# A Question About the Book
# **Categories and Sheaves**

Pierre-Yves Gaillard

**Abstract.** This is the beginning of an attempt at rewriting the book **Categories and Sheaves** by Kashiwara & Schapira without using Grothendieck's universes axiom.

The purpose of this short text is to ask a question about the book

[KS] **Categories and Sheaves** by Kashiwara & Schapira (Springer, 2006).

The question is: what's wrong with the way suggested below of rewriting [KS] without using Grothendieck's universes axiom?

Of course, I have rewritten only a tiny part of [KS], but, as I suspect there is a simple answer to the above question, I hope the present text is long enough for the reader to see what I have in mind.

[I don't claim I understand all the contents of [KS]. I'm planning to read the whole book, but I'm progressing very slowly, and, at the time of writing, I've read only the first two chapters. The present text only refers to the first chapter.]

## Preliminaries

We refer to [KS] for any unexplained notation or definition. In particular, we use the definition of a category given in [KS], the main point being that the objects of a category form a set.

Let us introduce the following additional notation:

$$\mathcal{C}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y),$$

$$\mathcal{B}^{\mathcal{A}} := \mathrm{Fct}(\mathcal{A},\mathcal{B})$$

whenever $\mathcal{A},\mathcal{B},\mathcal{C}$ are categories and $X,Y$ objects of $\mathcal{C}$.

Let $F$ be a functor from $\mathcal{A}$ to $\mathcal{B}$, and let $\mathcal{B}'$ be the full subcategory of $\mathcal{B}$ whose set of object is

$$\{F(X) \mid X \in \mathcal{A}\}.$$

Say that the functor from $\mathcal{A}$ to $\mathcal{B}'$ induced by $F$ is the **graph** of $F$.

The following trivial observations will be tacitly used in the sequel. Let $F$ and $G$ be functors from $\mathcal{A}$ to $\mathcal{B}$, and let $\varphi$ be a functorial morphism from $F$ to $G$. Then $\varphi$ is invertible if and only if

$$\varphi(X) : F(X) \to G(X)$$

is invertible for all $X$ in $\mathcal{A}$. Moreover, if $H$ is a full embedding of $\mathcal{B}$ in $\mathcal{C}$ , then so is the induced functor from $\mathcal{B}^{\mathcal{A}}$ to $\mathcal{C}^{\mathcal{A}}$. In particular, the set $\mathcal{B}^{\mathcal{A}}(F,G)$ depends only on the graphs of $F$ and $G$, and so does the condition that $\varphi$ in $\mathcal{B}^{\mathcal{A}}(F,G)$ is an isomorphism.

If $X$ and $Y$ are sets, we denote by $\mathrm{Set}(X,Y)$ the set of all maps from $X$ to $Y$; for any set $S$, we write

$$\mathrm{Set}\,|S$$

for the category defined by

$$\mathrm{Ob}(\mathrm{Set}\,|S) := S, \quad (\mathrm{Set}\,|S)(X,Y) := \mathrm{Set}(X,Y);$$

and we call a category of this form a **category of sets**. Any inclusion $S \subset T$ of sets induces a full embedding of $\mathrm{Set}\,|S$ in $\mathrm{Set}\,|T$.

For any category $\mathcal{C}$, write $\mathcal{C}^o$ for the category opposite to $\mathcal{C}$.

Let $\mathcal{C}$ be a category.

Let

$$F \in \mathcal{C}_\wedge$$

be a symbolic abbreviation for the condition that $F$ is the graph of a functor from $\mathcal{C}^o$ to a category of sets. We'll sometimes say, in a symbolic manner, that such an $F$ is an **"object"** of $\mathcal{C}_\wedge$ (and we'll try to never forget the quotation marks). For $F$ and $G$ "objects" of $\mathcal{C}_\wedge$, we define

$$\mathcal{C}_\wedge(F,G)$$

as the set of morphisms from $F$ to $G$ viewed as functors from $\mathcal{C}^o$ into a large enough category of sets. By the above observations, this set is well defined.

Let

$$F \in \mathcal{C}_\vee$$

be a symbolic abbreviation for the condition that $F$ is the graph of a functor from $\mathcal{C}$ to a category of sets. We'll sometimes say that such an $F$ is an "object" of $\mathcal{C}_\vee$. For $F$ and $G$ "objects" of $\mathcal{C}_\vee$ we define

$$\mathcal{C}_\vee(F,G)$$

as the set of morphisms from $G$ to $F$ viewed as functors from $\mathcal{C}$ into a large enough category of sets.

If $S$ is a set whose elements are "objects" of $\mathcal{C}_\wedge$ [resp. of $\mathcal{C}_\vee$], we define the category

$$\mathcal{C}_\wedge|S \quad \text{[resp. } \mathcal{C}_\vee|S]$$

by the conditions

$$\mathrm{Ob}(\mathcal{C}_\wedge|S) := S, \quad (\mathcal{C}_\wedge|S)(F,G) := \mathcal{C}_\wedge(X,Y)$$

$$\text{[resp. } \mathrm{Ob}(\mathcal{C}_\vee|S) := S, \quad (\mathcal{C}_\vee|S)(F,G) := \mathcal{C}_\vee(X,Y)].$$

It is easily checked that this is indeed a category.

## Yoneda's Lemma

For $X$ in $\mathcal{C}$, the formulas

$$h(X)(Y) := \mathcal{C}(Y,X), \quad k(X)(Y) := \mathcal{C}(X,Y)$$

define "objects" $h(X)$ and $k(X)$ of $\mathcal{C}_\wedge$ and $\mathcal{C}_\vee$. Moreover, $h$ [resp. $k$] induces a functor from $\mathcal{C}$ to $\mathcal{C}_\wedge|S$ [resp. $\mathcal{C}_\vee|S$], where $S$ is the set

$$\{h(X) \mid X \in \mathcal{C}\} \quad \text{[resp. } \{k(X) \mid X \in \mathcal{C}\}].$$

For any $X$ in $\mathcal{C}$, any $A$ in $\mathcal{C}_\wedge$, and any $\varphi$ in $\mathcal{C}_\wedge(h(X),A)$, put

$$F(X,A) := \mathcal{C}_\wedge(h(X),A),$$

$$G(X,A) := A(X),$$

$$\Phi(X,A)(\varphi) := \varphi(X)(\mathrm{id}_X).$$

Let $S$ be a set of "objects" of $\mathcal{C}_\wedge$, and put

$$T := \{F(Y,B), G(Y,B) \mid Y \in \mathcal{C},\ B \in S\}.$$

**Yoneda's Lemma.** *The above formulas define two functors*

$$F|S,\ G|S \ : \ \mathcal{C}^o \times \ \mathcal{C}_\wedge|S \ \to \ \mathrm{Set}\,|T$$

*and a functorial isomorphism $\Phi|S$ from $F|S$ to $G|S$.*

In particular, the element $\varphi$ of $\mathcal{C}_\wedge(h(X),A)$ is an isomorphism if and only if the corresponding element $u$ of $A(X)$ is such that, for all $Y$ in $\mathcal{C}$, the map

$$f \mapsto A(f)(u)$$

is a bijection of $\mathcal{C}(Y, X)$ onto $A(X)$.

If such a $u$ exists, we say that $X$ is a **representative** of $A$, and that the pair $(X, u)$ is a **representation** of $A$.

Of course, there are dual statements for $\mathcal{C}_\vee$ and $k$.

### Lemma 1.4.12 of [KS]

It seems to me (but I may be wrong) that Lemma 1.4.12 on page 26 of [KS] is a good test case for the question discussed here.

Let $\mathcal{C}$ be a category and $A$ an "object" of $\mathcal{C}_\wedge$.

We'll be guided by the diagram

$$\begin{tikzcd} & \mathcal{C}_A \arrow[dl, "\omega"'] \arrow[dr, "H"] & \\ \mathcal{C}_{\wedge A} \arrow[rr, shift left, "\lambda"] & & \mathcal{C}_{A\wedge}, \arrow[ll, shift left, "\mu"] \end{tikzcd} \tag{1}$$

which makes non sense for the time being.

Define an object of $\mathcal{C}_A$ as being a element of the set $\mathcal{C}_\wedge(h(X), A)$, where $X$ is an object of $\mathcal{C}$. Let

$$(x : h(X) \to A), \quad (y : h(Y) \to A)$$

be in $\mathcal{C}_A$. By Yoneda's Lemma $x$ and $y$ can be viewed as elements of $A(X)$ and $A(Y)$, respectively. Define

$$\mathcal{C}_A(x, y)$$

as being the set of those $f$ in $\mathcal{C}(X, Y)$ such that

$$x = y \circ h(f), \text{ or } x = A(f)(y),$$

according one's favorite side of Yoneda's isomorphism. Then $\mathcal{C}_A$ is a category.

Define an "object" of $\mathcal{C}_{\wedge A}$ as being a element of $\mathcal{C}_\wedge(B, A)$ for some "object" $B$ of $\mathcal{C}_\wedge$, and define the set

$$\mathcal{C}_{\wedge A}(b, c)$$

of morphisms between the "objects"

$$b : B \to A, \quad c : C \to A$$

of $\mathcal{C}_{\wedge A}$ as being the set of those $f$ in $\mathcal{C}_\wedge(B, C)$ such that $b = c \circ f$. If $S$ is a set of "objects" of $\mathcal{C}_{\wedge A}$, define the category $\mathcal{C}_{\wedge A}|S$ by

$$\mathrm{Ob}(\mathcal{C}_{\wedge A}|S) := S, \quad (\mathcal{C}_{\wedge A}|S)(F, G) := \mathcal{C}_{\wedge A}(X, Y).$$

Each object $x$ of $\mathcal{C}_A$ can be viewed as an "object" $\omega(x)$ of $\mathcal{C}_{\wedge A}$, and each morphism $f$ in $\mathcal{C}_A(x,y)$ can be viewed as an element

$$\omega(f) \in \mathcal{C}_{\wedge A}(\omega(x), \omega(y)).$$

Thus, if $S$ is a set such that $\omega(x) \in S$ for all $x$ in $\mathcal{C}_A$, there is a natural functor

$$\omega|S \; : \; \mathcal{C}_A \; \to \; \mathcal{C}_{\wedge A}|S.$$

Similarly, to each object $x$ of $\mathcal{C}_A$ we can attach the "object" $H(x)$ of $\mathcal{C}_{A\wedge}$ defined by

$$H(x)(y) := \mathcal{C}_A(y,x),$$

and if $S$ is a set such that $H(x) \in S$ for all $x$ in $\mathcal{C}_A$, then we can form the Yoneda embedding

$$H|S : \mathcal{C}_A \to \mathcal{C}_{A\wedge}|S.$$

To each "object" $b : B \to A$ of $\mathcal{C}_{\wedge A}$ we attach the "object" $\lambda(b)$ of $\mathcal{C}_{A\wedge}$ defined by

$$\lambda(b)(x) := \mathcal{C}_{\wedge A}(\omega(x), b),$$

and to each $f$ in $\mathcal{C}_{\wedge A}(b,c)$ we attach the obvious element

$$\lambda(f) \in \mathcal{C}_{A\wedge}(\lambda(b), \lambda(c)).$$

If $S$ is a set of "objects" of $\mathcal{C}_{\wedge A}$, and $T$ a set of "objects" of $\mathcal{C}_{A\wedge}$ such that $\lambda(b)$ is in $T$ for all $b$ in $S$, then there is an obvious functor

$$\lambda|TS : \mathcal{C}_{\wedge A}|S \to \mathcal{C}_{A\wedge}|T.$$

Similarly, to each "object" $F$ of $\mathcal{C}_{A\wedge}$ we attach the "object"

$$\mu(F) = (f : \nu(F) \to A) \in \mathcal{C}_{A\wedge}$$

defined by

$$\nu(F)(X) := \{(s,x) \mid x \in A(X),\ s \in F(x)\}, \quad f(X)(s,x) := x,$$

and to each $u$ in $\mathcal{C}_{A\wedge}(F,G)$ we attach the element

$$\mu(u) \in \mathcal{C}_{\wedge A}(\mu(F), \mu(G))$$

defined by

$$\mu(u)(X)(s,x) := u(x)(s).$$

If $S$ is a set of "objects" of $\mathcal{C}_{A\wedge}$, and $T$ a set of "objects" of $\mathcal{C}_{\wedge A}$ such that $\mu(F)$ is in $T$ for all $F$ in $S$, then there is an obvious functor

$$\mu|TS : \mathcal{C}_{A\wedge}|S \to \mathcal{C}_{\wedge A}|T.$$

We claim that the diagram (1) commutes in the following sense:

We have

$$(\lambda|TS) \circ (\omega|S) \simeq H|T, \quad (\mu|TS) \circ (H|S) \simeq \omega|T$$

whenever these functors are defined. Moreover, if $S \subset U$, and if the functor

$$(\mu|UT) \circ (\lambda|TS)$$

is defined, then it is isomorphic to the inclusion of $C_{\wedge A}|S$ in $C_{\wedge A}|U$. Finally, if $S \subset U$, and if the functor

$$(\lambda|UT) \circ (\mu|TS)$$

is defined, then it is isomorphic to the inclusion of $C_{A\wedge}|S$ in $C_{A\wedge}|U$.

The proof is obvious.