On the Physics in Fundamental General Relativity Theory.

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Abstract.

It is the intention here to examine in detail the way in which purely physical ideas are introduced into General Relativity to enable the initially pure mathematical theory to be applied to physical problems. The intention is to examine the approach used frequently in undergraduate presentations since this is the point in life when so many are crucially influenced by thoughts and ideas put before them.

Introduction.

The theory of general relativity grows out of what is initially pure mathematical reasoning associated with the field known as Riemannian Geometry. When first introduced to the area, I suspect many students - I myself being one such student some years ago - are seduced by the beautiful and powerful mathematics. This, in turn, often – probably usually – prevents the student asking questions that might occur to them under different circumstances. Such questions frequently revolve around the way in which highly physical concepts such as mass and potential are introduced into an essentially pure mathematical scenario to produce what is supposedly a theory of gravitation purporting to generalise that of Newton.

It might be noted that concerns about this very topic have been raised fairly recently [1]. Hence, it is the aim of this article to examine in some detail the way in which the pure mathematical theory is usually developed for undergraduates before noting how the physical concepts alluded to above are introduced. It should be admitted from the outset that none of the material here is new; the mathematical detail is being included purely to enable a full and complete examination of all the steps followed so that, having seen precisely how and where physical ideas are introduced, conclusions may be reached.

The Background Theory.

In most, if not all, textbooks introducing this subject, the general pattern is laid out very quickly. The first few chapters of such books are devoted to a fairly detailed study of the basics of tensor algebra and tensor calculus. This is seen quite clearly in, for example, the book by Adler, Bazin and Schiffer [2], where the first real mention of physics occurs in chapter 4 when tensors in physics are discussed. However, it should be noted from the outset that such an approach is highly desirable, even essential, considering that the basic foundations of the subject are firmly rooted in Riemannian Geometry and involve the constant use of tensors.

The starting point for everything is an introduction to tensors and manipulations involving these quantities. At this stage, tensors are purely mathematical entities which transform according to rigorously specified rules. After these preliminaries, one is plunged into the details of Riemannian Geometry and it is interesting to note that the path followed closely resembles that to be found in books devoted entirely to the study of differential geometry [3]. Again, this might be felt to be as it should be, but it does indicate just how deeply rooted the whole theory is in pure mathematics. Often, once attention is restricted to Riemannian space, a discussion of geodesics in such a space ensues. A geodesic is, of course, simply a curve of stationary length between two points and is easily visualised as a generalisation to general spaces of the straight line in Euclidean space. In a Euclidean space, the square of the distance between two neighbouring point with coordinates (x,y,z) and $(x+dx,y+dy,z+dz)$ is written

$$ds^2 = dx^2 + dy^2 + dz^2$$

and, in a Riemannian space, this is written

$$ds^2 = g_{ij}dx^i dx^j$$

for the square of the distance between the neighbouring points x^i and x^i+dx^i, where i,j may take the values 0,1,2,3, since a four dimensional space is usually considered. This expression for the distance between two neighbouring points is a quantity generally referred to as the metric and the quantity g_{ij} is called the metric tensor. The notation in this latter expression is such that, although the 2 refers to a power as usual, the i and j are indices over which one must sum for the values 0,1,2,3. In order to discuss these curves called geodesics, it is necessary to carry out the manipulation referred to as covariant differentiation. The notation used here is for an ordinary partial derivative to be denoted as follows:

$$\frac{\partial g_{ij}}{\partial x^k} = g_{ij,k}$$

where the index notation has been used again.

Once this has been introduced, further manipulations lead to the introduction of the so-called Christoffel symbols:

$$[jk,i] = \tfrac{1}{2}\left(g_{ki,j} + g_{ij,k} - g_{jk,i}\right),$$

which is the Christoffel symbol of the first kind, and

$$\Gamma_{jk}^{m} = g^{mi}[jk,i],$$

which is the Christoffel symbol of the second kind.

The next crucial step is to note that, if A_i is a vector, then the quantity

$$A_{i,j} - \Gamma_{ij}^{k} A_k$$

is a tensor and is called the covariant derivative of A_i with respect to x^j. It is often written $A_{i;j}$

Next the second order covariant derivative of a vector is evaluated and it is noted that the order of differentiation is crucial in this manipulation. In fact, after some work, it is seen that

$$A_{i;jk} - A_{i;kj} = \left(\Gamma_{ik,j}^{m} - \Gamma_{ij,k}^{m} + \Gamma_{ik}^{n}\Gamma_{nj}^{m} - \Gamma_{ij}^{n}\Gamma_{nk}^{m}\right)A_m \; = R_{ijk}^{m} A_m,$$

which serves to act as an introduction to, and definition of, the Riemann or Curvature tensor.

By simply examining the definition, several properties of the Riemann tensor may now be found. The most important for the present purpose is the so-called Bianchi identity, which is

$$R_{mkij;n} + R_{mkjn;j} + R_{mkni;j} = 0,$$

where the process of lowering the first index by using the fact that

$$R_{nijk} = g_{nm} R_{ijk}^{m}$$

has been employed.

Another technique of great use in connection with tensors is that of contraction. This is best explained via the example of contracting the Riemann tensor on its first and last indices:

$$g^{nk} R_{nijk} = R_{ij},$$

which is the Ricci tensor.

Now multiplying the Bianchi identity by g^{mj} and noting that $g_{,n}^{mj} = 0$,

$$R_{ki;n} - R_{kn;i} + R_{kni;j}^{j} = 0.$$

Then multiply by g^{kn} and again note that $g_{;n}^{kn} = 0$ to give

$$R_{i;n}^{n} - R_{;i} + R_{i;j}^{j} = 0$$

or

$$\left(R_i^n - \frac{1}{2}\delta_i^n R\right)_{;n} = 0$$

or

$$G_{i;n}^{n} = 0,$$

which shows that the tensor G_i^n has zero divergence, is given by $G_i^n = R_i^n - \frac{1}{2}\delta_i^n R$ and is known as the Einstein tensor.

It is important to note that everything commented on or derived up to now has been associated with an abstract mathematical space. There has been no mention of anything physical; no gravitational fields have been mentioned and no masses have been introduced. The entire discussion might be thought to have been geometrical in nature. However, basically at this point, Einstein took the equations

$$G_{ij} = 0$$

as his field equations for empty space-time. This choice was not made completely randomly, but on the basis of three principles:

(i) the principle of covariance which basically states that the laws of physics must be expressed in a form independent of any coordinate system,

(ii) the principle of equivalence which states that there is no difference between an acceleration and a gravitational field,

(iii) Mach's principle which states that the geometrical properties of space-time are determined completely by the material present in it.

The precise details of the argument involved may be found in many textbooks on the subject [2]. However, initially, a search is instigated for a tensor quantity which describes a matter distribution with respect to any frame in space-time and then an attempt is made to link this to the Einstein tensor defined above. The obvious candidate at the time had to be the so-called energy-momentum tensor, T_i^n, since both matter and electromagnetic energy contribute to its components. Since mass and energy are fundamentally identical, it was expected that all forms of energy – including electromagnetic – would contribute to the gravitational field. Accordingly, the equation

$$G_i^n = \kappa T_i^n$$

was adopted as Einstein's Law of Gravitation, where κ is a constant of proportionality yet to be determined and this equation reduces to

$$G_i^n = 0$$

in the absence of matter or, alternatively, for empty space-time. Do note that this is the same equation for empty space-time as that mentioned above but with the one suffix raised by the usual techniques of tensor manipulation.

The Introduction of Physical Concepts.

Although the above appears to show how genuinely physical quantities are introduced into general relativity, the reality is, if anything, even more bizarre. It should be noted that the process involves two separate steps. Firstly, by having recourse to Newtonian ideas, a value is found for the constant κ. Secondly, a solution for the field equations in empty space-time is sought for the case of a spherically symmetric static field. This second step was originally investigated by Schwarzschild in 1916 [4] but more of that later.

For the first step, well-established Newtonian theory is assumed a first approximation to the field equations. Therefore, the situation is considered in which velocities are assumed small and gravitational fields weak. If these assumptions are made and the case of a free particle in a potential field ϕ is considered, the component of the metric tensor denoted by g_{00} is found to be given by

$$g_{00} = 1 + 2\phi.$$

Hence, the potential field is introduced by comparing a result derived purely geometrically with a result from Newtonian mechanics.

The complete equations relevant for Einstein's Law of Gravitation are then considered and it is noted that, in the approximation being used, the component of the energy-momentum tensor denoted by T_{00} is equal to ρ, the density of matter involved. By using this together with the above expression for g_{00}, it is found that

$$\nabla^2\phi = \frac{1}{2}\kappa\rho$$

and, when this is compared with the well-known Newtonian expression

$$\nabla^2\phi = 4\pi\rho,$$

it is seen immediately that, for consistency, κ must be chosen to have the value 4π.

As stated already, the second step consists of searching for a solution to the equations appropriate for empty space-time but, following Schwarzschild, attention is customarily concentrated upon a solution for a spherically symmetric, static field. This problem actually involves seeking a suitable expression for the line element ds^2. After some general geometrical arguments, it is found that, in terms of spherical polar coordinates, the most general line element with the designated properties may be written

$$ds^2 = e^{\upsilon}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2).$$

Recognising the fact that the coefficients in this expression represent components of the relevant metric tensor, that is:

$$g_{00} = e^{\upsilon}, \qquad g_{11} = -e^{\lambda}, \qquad g_{22} = -r^2, \qquad g_{33} = -r^2 sin^2\theta,$$

the various components of the Einstein tensor may be evaluated. Since each of these components must be separately equal to zero for empty space-time, these equations may be solved to yield

$$e^{\upsilon} = g_{00} = b - {}^{a}/_{r},$$

where a and b are constants.

Referring back once again to the earlier result identifying g_{00} with $(1 + 2\phi)$, where ϕ is the Newtonian potential and noting that, for the gravitational field surrounding a spherically symmetric mass m,

$$\phi = -{}^{m}/_{r},$$

it follows that the constants a and b must be chosen to have the values

$$b = 1 \text{ and } a = 2m$$

for consistency.

Hence, a mass, m, is introduced into the situation which supposedly describes empty space-time purely to achieve consistency with well-established results of Newtonian mechanics.

Discussion.

It is immediately obvious from the above discussion that the ideas of potential and mass only enter the theory when the Newtonian approximation is considered. This does not seem totally unreasonable when contemplating the entry of the potential but a moment's reflection introduces an element of unease when it comes to the introduction of mass into the theory. In the initial discussion of the Newtonian approximation, the complete Einstein equation describing his Law of Gravitation was used. This involved the presence of the energy-momentum tensor appearing on the right-hand side of the equation, albeit in a simplified form. However, in the second part of the discussion, the empty space-time version of the equation was used. Hence, only the Einstein tensor appeared and the equation used placed this tensor equal to zero. From all that has gone before, it is abundantly clear that this tensor is a purely geometrical quantity, describing the purely geometrical properties of the space-time. Hence, when the investigator has sought a line element for such a space, how valid is it to call on a Newtonian result which involves a particle mass when the evaluation of certain constants is contemplated? Indeed, considering the original problem concerned an empty space-time, how valid is it to draw on results relating to the gravitational field surrounding a spherically symmetric mass? The immediate reaction must surely be that there cannot be a mass, spherically symmetric or otherwise, in an empty space and so, how can such an entity have any part to play in the said discussion? There are certainly some serious questions here which are in need of urgent answers. Those answers may be readily forthcoming but they haven't appeared as yet and undergraduates continue to be seduced by the beauty of the mathematics. If anything, the situation becomes all the more disturbing when it is remembered that Hermann Weyl once said "I always try to combine the true with the beautiful, but when I have to choose one or the other, I usually choose the beautiful". It might be wondered how far this attitude extends in the realms of academia?

Nevertheless, the questions posed here are deceptively simple but far reaching as far as physics is concerned and the anonymous quote – ‘A child can ask questions a wise man cannot answer’ – springs to mind very readily. The questions are simple but cannot be dismissed too readily.

Of course, whatever qualms may be harboured, the fact remains that the theory has been successful in explaining several observed phenomena. However, it has to be noted that these same phenomena do admit explanations by means other than recourse to the methods associated with general relativity and this fact might be felt to raise further queries. One example of such alternative explanations is provided by the work of Bernard Lavenda [5] and another by that of Harold Aspden [6]. Work such as this cannot be ignored or dismissed as irrelevant and is based on methods seemingly more directly linked with physical reality than general relativity appears to be. As noted earlier, the mathematical theory associated with general relativity is truly beautiful but such beauty is not, of itself, any guarantee of correctness. It does seem that the time has come for the topic to be reassessed and any such reassessment must be carried out by people with completely open minds.

Finally, it should be noted, once again, that the method for seeking a spherically symmetric solution to the Einstein equations was originally due to Schwarzschild. However, again it must be stressed that the version of the so-called Schwarzschild solution appearing in the vast majority of texts is not actually that due to Schwarzschild himself, as may be noticed by either checking his original article or the fairly recent translation of that article [4]. The crucial difference is the absence of the well-known singularity at $r = 2m$ in his original. Hence, when checking through the minute details of what is simply sketched out here, reference to the original is advisable.

References.

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