On Clifford Space and Higher Curvature Gravity

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Abstract

Clifford-space Gravity is revisited and new results are found. A derivation of the proper expressions for the connections (with *torsion*) in Clifford spaces (C-spaces) is presented. The introduction of hyper-determinants of hyper-matrices are instrumental in the derivation of the C-space generalized gravitational field equations from a variational principle and based on the extension of the Einstein-Hilbert-Cartan action. We conclude by pointing out the relations of Clifford space gravity to Lanczos-Lovelock-Cartan higher curvature gravity with torsion and extended gravitational theories based on $f(R), f(R_{\mu\nu}), ...$ actions, for polynomial-valued functions. Introducing nonmetricity furnishes higher curvature extensions of metric affine theories of gravity.

1 Introduction

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. This extended relativity in Clifford spaces theory should *not* be confused with the extended relativity theory (ER) proposed by Erasmo Recami and collaborators [3] many years ago which was based on the Special Relativity theory extended to Antimatter and Superluminal motions. Since the beginning of the seventies, an "Extended special Relativity" (ER) exists, which on the basis of the ordinary postulates of Special Relativity (chosen "com grano salis") describes also superluminal motions in a rather simple way, and without any severe causality violations. Reviews of that theory of ER can be found in [3].

The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are

Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set to be equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1].

The poly-vector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate). These poly-vector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background.

The C-space poly-vector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions

$$\mathbf{X} \;=\; \sigma\, \mathbf{1} + x^\mu\, \gamma_\mu + x^{\mu\nu}\, \gamma_\mu\wedge\gamma_\nu + x^{\mu\nu\rho}\, \gamma_\mu\wedge\gamma_\nu\wedge\gamma_\rho + x^{\mu\nu\rho\tau}\, \gamma_\mu\wedge\gamma_\nu\wedge\gamma_\rho\wedge\gamma_\tau \quad (1.1)$$

σ is the Clifford scalar component of the poly-vector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in C-space

$$(d\Sigma)^2 \;=\; (d\sigma)^2 \;+\; dx_\mu\, dx^\mu \;+\; dx_{\mu\nu}\, dx^{\mu\nu} \;+\; \ldots \quad (1.2)$$

is *not* equal to the proper time Lorentz-invariant interval ds in ordinary spacetime $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$.

The main purpose of this work is to construct a generalized gravitational theory in Clifford spaces based on the proper expression for the connections involving the generalized spin connection terms. Clifford space gravity has a relation to Lanczos-Lovelock-Cartan higher curvature gravity with torsion, and extended gravitational theories based on $f(R), f(R_{\mu\nu}), ...$ actions, for polynomial-valued functions. In essence, the Lanczcos-Lovelock-Cartan curvature tensors appear as Ricci-like traces of certain components of the C-space curvatures. Introducing nonmetricity furnishes higher curvature extensions of metric affine theories of gravity [9].

2 Clifford-space Gravity

At the beginning of this section we follow closely the work in [1] and then we depart from it by constructing Clifford space (C-space) gravity without making any a priori assumptions on the C-space connections. Let the vector fields γ_μ, $\mu = 1, 2, ..., n$ be a coordinate basis in V_n satisfying the Clifford algebra relation

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu} \tag{2.1}$$

where $g_{\mu\nu}$ is the metric of V_n. In curved space γ_μ and $g_{\mu\nu}$ cannot be constant but necessarily depend on position x^μ. An arbitrary vector is a linear superposition [4] $a = a^\mu \gamma_\mu$ where the components a^μ are *scalars* from the geometric point of view, whilst γ_μ are *vectors*.

Besides the basis $\{\gamma_\mu\}$ we can introduce the reciprocal basis[1] $\{\gamma^\mu\}$ satisfying

$$\gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu} \tag{2.2}$$

where $g^{\mu\nu}$ is the covariant metric tensor such that

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu, \;\; \gamma^\mu \gamma_\nu + \gamma_\nu \gamma^\mu = 2\delta^\mu{}_\nu \text{ and } \gamma^\mu = g^{\mu\nu}\gamma_\nu$$

Let us now consider C-space and very briefly recur to the procedure of [1], [5]. A basis in C-space is given by

$$E_A = \{\gamma, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, ...\} \tag{2.3}$$

where γ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In an r-vector $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ we take the indices so that $\mu_1 < \mu_2 < ... < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X = X^A E_A = s\,\mathbf{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... \tag{2.4}$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes. By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter L_p in the expansion of the poly-vector X we can set set to unity to simplify matters.

In a flat C-space the basis vectors E^A are constants. In a curved C-space this is no longer true. Each E_A is a function of the C-space coordinates

$$X^A = \{s, x^\mu, x^{\mu\nu}, ...\} \tag{2.5}$$

[1] In Appendix A of the Hesteness book [4] the frame $\{\gamma^\mu\}$ is called *dual* frame because the duality operation is used in constructing it.

which include scalar, vector, bivector,..., r-vector,..., coordinates.

We now *depart* from our previous work [1] and construct the connection in curved C-space based on the derivatives acting on the C-space beins E_M^A according to $\partial_K E_M^A = \hat{\Gamma}^L_{KM} E_L^A$ where $\partial_A \equiv \partial/\partial X^A$ is the ordinary derivative in C-space.

We begin by writing the covariant derivative of the $E_M^A(\mathbf{X})$ beins as

$$\nabla_K E_M^A \ = \ \partial_K (E_M^A) \ - \ \Gamma^L_{KM} \ E_L^A \ - \ \omega^A_{KC} \ E_M^C \tag{2.6}$$

after using the standard connection Γ^L_{KM} to contract curved space indices and the spin connection ω^A_{KC} to contract tangent space indices. One must assume also that the inverse bein E_A^M is well defined as well so that $E_M^A E_B^M = \delta_B^A$. The covariant derivative is zero $\nabla_K E_M^A = 0$ if the nonmetricity is *zero*. In this latter case

$$\nabla_K E_M^A \ = \ 0 \ \Rightarrow \ \partial_K (E_M^A) \ = \ \Gamma^L_{KM} \ E_L^A \ + \ \omega^A_{KC} \ E_M^C \tag{2.7a}$$

and one can then reabsorb the contribution of the spin connection $\omega_K^{AC} = -\omega_K^{CA}$ into the ordinary connection as follows

$$\hat{\Gamma}^L_{KM} \ = \ \Gamma^L_{KM} \ + \ \omega^A_{KC} \ E_M^C \ E_A^L, \quad \hat{\Gamma}_{KML} \ = \ \Gamma_{KML} \ + \ \omega_{KCA} \ E_M^C \ E_L^A \tag{2.7b}$$

and rewrite the ordinary derivatives of γ_M using now the *hatted* connections as

$$\partial_K \ \gamma_M \ = \ \partial_K (E_M^A \gamma_A) \ = \ \hat{\Gamma}^L_{KM} \ E_L^A \ \gamma_A \ = \ \hat{\Gamma}^L_{KM} \gamma_L \tag{2.8}$$

where as usual $A, B, C, D...$ are tangent space indices and J, K, M, N, are curved space ones. The presence of the hatted connection in the definition of the derivatives of the curved space basis generators in eq-(2.8) should be contrasted with our previous results [1] and with a spin gauge theory of gravity in Clifford Spaces [6]. The zero covariant derivative of the E_M^A is also compatible with the zero covariant derivative of the C-space metric. The latter curved C-space metric is defined[2] as the scalar part of the Clifford geometric product

$$g_{MN} \ = \ < \gamma_M \gamma_N > \ = \ E_M^A \ E_N^B \ < \gamma_A \gamma_B > \ = \ E_M^A \ E_N^B \ \eta_{AB} \tag{2.9}$$

The scalar part of the curved C-space generator basis γ_M is given by

$$< \gamma_M > = < E_M^A \gamma_A > = E_M^0 \gamma_0$$

where γ_A is the tangent space basis generator and γ_0 corresponds to the tangent space unit generator $\mathbf{1}$. The Clifford geometric product $\gamma_M \gamma_N$ can be written in terms of the Clifford algebra structure constants C_{AB}^C associated with the tangent space generators γ_A, and given in the Appendix, as

[2] We thank Matej Pavsic for insisting on this point.

$$\gamma_M\gamma_N \ = \ E^A_M \ E^B_N \ \gamma_A\gamma_B \ = \ E^A_M \ E^B_N \ C^C_{AB} \ \gamma_C \ =$$

$$(E^A_M \ E^B_N \ C^C_{AB} \ E^J_C) \ \gamma_J \ = \ C^J_{MN} \ \gamma_J, \quad E^A_M \ E^B_N \ C^C_{AB} \ E^J_C \ \equiv \ C^J_{MN} \qquad (2.10)$$

Taking derivatives of (2.10) and evaluating the scalar parts gives then

$$\partial_K \ g_{MN} \ = \ \partial_K \ < \gamma_M\gamma_N > \ = \ \partial_K \ < C^J_{MN}\gamma_J > \ =$$

$$\partial_K \ (C^J_{MN} \ E^0_J) \ = \ (\partial_K C^J_{MN}) \ E^0_J \ + \ C^J_{MN} \ \hat\Gamma^L_{KJ} \ E^0_L \ = \ < \partial_K \ (C^J_{MN} \ \gamma_J) > \ =$$

$$< \partial_K \ (\gamma_M\gamma_N) > \ = \ \hat\Gamma^L_{KM} \ g_{LN} \ + \ g_{ML} \ \hat\Gamma^L_{KN} \ = \ \hat\Gamma_{KMN} \ + \ \hat\Gamma_{KNM} \ =$$

$$\Gamma_{KMN} \ + \ \Gamma_{KNM} \ = \ \partial_K \ g_{MN} \qquad (2.11)$$

due to an explicit *cancellation* of the spin connection terms in the sum of the $\hat\Gamma$ terms

$$\omega_{KCA} \ E^C_M \ E^A_N \ + \ \omega_{KCA} \ E^C_N \ E^A_M \ = \ 0 \qquad (2.12)$$

in eq-(2.11) as a result of the anti-symmetry property $\omega_{KAC} = -\omega_{KCA}$.

Hence, from (2.11) one can infer that the nonmetricity is zero (as expected)

$$\nabla_K \ g_{MN} \ = \ \partial_K \ g_{MN} \ - \ \Gamma_{KMN} \ - \ \Gamma_{KNM} \ = \ 0 \qquad (2.13)$$

and that the derivative operation *commutes* with taking the scalar part of the Cliford geometric product

$$\partial_K < \gamma_M\gamma_N > \ = \ < \partial_K(\gamma_M\gamma_N) > \qquad (2.14)$$

One may notice that it is *not* necessary to focus on the scalar parts of the Clifford geometric product to arrive at (2.11) but one can start directly from the full Clifford geometric product written in terms of the Clifford algebra structure constants as

$$\gamma_M \ \gamma_N \ = \ E^A_M \ E^B_N \ \gamma_A\gamma_B \ = \ E^A_M \ E^B_N \ C^C_{AB}\gamma_C \ = \ C^C_{MN} \ \gamma_C \qquad (2.15)$$

C^C_{AB} are constants but $C^C_{MN} = E^A_M E^B_N C^C_{AB}$ are not constant due to the $\mathbf{X}$ dependence of the beins. Taking derivatives on both sides of (2.15), , gives

$$\partial_K \ (\gamma_M \ \gamma_N) \ = \ \partial_K \ (C^C_{MN}\gamma_C) \ \Rightarrow \ \hat\Gamma^L_{KM} \ C^C_{LN} \ \gamma_C + \hat\Gamma^L_{KN} \ C^C_{ML} \ \gamma_C \ = \ (\partial_K C^C_{MN}) \ \gamma_C \qquad (2.16a)$$

which allows the cancellation of the γ_C factors on both sides. Upon writing

$$C^C_{LN} \ = \ E^A_L \ E^B_N \ C^C_{AB}, \ C^C_{ML} \ = \ E^A_M \ E^B_L \ C^C_{AB}, \ C^C_{MN} \ = \ E^A_M \ E^B_N \ C^C_{AB} \quad (2.16b)$$

allows to cancel also the structure constants C^C_{AB} on both sides of the equation (2.16b), when $C^C_{AB} \neq 0$, leading to a rigorous expression involving the derivatives of the C-space beins and the *hatted* connections of the form

$$\hat\Gamma^L_{KM} \ E^A_L \ E^B_N \ + \ \hat\Gamma^L_{KN} \ E^B_L \ E^A_M \ = \ \partial_K(E^A_M \ E^B_N), \qquad (2.17)$$

Eq-(2.17) is another way to $define$ the hatted connections in terms of derivatives of the C-space beins (rather than the metric) and which *encodes* the full Clifford algebraic structure. We may notice that if one multiplies both sides of (2.17) by η_{AB} one arrives at the same relationship as in eq-(2.11) after defining $E_M^A E_N^B \eta_{AB} = g_{MN}$, and the remaining terms as well, due to the cancellation of the spin connection terms as displayed by eq-(2.12).

If the ordinary connection is *symmetric* $\Gamma_{KMN} = \Gamma_{MKN}$ (*not* to be confused with the $\hat{\Gamma}_{KMN}$) in the first two indices, from eq-(2.11) one can arrive at the Levi-Civita-like expression. This can be attained by writing

$$\partial_K \, g_{MN} \; - \; \Gamma_{KMN} \; - \; \Gamma_{KNM} \; = \; 0 \qquad (2.18a)$$

$$\partial_M \, g_{KN} \; - \; \Gamma_{MKN} \; - \; \Gamma_{MNK} \; = \; 0 \qquad (2.18b)$$

$$\partial_N \, g_{MK} \; - \; \Gamma_{NMK} \; - \; \Gamma_{NKM} \; = \; 0 \qquad (2.18c)$$

and after subtracting eqs-(2.18b, 2.18c) from eq-(2.18a) one arrives at the Levi-Civita-like expression for the connection

$$\Gamma_{MNK} \; = \; \frac{1}{2} \left(\partial_M g_{KN} \; + \; \partial_N g_{MK} \; - \; \partial_K g_{MN} \right) \qquad (2.19)$$

due to the symmetry property $\Gamma_{KMN} = \Gamma_{MKN}$ in the first two indices. In general, C-space admits torsion [1] and the connection $\Gamma^N_{KM} \neq \Gamma^N_{MK}$ is not symmetric. For example, if $\Gamma_{KMN} = \Gamma_{KNM}$ is symmetric in the *last* two indices from (2.11) one has $\Gamma_{KMN} = \frac{1}{2} \partial_K g_{MN} \neq \Gamma_{MKN} = \frac{1}{2} \partial_M g_{KN}$ and one has torsion.

The $torsion$ is defined as $T^N_{KM} = \Gamma^N_{KM} - \Gamma^N_{MK}$ in C-space, assuming the anholonomy coefficients f^N_{KM} are zero, $[\partial_K, \partial_M] = f^N_{KM} \partial_N$. If the latter coefficients are not zero one must include f^N_{KM} into the definition of Torsion as follows

$$T^N_{KM} \; = \; \Gamma^N_{KM} \; - \; \Gamma^N_{MK} \; - \; f^N_{KM} \qquad (2..21)$$

Also one can also introduce nonmetricity

$$\nabla_K E^A_M \; \neq \; 0 \; \Rightarrow \; \partial_K E^A_M \; \neq \; \hat{\Gamma}^L_{KM} \; E^A_L \; \Rightarrow \; \nabla_K g_{MN} \; = \; Q_{KMN} \; \neq \; 0 \quad (2.20)$$

However for the time being we will set the nonmetricity to zero.

In the case of nonsymmetric connections with torsion, the curvature obeys the relations under the exchange of indices

$$\mathbf{R}_{MNJK} \; = \; - \; \mathbf{R}_{NMJK}, \; \; \mathbf{R}_{MNKJ} \; = \; - \; \mathbf{R}_{MNJK}, \; \; but \; \mathbf{R}_{MNJK} \; \neq \; \mathbf{R}_{JKMN} \qquad (2.22)$$

and is defined, when $f^J_{MN} = 0$, in terms of the *hatted* connection components $\hat{\Gamma}^L_{KM} = \Gamma^L_{KM} + \omega^A_{KC} E^C_M E^L_A$ as follows

$$\mathbf{R}_{MNJ}{}^K \; = \; \partial_M \; \hat{\Gamma}_{MJ}{}^K \; - \; \partial_N \; \hat{\Gamma}_{MJ}{}^K \; + \; \hat{\Gamma}^K_{ML} \; \hat{\Gamma}^L_{NJ} \; - \; \hat{\Gamma}^K_{NL} \; \hat{\Gamma}^L_{MJ} \qquad (2.23)$$

The spin connection terms do not decouple from the expression in (2.23); this is one of the differences among the present construction of C-space gravity and those in [1], [6]. There are other differences, like the use of hyper-determinants, as we shall elaborate below. If $f^K_{MN} \neq 0$ one must also include these anholonomy coefficients into the definition of curvature (2.23) by adding terms of the form $-f^L_{MN}\hat{\Gamma}^K_{LJ}$.

The standard Riemann-Cartan curvature tensor in ordinary spacetime is *contained* in C-space as follows

$$\mathcal{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} = \partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} - \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} + \Gamma^{\rho_2}_{\mu_1\sigma}\,\Gamma^{\sigma}_{\mu_2\rho_1} - \Gamma^{\rho_2}_{\mu_2\sigma}\,\Gamma^{\sigma}_{\mu_1\rho_1} \subset$$

$$\mathbf{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} = \partial_{\mu_1}\hat{\Gamma}^{\rho_2}_{\mu_2\rho_1} - \partial_{\mu_2}\hat{\Gamma}^{\rho_2}_{\mu_1\rho_1} + \hat{\Gamma}^{\rho_2}_{\mu_1\ \mathbf{M}}\,\hat{\Gamma}^{\mathbf{M}}_{\mu_2\rho_1} - \hat{\Gamma}^{\rho_2}_{\mu_2\ \mathbf{M}}\,\hat{\Gamma}^{\mathbf{M}}_{\mu_1\rho_1} \tag{2.24}$$

due to the contractions involving the poly-vector valued indices $\mathbf{M}$ in eq-(2.24) and the presence of the ordinary connection $\Gamma^{\rho}_{\mu\nu}$ within the $\hat{\Gamma}^{\rho}_{\mu\nu}$. There is also the crucial difference that $\mathbf{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2}(s, x^\nu, x^{\nu_1\nu_2}, ...)$ has now an *additional* dependence on all the C-space poly-vector valued coordinates $s, x^{\nu_1\nu_2}, x^{\nu_1\nu_2\nu_3}, ...$ besides the x^ν coordinates. The curvature in the presence of torsion does not satisfy the same symmetry relations when there is no torsion, therefore the Ricci-like tensor is no longer symmetric

$$\mathbf{R}_{MNJ}{}^{N} = \mathbf{R}_{MJ}, \quad \mathbf{R}_{MJ} \neq \mathbf{R}_{JM}, \quad \mathbf{R} = g^{MJ}\,\mathbf{R}_{MJ} \tag{2.25}$$

The C-space Ricci-like tensor is

$$\mathbf{R}^N_M = \sum_{j=1}^{D} \mathbf{R}_M{}^{N\,[\nu_1\nu_2...\nu_j]}{}_{[\nu_1\nu_2...\nu_j]} + \mathbf{R}_M{}^{N}{}_{\mathbf{0}}{}^{\mathbf{0}} \tag{2.26}$$

and the C-space curvature scalar is

$$\mathbf{R} = \sum_{j,k=1}^{D} \mathbf{R}_{[\mu_1\mu_2...\mu_j]\ [\nu_1\nu_2...\nu_k]}{}^{[\mu_1\mu_2...\mu_j]\ [\nu_1\nu_2...\nu_k]} + \sum_{j=1}^{D} \mathbf{R}_{[\mu_1\mu_2...\mu_j]\ \mathbf{0}}{}^{[\mu_1\mu_2...\mu_j]\ \mathbf{0}} \tag{2.27}$$

One may construct an Einstein-Hilbert-Cartan like action based on the C-space curvature scalar. This requires the use of hyper-determinants. The hyper-determinant of a hyper-matrix [15] can be recast in terms of discriminants [16]. In this fashion one can define the hyper-determinant of g_{MN} as products of the hyper-determinants corresponding to the hyper-matrices[3]

$$g_{[\mu_1\mu_2]\ [\nu_1\nu_2]}, \ \ldots, \ g_{[\mu_1\mu_2...\mu_k]\ [\nu_1\nu_2...\nu_k]}, \quad for \quad 1 < k < D \tag{2.28}$$

and construct a suitable measure of integration $\mu_{\mathbf{m}}(s, x^\mu, x^{\mu_1\mu_2}, \ldots, x^{\mu_1\mu_2...\mu_D})$ in C-space which, in turn, would allow us to build the C-space version of the

[3]The hyper-determinant of a product of two hyper-matrices is *not* equal to the product of their hyper-determinants. However, one is not multiplying two hyper-matrices but decomposing the hyper-matrix g_{MN} into its different blocks.

Einstein-Hilbert-Cartan action with a cosmological constant

$$\frac{1}{2\kappa^2} \int ds \prod dx^\mu \prod dx^{\mu_1\mu_2} \ \ldots \ dx^{\mu_1\mu_2\ldots\mu_D} \ \mu_{\mathbf{m}}(s, x^\mu, x^{\mu_1\mu_2}, \ldots) \ (\mathbf{R} - 2\Lambda) \tag{2.29}$$

κ^2 is the C-space gravitational coupling constant. In ordinary gravity it is set to $8\pi G_N$, with G_N being the Newtonian coupling constant.

The measure must obey the relation

$$[\mathbf{DX}] \ \mu_{\mathbf{m}}(\mathbf{X}) \ = \ [\mathbf{DX}'] \ \mu'_{\mathbf{m}}(\mathbf{X}') \tag{2.30}$$

under poly-vector valued coordinate transformations in C-space. The C-space metric transforms as

$$g'_{JK} \ = \ g_{MN} \ \frac{\partial X^M}{\partial X'^J} \ \frac{\partial X^N}{\partial X'^K} \tag{2.31}$$

but now one has that

$$\sqrt{hdet \ g'} \ \neq \ \sqrt{hdet \ g} \ hdet \left(\frac{\partial X^M}{\partial X'^N} \right) \tag{2.32}$$

due to the multiplicative "anomaly" of the product of hyper-determinants. So the measure μ_m does not coincide with the square root of the hyper-determinant. It is a more complicated function of the hyper-determinant of g_{AB} and obeying eq-(2.63).[4] One could write $hdet(X \cdot Y) = Z_A hdet(X) hdet(Y)$, where $Z_A \neq 1$ is the multiplicative anomaly and in this fashion eq-(2.30) leads to an *implicit* definition of the measure $\mu_m(hdet g_{AB})$.

The ordinary determinant $g = det(g_{\mu\nu})$ obeys

$$\delta\sqrt{-g} = - \ \frac{1}{2} \ \sqrt{-g} \ g_{\mu\nu} \ \delta g^{\mu\nu} \tag{2.33}$$

which was fundamental in the derivation of Einstein equations from a variation of the Einstein-Hilbert action. However, when hyper-determinants of the C-space metric g_{AB} are involved it is *no* longer true that the relation (2.33) holds anymore in order to obtain the C-space gravity field equations in the presence of torsion and a cosmological constant.

Using the relation $\delta\mathbf{R}_{MN} = \nabla_J \delta\hat{\Gamma}^J_{MN} - \nabla_N \delta\hat{\Gamma}^J_{JM}$, a variation of the action

$$\frac{1}{2\kappa^2} \int ds \prod dx^\mu \prod dx^{\mu_1\mu_2} \ \ldots \ dx^{\mu_1\mu_2\ldots\mu_D} \ \mu_m(|hdet \ g_{MN}|) \ (\mathbf{R} - 2\Lambda) + S_{matter} \tag{2.34}$$

[4]There is no known generalization of the Binet-Cauchy formula $\det(AB) = \det(A)\det(B)$ for 2 arbitrary hypermatrices. However, in the case of particular types of hypermatrices, some results are known. Let X, Y be two hypermatrices. Suppose that Y is a $n \times n$ matrix. Then, a well-defined hypermatrix product XY is defined in such a way that the hyperdeterminant satisfies the rule $hdet(X \cdot Y) = hdet(X) hdet(Y)^{N/n}$. There, n is the degree of the hyperdeterminant and N is a number related to the format of the hypermatrix X.

with respect to the C-space metric g_{MN} yields the correct C-space field equations

$$\mathbf{R}_{(MN)} \; + \; (\mathbf{R} - 2\Lambda) \; \frac{\delta ln(\mu_m(|hdet \; g_{MN}|))}{\delta g^{MN}} \; = \; \kappa^2 \; \mathbf{T}_{MN} \tag{2.35}$$

If, and only if,

$$\frac{\delta ln(\mu_m(|hdet \; g_{MN}|))}{\delta g^{MN}} \; = \; - \; \frac{1}{2} \; g_{MN} \tag{2.36}$$

then the field equations (2.35) would coincide with the naive C-space extension of Einstein's equations with a cosmological constant. One should note that the field equations (2.35) *contain torsion* since $\mathbf{R}_{(MN)}, \mathbf{R}$ are defined in terms of the nonsymmetric connection $\Gamma^J_{MN} \neq \Gamma^J_{NM}$ which are contained within the $\hat{\Gamma}^J_{MN}$. Eqs-(2.35) are the correct C-space field equations one should use in general. The field equations (2.35), for example, are very different from those found in [19] based on a fourth-rank symmetric metric tensor.

The hyper-determinant of the C-space metric g_{MN} (a hyper matrix) involving all the components of the same and different grade is defined as

$$\begin{aligned} \mathrm{hdet}(g_{MN}) \equiv g_{\mathbf{00}} \, \det(g_{\mu\nu}) \mathrm{hdet}(g_{\mu_1\mu_2 \; \nu_1\nu_2}) \mathrm{hdet}(g_{\mu \; \nu_1\nu_2}) \cdots \\ \cdots \mathrm{hdet}(g_{\mu_1 \ldots \mu_{D-1} \nu_1 \ldots \nu_{D-1}}) g_{\mu_1 \ldots \mu_D \nu_1 \ldots \nu_D} \end{aligned} \tag{2.37}$$

where the hyper-determinant of $g_{\mu\nu}$ coincides with the ordinary determinant of $g_{\mu\nu}$. Notice once more that the hyper-determinant of a product of two hyper-matrices is *not* equal to the product of their hyper-determinants. However, in (2.37) one is not multiplying two hyper-matrices g_{AB}, g'_{AB}, but decomposing the hyper-matrix g_{AB} into different blocks.

To see how the components of g_{MN} can be realized as hyper-matrices one may choose for example the bivector-bivector metric entries $g_{12 \; 34} = g_{34 \; 12}$ such that these components are constrained to obey $g_{21 \; 34} = -g_{12 \; 34} = g_{12 \; 43}$. And $g_{11 \; 34} = g_{22 \; 34} = \ldots = g_{DD \; 34} = 0$. In this fashion one can realize $g_{\mu_1\mu_2 \; \nu_1\nu_2}$ as the entries of a hyper-matrix h_{ijkl} obeying certain symmetry (anti-symmetry) conditions. One may choose for example the vector-bivector metric entries $g_{1 \; 34} = g_{34 \; 1}$ such that $g_{1 \; 34} = -g_{1 \; 43}$. And $g_{1 \; 11} = g_{1 \; 22} = \ldots = g_{1 \; DD} = 0$. In this fashion one can realize $g_{\mu \; \nu_1\nu_2}$ as the entries of a hyper-matrix h_{ijk}, ... Hence, a variation of the action with respect to g_{MN} leads to a complicated expression (2.35) that does not necessarily coincide with the naive extension of Einstein's field equations in C-space. We are assuming also that the hyper-determinant exists and is non-vanishing. At this stage we should also remark that C-space gravity corresponding to a Clifford algebra in D-dimensions is physically very distinct from ordinary gravity in the corresponding 2^D-dimensional space.

The hyper-determinant of a third rank hyper-matrix A_{ijk} where the indices i, j, k run from $1, 2, 3, ..., D$ may be taken to be

$$\frac{1}{D!}\ \epsilon^{i_1 i_2 \ldots i_D}\ \epsilon^{j_1 j_2 \ldots j_D}\ \epsilon^{k_1 k_2 \ldots k_D}\ A_{i_1 j_1 k_1}\ A_{i_2 j_2 k_2}\ \ldots\ A_{i_D j_D k_D} \tag{2.38}$$

and similarly one can define the hyper-determinants for the other higher-rank hyper-matrices. We should emphasize that the hyper-determinant (2.38) when $D = 2$ is not the same as the Cayley hyper-determinant of a $2 \times 2 \times 2$ cubic hyper-matrix [15]. The latter Cayley determinant was instrumental in finding the relationship between the black hole entropy in extended supergravity and the entanglement of a three qubit system [17].

To sum up, the use of hyper-determinants is required to construct the analog of the Einstein-Hilbert-Cartan action in C-spaces. A variation of the action in C-space leads to the generalized field equations (2.35) (with torsion) that do *not* necessarily coincide with the naive C-space extension of Einstein's field equations. In ordinary Relativity, without torsion, one can construct the Einstein tensor by performing two successive contractions of the differential Bianchi identity. It also leads to the conservation of the stress energy tensor in the right hand side. Presumably this procedure based on the *modified* Bianchi identities could apply also to C-space leading to the field equations (2.35) which *contain* torsion since $\mathbf{R}_{(MN)}, \mathbf{R}$ are defined in terms of $\hat{\Gamma}^J_{[MN}$ which contain the non-symmetric connection $\Gamma^J_{MN} \neq \Gamma^J_{NM}$. An immediate question arises, does the Palatini formalism work also in C-spaces? Namely, does a variation with respect to the C-space connection $(\delta S / \delta \hat{\Gamma}^J_{MN}) = 0$ yield the *same* connections as those depicted above in this section? We leave this difficult question for future work.

A crucial question remains. In view of the expressions found in the above construction of C-space gravity, one can realize that one would have been able to obtain them without recurring to the Clifford algebraic structure, by simply extending the ordinary coordinate indices to poly-vector valued ones. For instance, by replacing $e^a_\mu, g_{\mu\nu}, \Gamma^\rho_{\mu\nu}, R^\sigma_{\mu\nu\rho}$... for $E^A_M, g_{MN}, \Gamma^K_{MN}, R^K_{MNJ}$, ..., respectively. So the key question is : where does the Clifford algebraic structure manifest itself in the above construction of C-space gravity? To simply answer this question we just write

$$\gamma^\sigma\ \gamma^\nu\ =\ \frac{1}{2}\ \{\gamma^\sigma, \gamma^\nu\}\ +\ \frac{1}{2}\ [\gamma^\sigma, \gamma^\nu]\ =\ g^{\sigma\nu}\ +\ \gamma^{\sigma\nu} \tag{2.39}$$

with symmetric $g^{\sigma\nu}$, antisymmetric $\gamma^{\sigma\nu} = -\gamma^{\nu\sigma}$ and after using the beins, and inverse beins, as follows

$$\gamma^\mu \gamma^\nu\ =\ e^\mu_a e^\nu_b \gamma^a \gamma^b\ =\ g^{\mu\nu}\ +\ \gamma^{\mu\nu} \tag{2.40}$$

where

$$g^{\mu\nu}\ =\ e^\mu_{(a} e^\nu_{b)}\ \eta^{ab}, \quad \gamma^{\mu\nu}\ =\ e^\mu_{[a} e^\nu_{b]}\ \gamma^{ab} \tag{2.41}$$

After taking derivatives of the anti-commutator with respect to x^ρ, using $\partial_\rho \gamma^\mu = \hat{\Gamma}^\mu_{\rho M} \gamma^M = \hat{\Gamma}^\mu_{\rho\sigma} \gamma^\sigma + ...$, one learns that

$$\partial_\rho \{\gamma^\mu,\ \gamma^\nu\}\ =\ 2\ \partial_\rho g^{\mu\nu}\ \Rightarrow$$

$$\hat{\Gamma}^{\mu}_{\rho\ \sigma}\ \gamma^{\sigma}\ \gamma^{\nu}\ +\ \gamma^{\mu}\ \hat{\Gamma}^{\nu}_{\rho\ \sigma}\ \gamma^{\sigma}\ +\ \hat{\Gamma}^{\nu}_{\rho\ \sigma}\ \gamma^{\sigma}\ \gamma^{\mu}\ +\ \gamma^{\nu}\ \hat{\Gamma}^{\mu}_{\rho\ \sigma}\ \gamma^{\sigma}\ =$$

$$2\ \hat{\Gamma}^{\mu\nu}_{\rho}\ +\ 2\ \hat{\Gamma}^{\nu\mu}_{\rho}\ =\ 2\ \partial_{\rho}g^{\mu\nu}\ \Rightarrow$$

$$\Gamma^{\mu\nu}_{\rho}\ +\ \Gamma^{\nu\mu}_{\rho}\ =\ \partial_{\rho}g^{\mu\nu} \tag{2.42}$$

and one obtains, as expected, the same relationship as that in eq-(2.11) after raising indices and due to the key cancellation of the spin connection terms in the summation of the $\hat{\Gamma}$ terms.

Taking derivatives of the commutator

$$\partial_{\rho}[\gamma^{\mu},\ \gamma^{\nu}]\ =\ 2\ \partial_{\rho}\gamma^{\mu\nu} \tag{2.43}$$

one learns that

$$\hat{\Gamma}^{\mu}_{\rho\ \sigma}\ \gamma^{\sigma\nu}\ -\ \hat{\Gamma}^{\nu}_{\rho\ \sigma}\ \gamma^{\sigma\mu}\ =\ \hat{\Gamma}^{[\mu\nu]}_{\rho\ \tau_1\tau_2}\ \gamma^{\tau_1\tau_2} \tag{2.44}$$

after using

$$\partial_{\rho}\gamma^{\mu\nu}\ =\ \hat{\Gamma}^{[\mu\nu]}_{\rho\ \tau_1\tau_2}\ \gamma^{\tau_1\tau_2}\ +\ ... \tag{2.45}$$

From eqs- (2.44) one obtains, after performing contractions of the form $<\gamma^{ab}\gamma_{cd}>=$ (constant) $\cdot\ \delta^{ab}_{cd}$, the following relationship

$$\hat{\Gamma}^{\mu}_{\alpha\ \sigma}\ \delta^{\sigma\nu}_{\rho_1\rho_2}\ -\ \hat{\Gamma}^{\nu}_{\alpha\ \sigma}\ \delta^{\sigma\mu}_{\rho_1\rho_2}\ =\ \hat{\Gamma}^{[\mu\nu]}_{\alpha\ \tau_1\tau_2}\ \delta^{\tau_1\tau_2}_{\rho_1\rho_2}\ =\ \hat{\Gamma}^{[\mu\nu]}_{\alpha\ \rho_1\rho_2} \tag{2.46a}$$

Raising and lowering indices in (2.46a) yields the algebraic relationship between the connection components of the form

$$\hat{\Gamma}^{\sigma}_{\alpha\ \mu}\ g_{[\sigma\nu]\ [\rho_1\rho_2]}\ -\ \hat{\Gamma}^{\sigma}_{\alpha\ \nu}\ g_{[\sigma\mu]\ [\rho_1\rho_2]}\ =\ \hat{\Gamma}_{\alpha\ [\mu\nu]\ [\rho_1\rho_2]} \tag{2.46b}$$

Here comes the key point. Since the expression for $\hat{\Gamma}^{\sigma}_{\alpha\ \mu}$ in the left hand side of (2.46b) involves terms with first derivatives of the ordinary rank-two metric $g_{\mu\nu}$, with vector-vector indices, and the expression for $\hat{\Gamma}_{\alpha\ [\mu\nu]\ [\rho_1\rho_2]}$ involves terms with first derivatives of a fourth-rank metric $g_{\mu_1\mu_2\ \nu_1\nu_2}$, with bivector-bivector indices, then eq-(2.46b) imposes an algebraic relationship (constraint) between the first order derivatives of the C-space metric components. Such algebraic relationship is enforced by the Clifford algebraic structure itself. This is precisely where the Clifford algebra manifests itself in the construction of C-space gravity. For this reason the latter is *not* a naive extension of vector indices to poly-vector valued ones.

If one wishes to avoid algebraic constraints among the first order derivatives of the C-space metric, a careful inspection of eq-(2.46b) reveals that one may convert eq-(2.46b) into an *identity*, instead of a constraint, by *decomposing* $g^{\mu_1\mu_2\ \nu_1\nu_2}$, into irreducible products of ordinary metrics, as well as the spin connection ω^{AC}_{K}, like

$$g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}\ =\ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ -\ g^{\mu_2\nu_1}\ g^{\mu_1\nu_2} \tag{2.47a}$$

and in general, for metric components of the same grade one has

$$g^{[\mu_1\mu_2\ldots\mu_k]\ [\nu_1\nu_2\ldots\nu_k]} = det\, G^{\mu_i\nu_j} = \epsilon_{j_1j_2\ldots j_k}\ g^{\mu_1\nu_{j_1}}\ g^{\mu_2\nu_{j_2}}\ \ldots\ g^{\mu_2\nu_{j_k}},\ \ k=1,2,3,\ldots D \qquad (2.47b)$$

The determinant of $G^{\mu_i\nu_j}$ can be written as

$$det \begin{pmatrix} g^{\mu_1\nu_1} & \ldots & \ldots & g^{\mu_1\nu_k} \\ g^{\mu_2\nu_1} & \ldots & \ldots & g^{\mu_2\nu_k} \\ ------------ & & & ---------------- \\ g^{\mu_k\nu_1} & \ldots & \ldots & g^{\mu_k\nu_k} \end{pmatrix}, \qquad (2.47b)$$

where each $g_{\mu_i\nu_j}$ is a function of all of the C-space coordinates $\mathbf{X}$ components. In the case of metric components of mixed grade, like $g_{\mu_1\mu_2\ \nu}, g_{\mu_1\mu_2\ \nu_1\nu_2\nu_3}, \ldots$ one may use the $\mathbf{0}$ index corresponding to the scalar component to write the decomposition as

$$g_{\mu_1\mu_2\ \nu} = g_{\mu_1\nu}\ g_{\mu_2\mathbf{0}} - g_{\mu_2\nu}\ g_{\mu_1\mathbf{0}} \qquad (2.48a)$$

$$g_{\mu_1\mu_2\ \nu_1\nu_2\nu_3} = g_{\mu_1\nu_1}\ g_{\mu_2\nu_2}\ g_{\mathbf{0}\nu_3} \pm \ permutations \qquad (2.48b)$$

$$g_{\mu_1\mu_2\mu_3\ \nu} = g_{\mu_1\nu}\ g_{\mu_2\mu_3\ \mathbf{0}} \pm \ permutations \qquad (2.48c)$$

and so forth.

The *decomposition* of the C-space metric into irreducible products is dictated by the full Clifford algebraic structure itself and cannot be inferred alone by just looking at the scalar part of the Clifford geometric product $< \gamma_M\gamma_N >$ and which lead to the equation (2.11) expressing the ordinary connections as derivatives of the metric g_{MN}. Similar decomposition of the C-space metric occurs for the C-space beins, inverse beins. The use of the C-space beins allows to rewrite the geometric product of curved base-space generators, like $\gamma^\mu\gamma^\nu = e^\mu_a e^\nu_b \gamma^a\gamma^b = g^{\mu\nu} + \gamma^{\mu\nu}$, after using $g^{\mu\nu} = e^\mu_{(a} e^\nu_{b)}\eta^{ab}$ and $\gamma^{\mu\nu} = e^\mu_{[a} e^\nu_{b]}\gamma^{ab}$ so the Clifford algebraic structure is also maintained in the curved-base manifold. In this way one can decompose the C-space inverse beins E^M_A into antisymmetrized sums of products of e^μ_a. For example, $e^{\mu_1\mu_2}_{a_1a_2} = e^{\mu_1}_{b_1} e^{\mu_2}_{b_2}\delta^{b_1b_2}_{a_1a_2}$; $e^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4} = e^{\mu_1\mu_2}_{b_1b_2} e^{\mu_3\mu_4}_{b_3b_4}\delta^{b_1b_2b_3b_4}_{a_1a_2a_3a_4}$; and so on.

One could relinquish the decomposition of the C-space metric, beins, and/or connections into its irreducible pieces if one enforces the relationships among their first order derivatives of the C-space metric g_{MN} components imposed by the full Clifford algebraic structure. In this case, when one is looking for solutions to the *second* order nonlinear differential C-space gravity field equations, one must take into account these relationships among the first order derivatives in the formulation of the initial valued Cauchy problem. This would be an undesirable program. For this reason, one should avoid these relations among the first order derivatives by decomposing the metric, beins, connections ... into its irreducible pieces and, which in turn, convert these a priori relations (constraints) into mere identities involving first derivatives of the irreducible metric components.

To finalize we add some remarks about the physical applications of C-space gravity to higher curvature theories of gravity. One of the key properties of Lanczos-Lovelock-Cartan gravity (with torsion) is that the field equations do not contain higher derivatives of the metric tensor beyond the second order due to the fact that the action does not contain derivatives of the curvature, see [8], [12], [14], [10] and references therein. They have numerous physical applications. For instance, gravitational actions of third order in the curvature leads to a conjecture about general Palatini-Lovelock-Cartan gravity [11] where the problem of relating torsional gravity to higher-order corrections of the bosonic string-effective action was revisited. In the torsionless case, black-strings and black-brane metric solutions in higher dimensions $D > 4$ play an important role in finding specific examples of solutions to Lanczos-Lovelock gravity [13].

The n-th order Lanczos-Lovelock-Cartan curvature tensor is defined as

$$\mathcal{R}^{(n)}{}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\rho_1\rho_2\ldots\rho_{2n}} = \delta^{\rho_1\rho_2\ldots\rho_{2n}}_{\tau_1\tau_2\ldots\tau_{2n}}\ \delta^{\nu_1\nu_2\ldots\nu_{2n}}_{\mu_1\mu_2\ldots\mu_{2n}}\ \mathcal{R}_{\nu_1\nu_2}{}^{\tau_1\tau_2}\ \mathcal{R}_{\nu_3\nu_4}{}^{\tau_3\tau_4}\ \ldots\ \mathcal{R}_{\nu_{2n-1}\nu_{2n}}{}^{\tau_{2n-1}\tau_{2n}} \tag{2.49}$$

the n-th order Lovelock curvature scalar is

$$\mathcal{R}^{(n)} = \delta^{\nu_1\nu_2\ldots\nu_{2n}}_{\tau_1\tau_2\ldots\tau_{2n}}\ \mathcal{R}_{\nu_1\nu_2}{}^{\tau_1\tau_2}\ \mathcal{R}_{\nu_3\nu_4}{}^{\tau_3\tau_4}\ \ldots.\ \mathcal{R}_{\nu_{2n-1}\nu_{2n}}{}^{\tau_{2n-1}\tau_{2n}} \tag{2.50}$$

the above curvature tensors are antisymmetric under the exchange of any of the μ (ρ) indices. The Lanczos-Lovelock-Cartan Lagrangian density is

$$\mathcal{L} = \sqrt{g}\ \sum_{n=0}^{[\frac{D}{2}]} c_n\ \mathcal{L}_n, \quad \mathcal{L}_n = \frac{1}{2^n}\ \mathcal{R}^{(n)} \tag{2.51}$$

where c_n are arbitrary coefficients; the first term corresponds to the cosmological constant. The integer part is $[\frac{D}{2}] = \frac{D}{2}$ when $D = even$, and $\frac{D-1}{2}$ when $D = odd$. The general Lanczos-Lovelock-Cartan (LLC) theory in D spacetime dimensions is given by the action

$$S = \int d^D x\ \sqrt{|g|}\ \sum_{n=0}^{[\frac{D}{2}]} c_n\, \mathcal{L}_n, \tag{2.52}$$

A simple ansatz relating the LLC higher curvatures to C-space curvatures is based on the following contractions

$$\mathcal{R}^{(n)}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}} \sim \sum_{k=1}^{D} \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}}{}_{\rho_1\rho_2\ldots\rho_k}{}^{\rho_1\rho_2\ldots\rho_k} + \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}}{}_{\mathbf{0}}{}^{\mathbf{0}} \tag{2.53}$$

Even simpler, one may still propose for an ansatz the following

$$\mathcal{R}^{(n)}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}} \sim \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}}{}_{\mathbf{0}}{}^{\mathbf{0}} \tag{2.54}$$

where one must take a *slice* in C-space which requires to evaluate all the terms in the right hand side of eqs-(2.53,2.54) at the "points" $s = x^{\mu_1\mu_2} = \ldots =$

$x^{\mu_1\mu_2\ldots\mu_D} = 0$, for all x^μ, since the left hand side of eqs-(2.53,2.54) solely depends on the vector coordinates x^μ.

The plausible relation to extended gravitational theories based on the scalar curvature and higher order terms like the $f(R), f(R_{\mu\nu}), \ldots$ actions for polynomial-valued functions, and which obviate the need for dark matter, warrants also further investigation [18]. After evaluating the C-space scalar curvature, setting the values of all the poly-vector coordinates to zero, except the x^μ coordinates, and decomposing the C-space metric g_{MN} into its irreducible products of $g_{\mu\nu}$, (and the C-space spin connection ω_K^{AC}) one has

$$\mathbf{R} \ = \ \sum \ldots \ = \ a_1 R \ + \ a_2 R^2 \ + \ldots \ + \ a_N R^N \tag{2.55}$$

$N = [D/2]$. where the scalar curvature with torsion in Riemann-Cartan space decomposes as $R = \hat{R} - \frac{1}{4} T_{abc} T^{abc}$ [10] . Thus in eq- (2.55) one has a special case of $f(R,T)$ for polynomial-valued functions involving curvature and torsion. A conformal fourth-rank gravity theory based on curvature square actions and polynomial identities for hyper-matrices has been studied earlier by [19]. Also worth of exploring is the introduction of nonmetricity and its role in the construction of higher curvature gravitational theories based on non-Riemannian Metric Affine theories of Gravity (MAG) studied by [9].

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APPENDIX

The Clifford geometric product corresponding to the tangent space generators can be written as

$$\gamma_A \ \gamma_B \ = \ \frac{1}{2} \ \{\gamma_A, \gamma_B\} \ + \frac{1}{2} \ [\gamma_A, \gamma_B] \tag{A.1}$$

The commutators $[\gamma_A, \gamma_B]$ for $pq = odd$ one has [7]

$$[\ \gamma_{b_1 b_2 \ldots b_p}, \ \gamma^{a_1 a_2 \ldots a_q} \ ] \ = \ 2\gamma^{a_1 a_2 \ldots a_q}_{b_1 b_2 \ldots b_p} \ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \ \delta^{[a_1 a_2}_{[b_1 b_2} \ \gamma^{a_3 \ldots a_q]}_{b_3 \ldots b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!} \ \delta^{[a_1 \ldots a_4}_{[b_1 \ldots b_4} \ \gamma^{a_5 \ldots a_q]}_{b_5 \ldots b_p]} \ - \ \ldots \tag{A.2}$$

for $pq = even$ one has

$$[\ \gamma_{b_1 b_2 \ldots b_p}, \ \gamma^{a_1 a_2 \ldots a_q} \ ] \ = \ - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 \ldots a_q]}_{b_2 b_3 \ldots b_p]} \ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \; \delta^{[a_1 \ldots a_3}_{[b_1 \ldots b_3} \; \gamma^{a_4 \ldots a_q]}_{b_4 \ldots b_p]} \; + \; \ldots \qquad (A.3)$$

The anti-commutators for $pq = even$ are

$$\{ \; \gamma_{b_1 b_2 \ldots b_p}, \gamma^{a_1 a_2 \ldots a_q} \; \} \; = \; 2\gamma^{a_1 a_2 \ldots a_q}_{b_1 b_2 \ldots b_p} \; -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \; \delta^{[a_1 a_2}_{[b_1 b_2} \; \gamma^{a_3 \ldots a_q]}_{b_3 \ldots b_p]} \; + \; \frac{2p!q!}{4!(p-4)!(q-4)!} \; \delta^{[a_1 \ldots a_4}_{[b_1 \ldots b_4} \; \gamma^{a_5 \ldots a_q]}_{b_5 \ldots b_p]} \; - \; \ldots \qquad (A.4)$$

and the anti-commutators for $pq = odd$ are

$$\{ \; \gamma_{b_1 b_2 \ldots b_p}, \; \gamma^{a_1 a_2 \ldots a_q} \; \} \; = \; - \; \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \; \delta^{[a_1}_{[b_1} \; \gamma^{a_2 a_3 \ldots a_q]}_{b_2 b_3 \ldots b_p]} \; -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \; \delta^{[a_1 \ldots a_3}_{[b_1 \ldots b_3} \; \gamma^{a_4 \ldots a_q]}_{b_4 \ldots b_p]} \; + \; \ldots \qquad (A.5)$$

Eqs-(A.1-A.5) allows to construct explicitly the Clifford geometric product of the curved C-space basis generators $\gamma_M \gamma_N = E^A_M E^B_N \gamma_A \gamma_B$ via the introduction of the C-space beins.

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