Definition à la Bourbaki of the Basic Notions of Category Theory

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Abstract. We unsuccessfully try to give definitions in the spirit of Bourbaki's set theory for the basic notions of category theory. The goal is to avoid using either Grothendieck's universes axiom, or "classes" (or "collections") of sets which are not sets. We explain why our attempt fails.

Realizing that there was a mistake in the previous versions of this text, I wanted to withdraw it from viXra. So, I read the appropriate documentation, and found the sentence

If you are withdrawing because your paper has errors please consider submitting a new version as an errata instead so that other people can avoid the same mistakes.

Following this wise advice, here is firstly the previous version of the text, and secondly an explanation of the mistake.

Part 1. Erroneous Version

We refer to

[B] **Théorie des Ensembles**, Bourbaki, Springer 2006,

for the basic definitions, but we allow ourselves to use sometimes a more flexible notation. In particular, the following convention will be in force:

Notational Convention: If R is a relation [resp. a term], if X is a letter, and if we are planning to substitute in R a term T for the letter X, then we write $R(X)$ instead of R, and denote by $R(T)$ the relation [resp. the term] resulting from the indicated substitution.

The main idea can be summarized as follows. We mimic definitions such as that of an equivalence relation given in [B]. More precisely, Bourbaki defines in Section II.6.1 an equivalence relation as a relation satisfying certain properties. For instance, $X = Y$ is an equivalence relation [with respect to the letters X and Y]. In particular, an equivalence relation is **not** a mathematical object. The mathematical objects [or, equivalently, the sets] are the *terms* of the theory, whereas an equivalence relation is a *relation*. Bourbaki introduces later the notion of a set equipped with an equivalence relation, but this is a different concept. One might say that an equivalence relation is a "metamathematical object", or a "typographical object" [typographical because, in [B], a relation is a particular type of "assemblage"].

Let X, Y, Z, U, f, g, h be distinct letters [in the sense of [B]].

Definition. A **category** $\mathcal{C}$ is given by the following data:

(a) A relation $\Omega(X)$.

It will be more suggestive to denote $\Omega(X)$ by $X \in \mathrm{Ob}(\mathcal{C})$. Note that the chain of symbols $X \in \mathrm{Ob}(\mathcal{C})$ is just a suggestive alternative for $\Omega(X)$ [$\Omega(X)$ being itself a convenient way of denoting Ω], but the symbols $\in$, $\mathcal{C}$, and $\mathrm{Ob}(\mathcal{C})$ have no meaning of their own in this situation. We sometimes even write $X \in \mathcal{C}$ for $X \in \mathrm{Ob}(\mathcal{C})$. [We insist: in general there is no set S such that $X \in S$ if $\Omega(X)$.]

(b) A relation $H(f, X, Y)$, which we denote also by $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, or even by $f \in \mathcal{C}(X, Y)$. [Again, $\mathcal{C}(X, Y)$ is **not** a set in general.]

(c) A term $C(g, f)$, which we denote also by $g \circ_{\mathcal{C}} f$, or even by $g \circ f$.

(d) A term $I(X)$, which we denote also by id_X.

The above items are subject to the following requirements:

(e) $f \in \mathcal{C}(X, Y)$ implies $X \in \mathcal{C}$ and $Y \in \mathcal{C}$,

(f) $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ imply $g \circ f \in \mathcal{C}(X, Z)$,

(g) $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ and $h \in \mathcal{C}(Z, U)$ imply

$$(h \circ g) \circ f = h \circ (g \circ f),$$

(h) $\mathrm{id}_X \in \mathcal{C}(X, X)$,

(i) $f \in \mathcal{C}(X, Y)$ implies $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories.

Definition. A **functor** F from $\mathcal{A}$ to $\mathcal{B}$ is given by two terms $F_0(X)$ and $F_1(f)$ satisfying the following conditions:

(a) $X \in \mathcal{A}$ implies $F_0(X) \in \mathcal{B}$,

(b) $f \in \mathcal{A}(X, Y)$ implies $F_1(f) \in \mathcal{B}(F_0(X), F_0(Y))$,

(c) $X \in \mathcal{C}$ implies $F_1(\mathrm{id}_X) = \mathrm{id}_{F_0(X)}$,

(d) $f \in \mathcal{A}(X, Y)$ and $g \in \mathcal{A}(Y, Z)$ imply $F_1(g \circ f) = F_1(g) \circ F_1(f)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and let F and G be functors from $\mathcal{A}$ to $\mathcal{B}$.

Definition. A **morphism of functors** θ from F to G is given by a term $\theta(X)$ such that:

(a) $X \in \mathcal{A}$ implies $\theta(X) \in \mathcal{B}(F(X), G(X))$,

(b) $f \in \mathcal{A}(X, Y)$ implies $G(f) \circ \theta(X) = \theta(Y) \circ F(f)$.

Part 2. Explanation of the mistake

Any set $\mathcal{C}$ can be viewed as a discrete category. More precisely, the objects of the category $\mathcal{C}$ are the elements of the set $\mathcal{C}$, the morphisms of $\mathcal{C}$ are the id_X with X in $\mathcal{C}$, and there are no other morphisms.

If $\mathcal{A}$ and $\mathcal{B}$ are two sets viewed as discrete categories, then the functors from $\mathcal{A}$ to $\mathcal{B}$ should be the set theoretical maps from $\mathcal{A}$ to $\mathcal{B}$.

Say that a "functor" (with quotations marks) from $\mathcal{A}$ to $\mathcal{B}$ is a functor from $\mathcal{A}$ to $\mathcal{B}$ in the sense of the definition given in Part 1.

So, a "functor" from $\mathcal{A}$ to $\mathcal{B}$ is a term $F(X)$, where X is a letter (occurring neither in $\mathcal{A}$ nor in $\mathcal{B}$) such that $X \in \mathcal{A}$ implies $F(X) \in \mathcal{B}$.

Assume for instance that $\mathcal{A}, \mathcal{B}$ and X are distinct letters. Then there is no term $F(X)$ as above. Indeed, if $F(X)$ was such a term, then the relation

$$X \in \mathcal{A} \implies F(X) \in \mathcal{B}$$

would hold if we substitute any terms for the letters $\mathcal{A}$ and $\mathcal{B}$. In particular, we could take for $\mathcal{A}$ a nonempty set and for $\mathcal{B}$ the empty set, which would yield a contradiction.

Assume now that Bourbaki's theory is consistent. Then it is impossible to prove that there are no maps from $\mathcal{A}$ to $\mathcal{B}$, because otherwise the previous argument would imply that this is so for *any* sets $\mathcal{A}$ and $\mathcal{B}$, and in particular if $\mathcal{A}$ is empty.

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