGR.

Now in order to evaluate $\gamma_g(r')^{-2} = (1 - 2GM_{0\text{a}}/r'c_g^2)$, we must have the value of the elusive gravitational charge $M_{0\text{a}}$ or the value of the rest mass M_0, since M_0 and $M_{0\text{a}}$ are equal in magnitude, as well as the value of the elusive proper radial distance r' from the center of the rest mass M_0 to the location of the rest mass m_0 of the test particle in the elusive proper Euclidean 3-space Σ'. Given that the rest mass M_0 can be calculated, then r' can be obtained by integrating the gravitational length contraction formula, as done in Eqs. (120) and (121).

Given the calculated value of M_0, then r_s can be evaluated from Eq. (121). And

$$\begin{aligned}
\int dr &= \int (1 - \frac{2GM_{0\mathrm{a}}}{r' c_g^2})^{1/2} dr' \\
r &= -\frac{1}{2} \frac{\sqrt{\dfrac{r'-r_s}{r'}}\, r' \left[-2\sqrt{r'^2 - r' r_s} + r_s \ln\left(-\dfrac{1}{2} r_s + r' + \sqrt{r'^2 - r' r_s} \right) \right]}{\sqrt{(r'-r_s) r'}}
\end{aligned} \tag{120}$$

where the integration has been obtained with the Mapplesoft's Mapple 11 software and

$$r_s = 2GM_{0\mathrm{a}}/c_g^2 \equiv 2GM_0/c_g^2 \ , \tag{121}$$

is the Schwarzschild (or gravitational) radius of the gravitational field source.

given the measured value of r (radial distance from the center of the inertial mass M of the gravitational field source to the location of the inertial mass m of the test particle in Σ) and the calculated value of r_s, then r' can be evaluated from Eq. (120). The first thing to do then is to calculate the rest mass M_0 of the gravitational field source.

Let us now calculate the rest mass and classical radius of a gravitational field source. Let us consider a spherically-symmetric gravitational field source of rest mass M_0 and classical radius R_0 (of M_0) on the flat proper spacetime (Σ', ct'), which is isolated from the gravitational field of every other source. The rest mass of the field source will interact with its own gravitational field to become the inertial mass M in the flat relativistic spacetime (Σ, ct) of TGR. Likewise the classical radius R_0 of M_0 will suffer gravitational contraction to become the gravitational-relativistic radius R of M in Σ in the context of TGR, along all radial directions from the center of M_0.

Two equations are required to calculate M_0 and R_0. The first is the mass relation (107) and the second is given by integrating the gravitational length contraction formula at the interior of the gravitational field source. Now the gravitational length contraction formula at the interior of a spherical solid non-black-hole body of rest mass M_0 and radius R_0 with assumed uniform mass-density, which shall be derived elsewhere with further development is the following

$$dr = \gamma_g(r')^{-1} dr' = (1 - \frac{2GM_{0\mathrm{a}}}{R_0 c_g^2} \frac{r'^2}{R_0^2})^{1/2};$$

$$r' \le R_0;\ R_0 > 2GM_{0\mathrm{a}}/c_g^2 = r_s \tag{122}$$

Whereas the first equation of system (12a) is valid for positions exterior to the gravitational field source (i.e. for $r' > R_0$).

By integrating Eq. (122) from the center to the surface of the spherical body and replacing $M_{0\mathrm{a}}$ by M_0 we have

$$\int_0^R dr = \int_0^{R_0} (1 - \frac{2GM_0}{R_0^3 c_g^2 r'^2})^{1/2} dr'$$

or

$$R = \frac{R_0}{2}\left(\cos\left(\sin^{-1}\sqrt{\frac{2GM_0}{R_0 c_g^2}}\right) + \frac{\sin^{-1}\sqrt{\frac{2GM_0}{R_0 c_g^2}}}{\sqrt{\frac{2GM_0}{R_0 c_g^2}}}\right);$$

$$R_0 > 2GM_0/c_g^2 \tag{123}$$

Eq. (107) must likewise be re-written with $M_{0\mathrm{a}}$ replaced by M_0 as follows

$$M = M_0(1 - \frac{2GM_0}{R_0 c_g^2});\ \ R_0 > 2GM_0/c_g^2 \tag{124}$$

Eqs. (123) and (124) must be solved simultaneously for M_0 and R_0, with the observed numerical values of M and R substituted. The numerical value of M_0 obtained must be substituted into Eq. (120) and (121) in order to evaluate r', given the observed (or measured) value of r. The value of $\gamma_g(r')$ can then be evaluated from the calculated values of M_0 and r'.

Eqs. (123) and (124) and Eqs. (120) and (121) apply for all gravitational field sources, safe black holes. It can be shown from Eqs. (123) and (124) that $M = M_0$ and $R = R_0$ in the Newtonian limit $2GM_0/R_0 c_g^2 = 0$. The approximation $M \approx M_0$ and $R \approx R_0$ can be made for sources of weak gravitational fields, such as the earth, the other planets and the Sun, but not for a source of strong gravitational field, such as a neutron star or a white dwarf.

5 Summary and conclusion

Every result in the contexts of TGR and combined TGR and SR on the flat four-dimensional relativistic spacetime (Σ, ct), has its corresponding result in the contexts of the intrinsic theory of gravitational relativity (ϕTGR) and combined intrinsic theory of gravitational relativity and intrinsic special theory of gravity (ϕTGR +

ϕSR) on the flat two-dimensional relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ underlying (Σ, ct). The entire TGR and combined TGR and SR on (Σ, ct) can be derived by the graphical approach in the two-world picture *via* ϕSG and ϕSG+ϕSR, as demonstrated for some results in part one of this paper [4].

The intrinsic gravitational local Lorentz transformation (ϕGLLT), the intrinsic gravitational local Lorentz invariance (ϕGLLI) and the intrinsic mass relation in the context of ϕTGR, as well as intrinsic local Lorentz transformation (ϕLLT), intrinsic Lorentz invariance (ϕLLI) and intrinsic mass relation in the context of combined ϕTGR and ϕSR on the flat two-dimensional relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$, which were derived graphically, were converted to the corresponding results namely, GLLT, GLLI and mass relation in the context of TGR and LLT, LLI and mass relation in the context of combined TGR and SR on the flat four-dimensional relativistic spacetime (Σ, ct), by simply removing the symbol ϕ from the results of ϕTGR and ϕTGR+ϕSR on flat $(\phi\rho, \phi c\phi t)$ essentially in the first part of this paper.

The results of TGR and TGR+SR derived by the analytical approach on the flat four-dimensional relativistic spacetime (Σ, ct) in this paper, can likewise be converted to the corresponding results of ϕTGR and ϕTGR+ϕSR on the flat two-dimensional relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$, by incorporating the symbol ϕ into the results of TGR and TGR+SR. For example, the gravitational potential relation,

$$\Phi(r') = (-GM_{0\mathrm{a}}/r')(1 - 2GM_{0\mathrm{a}}/r'c_g^2)^{1/2},$$

in TGR becomes,

$$\phi\Phi(\phi r') = (-G\phi M_{0\mathrm{a}}/\phi r')(1 - 2G\phi M_{0\mathrm{a}}/\phi r'\phi c_g^2)^{1/2},$$

in the context of ϕTGR; the force relation,

$$\vec{F} = \vec{F}'(1 - 2GM_{0\mathrm{a}}/r'c_g^2)^{1/2},$$

in TGR becomes,

$$\phi F = \phi F'(1 - 2G\phi M_{0\mathrm{a}}/\phi r'\phi c_g^2)^{1/2},$$

in the context of ϕTGR; the special-relativistic kinetic energy relation,

$$\tilde{T} = m_0 c_\gamma^2(1 - 2GM_{0\mathrm{a}}/r'c_g^2)^{1/2}[(1 - v^2/c_\gamma^2)^{-1/2} - 1],$$

in the context of combined TGR and SR becomes

$$\phi\tilde{T} = \phi m_0\phi c_\gamma^2(1 - 2G\phi M_{0\mathrm{a}}/\phi r'\phi c_g^2)^{1/2}\,[(1 - \phi v^2/\phi c_\gamma^2)^{-1/2} - 1],$$

in the context of combined ϕTGR and ϕSR; etc.

The flat four-dimensional relativistic spacetime (Σ, ct) and the inertial masses m, M of particles and bodies in it, are the outward (or physical) manifestations of the flat two-dimensional relativistic intrinsic $(\phi\rho, \phi c\phi t)$ and the intrinsic inertial masses $\phi m, \phi M$ of particles and bodies in it. Likewise the TGR, SR, TGR+SR, CG, CM, etc, on (Σ, ct) are the outward (or physical) manifestations of ϕTGR, ϕSR, ϕTGR+ϕSR, ϕCG, ϕCM, etc, in the underlying $(\phi\rho, \phi c\phi t)$. The following remarks follow:

1. The flat four-dimensional spacetime (Σ, ct) is impossible without the flat two-dimensional intrinsic spacetime $(\phi\rho, \phi c\phi t)$.
2. The theories of gravity and motion namely, TGR, SR, TGR+SR, CG, CM, as well as electromagnetism (EM) and other natural laws on the flat four-dimensional spacetime (Σ, ct), are impossible without the corresponding intrinsic theories on the flat two-dimensional intrinsic spacetime $(\phi\rho, \phi c\phi t)$.
3. Intrinsic physics, (i.e. ϕTGR, ϕSR, ϕTGR+ϕSR, ϕCG, ϕCM, intrinsic electromagnetism (ϕEM) and other intrinsic natural laws in $(\phi\rho, \phi c\phi t)$, help to determine the observed physics, (i.e. TGR, SR, TGR+SR, CG, CM, (EM) and other natural laws), in spacetime

The results of the graphical approach to the theory of gravitational relativity (TGR), combined theory of gravitational relativity and special theory of relativity (TGR+SR) and the other associated theories in the first part of this paper and of the analytical approach to the theories in this second part, shall be deemed to be sufficient for these initial papers, while other ramifications of the theories shall unfold with further development.

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Appendix

Transformations of differential coefficients and differential operators in natural laws on flat spacetime in a gravitational field in the context of the theory of gravitational relativity

The natural laws retain their usual instantaneous differential forms on the flat relativistic spacetime in a gravitational field in the context of the theory of gravitational relativity (TGR). Thus a classical or special-relativistic natural law (non-gravitational or gravitational) may take on the following general form on the flat proper spacetime (Σ', ct') in the absence of relative gravity at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field of arbitrary strength

$$A'\nabla'^2\vec{W}' + B'\vec{\nabla}' \times \vec{W}' = C'\frac{\partial\vec{W}'}{\partial t'} + D'\frac{\partial^2\vec{W}'}{\partial t'^2} \qquad (A1)$$

The twice differentiable function $\vec{W}'(r', \theta', \varphi', t')$ and the differential coefficients A', B', C' and D' are proper (or primed) physical parameters or physical constants on the flat proper spacetime (Σ', ct') (in Fig. 3 or 11 of [1]) in the absence of relative gravity at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field. The primed differential operators $\nabla'^2, \vec{\nabla}', \partial^2/\partial t'^2$ and $\partial/\partial t'$ are in terms of the coordinates of the flat proper spacetime (Σ', ct').

The transformations of the primed classical or primed special-relativistic natural law (A1) on flat proper spacetime (Σ', ct') in the assumed absence of relative gravitational field, at the first stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field into the unprimed form of the law on flat relativistic spacetime (Σ, ct) in the context of the theory of gravitational relativity (TGR) in relative gravitational field, at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field, involves two steps described as follows

Step 1:

The proper (or primed) function $\vec{W}'$ and the proper (or primed) differential coefficients A', B', C' and D', which are physical parameters or physical constants on the flat proper spacetime (Σ', ct'), are transformed into their gravitational-relativistic (or unprimed) values $\vec{W}, A, B, C$ and D respectively, on the flat relativistic spacetime (Σ, ct) in a gravitational field in the context of TGR, as done for some physical parameters in this paper and as shall be done further in the other papers.

Step 2:

The primed differential operators $\nabla'^2, \vec{\nabla}', \partial/\partial t'$ and $\partial^2/\partial t'^2$ in terms of the coordinates of the flat proper spacetime (Σ', ct') are transformed into the respective unprimed differential operators $\nabla^2, \vec{\nabla}, \partial/\partial t$ and $\partial^2/\partial t^2$ in terms of the coordinates of the flat relativistic spacetime (Σ, ct) in the context of TGR as described hereunder.

Now the transformations of the proper (or primed) parameters $\vec{W}', A', B', C'$ and D' to the relativistic (or unprimed) parameters $\vec{W}, A, B, C$ and D respectively in the context of TGR will take the following forms

$$\vec{W} = f_W(V'_g(r))\vec{W}' \text{ or } \vec{W}' = f_W^{-1}(V'_g(r))\vec{W} \qquad (A2a)$$

$$A = f_A(V'_g(r'))A' \text{ or } A' = f_A^{-1}(V'_g(r'))A \qquad (A2b)$$

$$B = f_B(V'_g(r'))B' \text{ or } B' = f_B^{-1}(V'_g(r'))B \qquad (A2c)$$

$$C = f_C(V'_g(r'))C' \text{ or } C' = f_C^{-1}(V'_g(r'))C \qquad (A2d)$$

$$D = f_D(V'_g(r'))D' \text{ or } D' = f_D^{-1}(V'_g(r'))D \qquad (A2e)$$

By using the inverse relations in equations (A2a) – (A2e) to eliminate $\vec{W}', A', B', C'$ and D' in Eq. (A1) we have,

$$\begin{aligned} & f_A^{-1}(V'_g(r'))A\nabla'^2(f_W^{-1}(V'_g(r'))\vec{W}) \\ & \quad + f_B^{-1}(V'_g(r'))B\vec{\nabla}' \times (f_W^{-1}(V'_g(r'))\vec{W}) \\ & = f_C^{-1}(V'_g(r'))C\frac{\partial}{\partial t'}(f_W^{-1}(V'_g(r'))\vec{W}) \\ & \quad + f_D^{-1}(V'_g(r'))D\frac{\partial^2}{\partial t'^2}(f_W^{-1}(V'_g(r'))\vec{W}) \qquad (A3) \end{aligned}$$

However the inverse function $f_W^{-1}(V'_g(r'))$, as well as the others are constant in space within every local Lorentz frame, within which the transformations of natural laws are derived in a gravitational field. They are also time-independent for static gravitational fields that we shall be concerned with. Hence the inverse function $f_W^{-1}(V'_g(r'))$ can be factored out, thereby simplifying Eq. (A3) as follows

$$\begin{aligned} & f_A^{-1}(V'_g(r'))A\nabla'^2\vec{W} + f_B^{-1}(V'_g(r'))B\vec{\nabla}' \times \vec{W} \\ & = f_C^{-1}(V'_g(r'))C\frac{\partial}{\partial t'}\vec{W} + f_D^{-1}(V'_g(r'))D\frac{\partial^2}{\partial t'^2}\vec{W} \qquad (A4) \end{aligned}$$

The first step transforms the natural law (gravitational or non-gravitational) of Eq. (A1) into Eq. (A4).

The transformations of the primed operators $\nabla'^2, \vec{\nabla}', \partial/\partial t'$ and $\partial^2/\partial t'^2$, in terms of the coordinates of the flat proper spacetime (Σ', ct') that appear in Eq. (A4), into the respective unprimed operators in terms of the coordinates of the flat relativistic spacetime (Σ, ct), in the context of TGR, must then be derived at the second step. This must be done with the aid of the gravitational local Lorentz transformation (GLLT) and its inverse of systems (3) and (4), which shall be re-written here as follows

$$\begin{aligned} dt' &= \gamma_g(r')(dt - (V'_g(r')/c_g^2)dr); \\ dr' &= \gamma_g(r')(dr - V'_g(r')dt); \\ r'd\theta' &= rd\theta \text{ and } r'\sin\theta' d\varphi' = r\sin\theta d\varphi \qquad (A5) \\ dt &= \gamma_g(r')(dt' + (V'_g(r')/c_g^2)dr'); \end{aligned}$$

$$dr = \gamma_g(r')(dr' + V'_g(r')dt');$$
$$rd\theta = r'd\theta' \text{ and } r\sin\theta d\varphi = r'\sin\theta' d\varphi' \qquad (A6)$$

where

$$\gamma_g(r') = (1 - V'_g(r')^2/c_g^2)^{-1/2} = (1 - 2GM_{0\mathrm{a}}/r'c_g^2)^{-1/2} \qquad (A7)$$

I. Transformations of differential operators that are relevant for the transformations of non-gravitational laws in the context of the theory of gravitational relativity

The special theory of relativity (SR) and other non-gravitational dynamical laws, such as the law of propagation of massless electric and magnetic fields (electromagnetism), law of propagation of massless waves, law of propagation of heat energy through media, law of propagation of pressure waves through media, etc, involve dynamical velocities $\vec{u}$ of massless waves, massless fields and other massless non-gravitational parameters within local Lorentz frames on the flat relativistic spacetime (Σ, ct) in every gravitational field relative to observers. They also acquire relative gravitational velocity $\vec{V}'_g(r')$ within local Lorentz frames on the flat relativistic spacetime (Σ, ct) in every gravitational field.

However the relative gravitational velocity is not made manifest in actual translation in spacetime, hence any magnitude of $\vec{V}'_g(r')$ is equivalent to zero magnitude of dynamical velocity. Consequently a massless non-gravitational parameter, although possesses both relative dynamical velocities $\vec{u}$ and relative gravitational velocities $\vec{V}'_g(r')$ on flat spacetime in a gravitational field, it possesses resultant relative dynamical velocity $\vec{u}$, since $\vec{V}'_g(r')$ must be set to zero in the composition of $\vec{u}$ and $\vec{V}'_g(r')$, from the point of view of dynamical relativity.

Now in deriving the transformations of the non-gravitational laws in the context of the theory of gravitational relativity (TGR), we must employ the gravitational local Lorentz transformation (GLLT) and its inverse, while allowing the gravitational speed $V'_g(r')$ to vanish. Consequently the form of the GLLT and its inverse to apply in deriving the transformations of non-gravitational laws in the context of TGR (obtained by letting $V_g(r') = 0$, hence $\gamma_g(r') = 1$ in Eqs. (A5) and (A6)), is the following

$$dt' = dt;\ dr' = dr';\ r'd\theta' = rd\theta;\ r'\sin\theta' d\varphi' = r\sin\theta d\varphi \qquad (A8)$$

The trivial coordinate interval transformations (A8) imply the following trivial differential operator transformations

$$\nabla'^2 = \nabla^2;\ \vec{\nabla}' = \vec{\nabla};\ \partial/\partial t' = \partial/\partial t;$$

$$\partial^2/\partial t'^2 = \partial^2/\partial t^2;\ \text{e.t.c.} \tag{A9}$$

It is the trivial coordinate interval transformations (A8) and the implied trivial differential operator transformations (A9) that must be employed in deriving the transformations of the non-gravitational laws in the context of TGR.

If we assume that the natural law (A1) is a non-gravitational law and use system (A9) in Eq. (A4) we have

$$f_A^{-1}(V_g'(r'))A\nabla^2\vec{W} + f_B^{-1}(V_g'(r'))B\vec{\nabla}\times\vec{W}$$

$$= f_C^{-1}(V_g'(r'))C\frac{\partial}{\partial t}\vec{W} + f_D^{-1}(V_g'(r'))D\frac{\partial^2}{\partial t^2}\vec{W} \tag{A10}$$

Equation (A10) is the transformation of the assumed non-gravitational law (A1) in a gravitational field in the context of TGR. The inverse transformation functions $f_A^{-1}(V_g'(r'))$, $f_B^{-1}(V_g'(r'))$, $f_C^{-1}(V_g'(r'))$ and $f_D^{-1}(V_g'(r'))$ may not cancel out in (A10). It must then be said that the non-gravitational law (A1) is not invariant with transformation in the context of TGR and that the law depends on position in a gravitational field as a consequence, (by virtue of the inverse transformation functions that do not cancel out).

On the other hand, the inverse transformation functions $f_A^{-1}(V_g'(r'))$, $f_B^{-1}(V_g'(r'))$, $f_C^{-1}(V_g'(r'))$ and $f_D^{-1}(V_g'(r'))$ may cancel out naturally in Eq. (A10), thereby simplifying that equation as follows

$$A\nabla^2\vec{W} + B\vec{\nabla}\times\vec{W} = C\frac{\partial}{\partial t}\vec{W} + D\frac{\partial^2}{\partial t^2}\vec{W} \tag{A11}$$

Then the assumed non-gravitational natural law (A1) must be said to be invariant with transformation in the context of TGR. It does not depend on position in a gravitational field as a consequence.

The inverse transformation functions $f_A^{-1}(V_g'(r'))$, $f_B^{-1}(V_g'(r'))$, $f_C^{-1}(V_g'(r'))$ and $f_D^{-1}(V_g'(r'))$, can be obtained by deriving the transformations of the primed parameters A′, B′, C′ and D′ and primed function $\vec{W}'$ into the unprimed parameters A, B, C and D and unprimed function $\vec{W}$ respectively in the context of TGR or they can simply be read off from a table containing the transformations of physical parameters and physical constants in the context of TGR, which shall be developed to a large extent in a paper shortly in this volume.

II. Transformations of differential operators that are relevant for the transformations of gravitational laws in the context of the theory of gravitational relativity

If the natural law (A1) is a gravitational law, then the step from Eq. (A1) to Eq. (A4) must be carried out. However the transformations of the primed differential operators $\nabla'^2, \vec{\nabla}', \partial/\partial t'$ and $\partial^2/\partial t'^2$ into the unprimed operators $\nabla^2, \vec{\nabla}, \partial/\partial t$ and $\partial^2/\partial t^2$ respectively, for the purpose of transforming gravitational laws in the context of TGR, are elaborate and grossly different from the trivial transformations of differential operators of system (A9) to be used in transforming the non-gravitational laws in the context of TGR.

While the gravitational speed $V'_g(r')$ that appears in the GLLT (A5) and its inverse (A6) must be set to zero from the point of view of dynamical relativity, thereby reducing the GLLT and its inverse to the trivial coordinate interval transformations (A8) and the implied trivial differential operator transformations of system (A9), for the purpose of deriving the transformations of the non-gravitational laws in the context of TGR, it (the gravitational speed) is a non-zero relative speed in the context of the gravitational laws (or from the point of view of gravitational relativity). The relativity of gravitational velocity and the associated relativity of gravity, as has been discussed at different points in the previous papers, starting from sub-sub-section 2.2.1 of [2], refers to relativity of position, that is, variations of $V'_g(r')$ and the other gravitational parameters $\Phi(r')$ and $\vec{g}(r')$ with position in a gravitational field.

It follows from the foregoing paragraph that the full GLLT and its inverse must be employed in deriving the transformations of differential operators to be used in transforming the gravitational laws in the context of the theory of gravitational relativity (TGR). Now let us obtain the total differential of the unprimed twice differentiable function $\vec{W}(r, \theta, \varphi, t)$ that appears in (A4) as follows

$$d\vec{W} = \frac{\partial \vec{W}}{\partial t}dt + \frac{\partial \vec{W}}{\partial r}dr + \frac{\partial \vec{W}}{r\partial \theta}rd\theta + \frac{\partial \vec{W}}{r\sin\theta\partial\varphi}r\sin\theta d\varphi \qquad (A13)$$

Division of (A13) by dt' gives the following

$$\frac{d\vec{W}}{dt'} = \frac{\partial \vec{W}}{\partial t}\frac{dt}{dt'} + \frac{\partial \vec{W}}{\partial r}\frac{dr}{dt'} + \frac{\partial \vec{W}}{r\partial \theta}\frac{rd\theta}{dt'} + \frac{\partial \vec{W}}{r\sin\theta\partial\varphi}\frac{r\sin\theta d\varphi}{dt'} \qquad (A14)$$

Upon evaluating $dt/dt', dr/dt', rd\theta/dt'$ and $r\sin\theta d\varphi/dt'$ with the aid of the inverse

GLLT of system (A6), Eq. (A14) becomes the following

$$\frac{d\vec{W}}{dt'} = \frac{\partial\vec{W}}{\partial t'} = \gamma_g(r')\frac{\partial\vec{W}}{\partial t} + \gamma_g(r')\frac{V_g(r')}{c_g^2}\frac{\partial\vec{W}}{\partial r}$$

Hence

$$\frac{\partial}{\partial t'} = \gamma_g(r')\frac{\partial}{\partial t} + \gamma_g(r')\frac{V_g(r')}{c_g^2}\frac{\partial}{\partial r} \qquad (A15)$$

and

$$\frac{\partial^2}{\partial t'^2} = \left(\gamma_g(r')\frac{\partial}{\partial t} + \gamma_g(r')\frac{V_g(r')}{c_g^2}\frac{\partial}{\partial r}\right)^2$$

$$\frac{\partial^2}{\partial t'^2} = \gamma_g(r')^2\frac{\partial^2}{\partial t^2} + 2\gamma_g(r')^2\frac{V_g(r')}{c_g^2}\frac{\partial^2}{\partial r\partial t} + \gamma_g(r')^2\frac{V_g(r')^2}{c_g^4}\frac{\partial^2}{\partial r^2} \qquad (A16)$$

Division of (A13) by dr' gives the following

$$\frac{d\vec{W}}{dr'} = \frac{\partial\vec{W}}{\partial t}\frac{dt}{dr'} + \frac{\partial\vec{W}}{\partial r}\frac{dr}{dr'} + \frac{\partial\vec{W}}{r\partial\theta}\frac{rd\theta}{dr'} + \frac{\partial\vec{W}}{r\sin\theta\partial\varphi}\frac{r\sin\theta d\varphi}{dr'} \qquad (A17)$$

Again upon evaluating dt/dr' , dr/dr' , $rd\theta/dr'$ and $r\sin\theta d\varphi/dr'$ with the aid of the inverse GLLT of system (A6), Eq. (A17) becomes the following

$$\frac{d\vec{W}}{dr'} = \frac{\partial\vec{W}}{\partial r'} = \gamma_g(r')\frac{V_g(r')}{c_g^2}\frac{\partial\vec{W}}{\partial t} + \gamma_g(r')\frac{\partial\vec{W}}{\partial r}$$

Hence

$$\frac{\partial}{\partial r'} = \gamma_g(r')\frac{\partial}{\partial r} + \gamma_g(r')\frac{V_g(r')}{c_g^2}\frac{\partial}{\partial t} \qquad (A18)$$

and

$$\frac{\partial^2}{\partial r'^2} = \left(\gamma_g(r')\frac{\partial}{\partial r} + \gamma_g(r')\frac{V_g(r')}{c_g^2}\frac{\partial}{\partial t}\right)^2$$

$$\frac{\partial^2}{\partial r'^2} = \gamma_g(r')^2\frac{\partial^2}{\partial r^2} + 2\gamma_g(r')^2\frac{V_g(r')}{c_g^2}\frac{\partial^2}{\partial r\partial t}$$

$$+\gamma_g(r')^2\frac{V_g(r')^2}{c_g^4}\frac{\partial^2}{\partial t^2} \tag{A19}$$

Division of Eq. (A13) by $r'd\theta'$ gives the following

$$\frac{d\vec{W}}{r'd\theta'}=\frac{\partial\vec{W}}{\partial t}\frac{dt}{r'd\theta'}+\frac{\partial\vec{W}}{\partial r}\frac{dr}{r'd\theta'}+\frac{\partial\vec{W}}{r\partial\theta}\frac{rd\theta}{r'd\theta'}$$
$$+\frac{\partial\vec{W}}{r\sin\theta\partial\varphi}\frac{r\sin\theta d\varphi}{r'd\theta'} \tag{A20}$$

But $dt/r'd\theta'=dr/r'd\theta'=r\sin\theta d\varphi/r'd\theta'=0$ and $rd\theta/r'd\theta'=1$ from system (A6). Hence Eq. (A20) simplifies as follows

$$\frac{d\vec{W}}{r'd\theta'}=\frac{\partial\vec{W}}{r'\partial\theta'}=\frac{\partial\vec{W}}{r\partial\theta}$$

Hence

$$\frac{\partial}{r'\partial\theta'}=\frac{\partial}{r\partial\theta} \tag{A21}$$

and

$$\frac{\partial^2}{r'^2\partial\theta'^2}=\frac{\partial^2}{r^2\partial\theta^2} \tag{A22}$$

Finally division of Eq. (A13) by $r'\sin\theta'd\varphi'$ gives the following

$$\frac{d\vec{W}}{r'\sin\theta'd\varphi'}=\frac{\partial\vec{W}}{\partial t}\frac{dt}{r'\sin\theta'd\varphi'}+\frac{\partial\vec{W}}{\partial r}\frac{dr}{r'\sin\theta'd\varphi'}$$
$$+\frac{\partial\vec{W}}{r\partial\theta}\frac{rd\theta}{r'\sin\theta'd\varphi'}+\frac{\partial\vec{W}}{r\sin\theta\partial\varphi}\frac{r\sin\theta d\varphi}{r'\sin\theta'd\varphi'} \tag{A23}$$

But,

$$dt/r'\sin\theta'd\varphi'=dr/r'\sin\theta'd\varphi'=rd\theta/r'\sin\theta'd\varphi'=0 \quad \text{and}$$
$$r\sin\theta d\varphi/r'\sin\theta'd\varphi'=1,$$

from system (A6). Hence Eq. (A23) simplifies as follows

$$\frac{d\vec{W}}{r'\sin\theta'd\varphi'}=\frac{\partial\vec{W}}{r'\sin\theta'\partial\varphi'}=\frac{\partial\vec{W}}{r\sin\theta\partial\varphi}$$

Hence

$$\frac{\partial}{r'\sin\theta'\partial\varphi'}=\frac{\partial}{r\sin\theta\partial\varphi} \tag{A24}$$

and

$$\frac{\partial^2}{r'^2 \sin^2 \theta' \partial \varphi'^2} = \frac{\partial^2}{r^2 \sin^2 \theta \partial \varphi^2} \tag{A25}$$

By collecting equations (A18), (A21) and (A24) we have

$$\vec{\nabla}' = \frac{\partial}{\partial r'}\hat{r}' + \frac{\partial}{r'\partial \theta'}\hat{\theta}' + \frac{\partial}{r' \sin\theta' \partial \varphi'}\hat{\varphi}'$$

$$= \left(\gamma_g(r')\frac{\partial}{\partial r} + \gamma_g(r')\frac{V_g(r')}{c_g^2}\frac{\partial}{\partial t}\right)\hat{r} + \frac{\partial}{r\partial\theta}\hat{\theta}$$

$$+\frac{\partial}{r\sin\theta\partial\varphi}\hat{\varphi} \tag{A26}$$

And by collecting equations (A19), (A22) and (A25) we have

$$\nabla'^2 = \frac{\partial^2}{\partial r'^2} + \frac{\partial^2}{r'^2 \partial \theta'^2} + \frac{\partial^2}{r'^2 \sin^2\theta' \partial\varphi'^2}$$

$$= \gamma_g(r')^2\frac{\partial^2}{\partial r^2} + 2\gamma_g(r')^2\frac{V_g(r')}{c_g^2}\frac{\partial^2}{\partial r \partial t}$$

$$+\gamma_g(r')^2\frac{V_g(r')^2}{c_g^4}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{r^2\partial\theta^2} + \frac{\partial^2}{r^2\sin^2\theta\partial\varphi^2} \tag{A27}$$

Equations (A15), (A16), (A26) and (A27) are the most general forms of the differential operator transformations to be used in transforming gravitational laws in the context of TGR. They shall find useful application in the transformations of the equations of the Maxwellian theory of gravity (MTG), to be developed elsewhere with further development.

II. 1. Transformation of the 3-geometry Newton's law of gravity in the context of the theory of gravitational relativity

For the transformation of the three-dimensional Newton's law of gravity in the context of the TGR, on the other hand, the GLLT must be truncated to a three-dimensional system by making the time dimension invariant (or absolute). This gives non-trivial spatial coordinate interval transformations only, which are the local coordinate transformations in the context of TGR that are relevant for the 3-geometry Newton's law of gravity namely,

$$dr' = \gamma_g(r')dr\,; \quad r'd\theta' = rd\theta\,;$$

$$r' \sin\theta' d\varphi' = r \sin\theta d\varphi \text{ and } dt' = dt \tag{A28}$$

The transformations of the gradient and Laplacian operators in the context of TGR implied by system (A28) are the following

$$\begin{aligned} \vec{\nabla}' &= \frac{\partial}{\partial r'}\hat{r}' + \frac{\partial}{r'\partial\theta'}\hat{\theta}' + \frac{\partial}{r'\sin\theta'\partial\varphi'}\hat{\varphi}' \\ &= \gamma_g(r')^{-1}\frac{\partial}{\partial r}\hat{r} + \frac{\partial}{r\partial\theta}\hat{\theta} + \frac{\partial}{r\sin\theta\partial\varphi}\hat{\varphi} \end{aligned} \tag{A29}$$

and

$$\begin{aligned} \nabla'^2 &= \frac{\partial^2}{\partial r'^2} + \frac{\partial^2}{r'^2\partial\theta'^2} + \frac{\partial^2}{r'^2\sin^2\theta'\partial\varphi'^2} \\ &= \gamma_g(r')^{-2}\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{r^2\partial\theta^2} + \frac{\partial^2}{r^2\sin^2\theta\partial\varphi^2} \end{aligned} \tag{A30}$$

And the first-order and second-order derivatives with respect to time implied by system (A28) are

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \text{ and } \frac{\partial^2}{\partial t'^2} = \frac{\partial^2}{\partial t^2} \tag{A31}$$

While Eqs. (A29) – (A30) are relevant for the transformation of Newton's law of gravity in non-spherically-symmetry gravitational fields, they simplify as follows in the case of spherically-symmetric gravitational fields

$$\frac{d}{dr'} = \gamma_g(r')^{-1}\frac{d}{dr} \text{ and } \frac{d^2}{dr'^2} = \gamma_g(r')^{-2}\frac{d^2}{dr^2} \tag{A31}$$