A FUNCTIONAL DETERMINANT EXPRESSION FOR THE RIEMANN XI-FUNCTION

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• ABSTRACT: We give an interpretation of the Riemann Xi-function $\xi(s)$ as the quotient of two functional determinants of an Hermitian Hamiltonian $H = H^{\dagger}$. To get the potential of this Hamiltonian we use the WKB method to approximate and evaluate the spectral Theta function $\Theta(t) = \sum_n \exp(-t\gamma_n^2)$ over the Riemann zeros on the critical strip $0 < \mathrm{Re}(s) < 1$. Using the WKB method we manage to get the potential inside the Hamiltonian H , also we evaluate the functional determinant $\det(H + z^2)$ by means of Zeta regularization, we discuss the similarity of our method to the method applied to get the Zeros of the Selberg Zeta function. In this paper and for simplicity we use units so $2m = 1 = \hbar$
• *Keywords:* = Riemann Hypothesis, Functional determinant, WKB semiclassical Approximation , Trace formula ,Bolte's law, Quantum chaos.

1. Riemann Zeta function and Selberg Zeta function

Let us consider a Riemann surface with constant negative curvature, this surface can be constructed as the upper complex plane divided by a discrete subgroup of the modular group $PSL(2, R)$, Selberg [14] studied the problem of the 2-dimensional Laplacian with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$,discrete boundary conditions are imposed for the discrete group $PSL(2, R)$

$$\Delta\Psi_n(x, y) = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \Psi_n(x, y) = E_n\Psi_n(x, y) \; E_n = \frac{1}{4} + k_n^2 \quad (1)$$

These momenta k_n are the non-trivial zeros of the Selberg Zeta function, which can be defined by an Euler product over the Geodesic of the surface in an analogy with the Riemann Zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{n}\frac{1}{(1-p_n^{-s})} \qquad Z(s)=\prod_{P}\prod_{k=0}^{\infty}\left(1-N(P)^{-(s+k)}\right) \tag{2}$$

In both cases the Riemann Zeta function and the Selberg Zeta function can be expressed by an infinite product

Selberg also studied a Trace formula which relates the Zeros (momenta of the Laplacian Δ) on the critical line $Z\left(\frac{1}{2}+ik_n\right)=0$ and the length of the Geodesic of the Surface in the form

$$\sum_{n}h(k_n)=\frac{\mu(D)}{4\pi}\int_{0}^{\infty}dkkh(k)\tanh(\pi k)+\sum_{P\in p.p.o}\frac{\ln N(P)}{N(P)^{1/2}-N(P)^{1/2}}g(\ln N(P) \tag{3}$$

Here, p.p.o means that we are taking the sum over the length of the Geodesic, $h(k)$ is a test function and $g(k)$ is the Fourier cosine transform of $h(k)$ $g(k)=\frac{1}{2\pi}\int_0^\infty dxh(x)\cos(kx)$ $\mu(D)$ is the area of the fundamental domain describing the Riemann surface . In case we had a surface with the length of the Geodesic $\ln N(P)=\ln p$ for 'p' on the second side of the equation a prime number,then the Selberg Trace is very similar to the Riemann-Weil sum formula [12]

$$\sum_{\gamma}h(\gamma)=2h\left(\frac{i}{2}\right)-g(0)\ln\pi-2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\ln n)+\frac{1}{2\pi}\int_{-\infty}^{\infty}dsh(s)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{is}{2}\right) \tag{4}$$

This formula (4) related a sum over the imaginary part of the Riemann zeros to another sum over the primes, here $\Lambda(n)=\begin{cases}\ln p \ n=p^k \\ 0 \text{ otherwise}\end{cases}$ with 'k' a positive integer is the Mangoldt function, in case $\ln N(P)=\ln p$ both zeta function of Selberg and Riemann are related by $\frac{1}{Z(s)}=\prod_{n=0}^{\infty}\zeta(n+s)$ and their logarithmic derivative is quite similar if we set the function $\Lambda_{geodesic}(P)=\frac{\ln N(P)}{1-N(P)^{-1}}$

$$\frac{\zeta'}{\zeta}(s)=-\sum_{n=1}^{\infty}\Lambda(n)n^{-s} \qquad \frac{Z'}{Z}(s)=\sum_{P\in p.p.o}\Lambda_{geodesic}(P)N(P)^{-s} \tag{5}$$

In both cases the Riemann and Selberg zeta functions obey a similar functional equation which relates the value at s and 1-s

$$\zeta(1-s)=X(s)\zeta(s) \qquad Z(1-s)=\exp\left(-\frac{\mu(D)}{4\pi}\int_{0}^{s-1/2}vtan(\pi v)dv+c\right)Z(s) \tag{6}$$

The constant of integration 'c' is determined by setting $s=1/2$, and $X(s)=2\left(2\pi\right)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$ for the case of the Riemann zeta function.

With the aid of the Selberg Trace formula (3) , we can evaluate the Eigenvalue staircase for the Laplacian $\Delta = y^2\left(\partial_x^2 + \partial_y^2\right)$

$$N\left(E = \frac{1}{4} + p^2\right) = \sum_{E_n \le E} 1 = \sum_n \frac{\mu(D)}{4\pi}\int_0^p dkkh(k)\tanh(\pi k) + \frac{1}{\pi}\arg Z\left(\frac{1}{2} + ip\right) \tag{7}$$

Here $p = \sqrt{E - \frac{1}{4}}$, we can immediatly see that the smooth part of (7) satisfy Weyl's law in dimension 2 $N_{smooth}(E) \approx \frac{\mu(D)}{4\pi}E$, the oscillatory part of (7) satisfy Bolte's semiclassical law [4] (page 34, theorem 2.10) $\frac{1}{\pi}\arg Z\left(\frac{\lambda}{2} + i\sqrt{E}\right)$ with $\lambda = 1$, the branch of the logarithm inside (7) is chosen, so $\arg Z\left(\frac{1}{2}\right) = 0$ in this case the Selberg Zeta function is the dynamical zeta function of a Quantum system and the Energies are related to the zeros of $Z(s)$.

2. A functional determinant for the Riemann Xi function $\xi(s)$

From the analogies between the Riemann Zeta function and the Selberg Zeta function, we could ask ourselves if there is a Hamiltonian operator (the simplest second order differential operator which has a classical and quantum meaning and it is well studied) in the form

$$H\Psi_n(x) = -\frac{d^2\Psi_n(x)}{dx^2} + V(x)\Psi_n(x) = E_n\Psi_n(x) \;\; \Psi_n(0) = 0 = \Psi_n(\infty) \;\; E_n = \gamma_n^2$$

$$V(x) = \left\{ \begin{array}{c} f(x) \;\; x > 0 \\ \infty \;\; x \le 0 \end{array} \right. \tag{8}$$

The function $f(x)$ defined inside the equation (8) for the potential $V(x)$ and the Hamiltonian $H = p^2 + V(x)$ must be evaluated.
The idea here is to choose $f(x)$ so the Energies of the Hamiltonian are the square of the imaginary part of the Riemann zeros $E_n = \gamma_n^2$.,
In this paper we will prove that this function can be obtained within the WKB approach as $f^{-1}(x) = 2\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}N(x)$,here $N(x)$ is the Eigenvalue staircase of the Hamiltonian defined in (8) .
For the case of this Hamiltonian, which involves the imaginary part of the zeros the exact eigenvalue staircase $N(x)$ can be evaluated [9]

$$N(E) = \sum_n H\left(E - E_n\right) = \frac{1}{\pi}\arg\xi\left(\frac{1}{2} + i\sqrt{E}\right) = 1 + \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + i\sqrt{E}\right) + \frac{\vartheta(\sqrt{E})}{\pi} \tag{9}$$

With $H(x) = \left\{ \begin{array}{c} 1 \text{ x>0} \\ 0 \text{ x<0} \end{array} \right.$, $\vartheta(T) = \arg\Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - \frac{T}{2}\ln\pi \approx \frac{T}{2}\ln\left(\frac{T}{2\pi e}\right) - \frac{\pi}{8} + \frac{1}{48T} + ...$

Also we will prove how the Riemann Xi function $\xi(s)=\frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\xi(1-s)$ is proportional to the functional determinant $\det\left(H-s(1-s)\right)$, and we will also show that the density of states can be evaluated from the argument of the Xi-function $E=p^2 \ \frac{1}{2\pi p}\frac{d}{dp}\Im m\log\det\left(H+i\epsilon-p\right)=\rho(E)=\sum\limits_{\gamma_n}\delta\left(p^2-\gamma_n^2\right)$

As a simple example of how Quantum Mechanics can help to solve problems of finding the roots of functions , let be a particle moving inside an infinite potential well , the energy is given by $E=p^2$ and the one dimensional Schröedinger equation [7] in units $\hbar=2m=1$ ($\hbar$ is the reduced Planck's constant with value $\hbar=1.05.10^{-34}\ J.T^{-1}$)

$$H_0u_n(x)=-\frac{d^2u_n(x)}{dx^2}+V(x)u_n(x)=E_nu_n(x)\ \mathrm{u}_n(0)=0=u_n(\pi)\ E_n=n^2 \\ u_n(x)=A\sin\left(\pi x\right), \tag{10}$$

n this case the Euler's product formula for the sine function is the quotient between 2 functional determinants

$$\frac{\sin\left(\pi\sqrt{x}\right)}{\pi\sqrt{x}}=\prod_{n=1}^{\infty}\left(1-\frac{x}{E_n}\right)=\frac{\det\left(H_0-x\right)}{\det\left(H_0\right)}$$

We can also compute the density of states to get the Poisson sum formula

$$\rho(E)=\sum_{n=1}^{\infty}\delta\left(E-E_n\right)=\frac{1}{2p}\left(\sum_n\delta\left(p-n\right)+\sum_n\delta\left(p-n\right)\right)=\frac{1}{2p}\sum_{n=-\infty}^{\infty}e^{2i\pi np} \tag{12}$$

o *Zeta regularized determinant for* $\xi(s)$ *:*

Given an Operator P with real Eigenvalues $\{E_n\}$, we can define its Zeta regularized determinant [6] in the form

$$\det\left(P+k^2\right)=\exp\left(-\frac{d}{ds}\zeta_P(s,k^2)\left.\right|_{s=0}\right) \tag{13}$$

Here $\zeta_P(s,k^2)=Tr\left\{(P+k^2)^{-s}\right\}=\sum\limits_n\left(E_n+k^2\right)^{-s}$ is the Spectral Zeta function of the operator taken over all the Eigenvalues, the relationship between this spectral zeta function and the Theta function $\Theta(t)=\sum\limits_n\exp(-tE_n)=\int_0^\infty dN(x)e^{-xt}$,t>0 always , is given by the Mellin transform $\sum\limits_{n=0}^{\infty}\frac{1}{(E_n+k^2)^s}=\frac{1}{\Gamma(s)}\int_0^\infty\frac{dt}{t}e^{-tk^2}\Theta(t)t^{s-1}$. If P is a Hamiltonian we can obtain the Theta function $\Theta(t)=\sum\limits_n\exp(-tE_n)$ (approximately) by an integral over the Phase space

[7]

$$\Theta(t)=\sum_{n=0}^{\infty}\exp\left(-tE_n\right)\approx\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\int_{0}^{\infty}dxe^{-tp^2-tf(x)}=\frac{1}{2\sqrt{\pi t}}\int_{0}^{\infty}dxe^{-tf(x)}=\Theta_{WKB}(t) \tag{14}$$

If we compare the semiclassical Theta function (14) and the spectral Theta function $\Theta(t)=\sum_n\exp(-tE_n)$ then we find

$$\Theta(t)=\sum_{n=0}^{\infty}\exp\left(-tE_n\right)=-s\int_0^{\infty}dtN(t)e^{-st}\approx\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_0^{\infty}dp\exp(-tp^2-tf(x)) \tag{15}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}dp\exp(-tp^2-tf(x))=\frac{1}{2\sqrt{\pi t}}\int_0^{\infty}dxe^{-tf(x)}=\frac{1}{2\sqrt{\pi t}}\int_0^{\infty}dre^{-tr}\frac{dV^{-1}(r)}{dr} \tag{16}$$

From expressions (14) and (15) and setting $N(0)=0$ (after changes of variable)

$$\sqrt{s}\int_0^{\infty}dxN(x)e^{-sx}=\frac{1}{2\sqrt{\pi}}\int_0^{\infty}dxf^{-1}(x)e^{-sx}\ \rightarrow f^{-1}(x)=2\sqrt{\pi}\frac{d^{1/2}}{dx^{1/2}}N(x) \tag{17}$$

To prove (16) and (17) we have used the properties of the integral representation for the Laplace inverse transform

$$\forall\alpha\in R\qquad D^{\alpha}e^{kt}=k^{\alpha}e^{kt}\qquad D^{\alpha}f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dsF(s)e^{st}s^{\alpha} \tag{18}$$

nd the fact that if two Laplace transforms are equal then $L\left\{f(t)\right\}=L\left\{g(t)\right\}$ implies that $f(t)=g(t)$, for the case of the Riemann Zeros $N(E)=\frac{1}{\pi}\arg\xi\left(\frac{1}{2}+i\sqrt{E}\right)$ (Bolte's semiclassical law in one dimension) so $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$, since we want our potential inside (8) to be positive whenever we take the inverse we must choose the POSITIVE branch of the inverse in order to get $f(x)\geq 0$ on the interval $[0,\infty)$, the half derivative and the half integral for any well behaved function are given in [13]

$$\frac{d^{-\frac{1}{2}}f(x)}{dx^{-\frac{1}{2}}}=\frac{1}{\Gamma(1/2)}\int_0^x dt\frac{f(t)}{\sqrt{x-t}}\qquad\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}}=\frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_0^x\frac{dtf(t)}{\sqrt{x-t}} \tag{19}$$

We have written implicitly the potential inside (8) , if the function $f(x)$ is defined by the functional equation $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)=2\sum_n\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}$, then we may evaluate the Spectral Zeta function of the Quantum system given in (8), then

$$\frac{\det\left(H+z^2\right)}{det\left(H\right)}=\exp\left(-\frac{d}{ds}\zeta_P(s,z^2)\left.\right|_{s=0}+\frac{d}{ds}\zeta_P(s,0)\left.\right|_{s=0}\right)=\frac{\xi(z+1/2)}{\xi(1/2)} \tag{20}$$

For the potential defined by $f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right)$, we can evaluate the Theta kernel using (14) ,(15) and (16) $\Theta(t) = \sum_n e^{-tE_n} = \frac{1}{2\sqrt{\pi t}} \int_0^\infty dx \frac{df^{-1}(x)}{dx} e^{-tx}$, for this potential the spectral theta function and its derivative are

$$\zeta_H(s, z^2) = \sum_{n=0}^{\infty} \frac{1}{(\gamma_n^2 + z^2)^s} \qquad -\frac{d}{ds}\zeta_H(0, z^2) = \sum_{n=0}^{\infty} \ln\left(\gamma_n^2 + z^2\right) \qquad \zeta\left(\frac{1}{2} + i\gamma_n\right) = 0 \tag{21}$$

Taking exponentials we reach to the infinite product for the Riemann Xi-function as an spectral determinant (functional determinant over the Eigenvalues of H)

$$\frac{\det\left(H + z^2\right)}{\det(H)} = \frac{\prod\limits_{n=0}^{\infty} (\gamma_n^2 + z^2)}{\prod\limits_{n=0}^{\infty} \gamma_n^2} = \prod_{n=0}^{\infty} \left(1 + \frac{z^2}{E_n}\right) = \frac{\xi\left(1/2 + z\right)}{\xi\left(1/2\right)} \tag{22}$$

If we choose the positive branch $f(x) \geq 0$ of the inverse $f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right)$ then the potential will be always positive so the Energies of the Hamiltonian inside (8) will be all positive $E_n = \gamma_n^2 \in R^+$, then all the non-trivial zeros of the Riemann Zeta function will be on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, with a simple change of variable $z = s - \frac{1}{2}$ we obtain

$$\frac{\xi(s)}{\xi(0)} = \frac{\det\left(H - s(1-s) + \frac{1}{4}\right)}{\det\left(H + \frac{1}{4}\right)} = \frac{\xi(1-s)}{\xi(0)} = \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \tag{23}$$

Equation (22) is the Hadamard product for the Riemann Xi-function in terms of the quotient of 2 functional determinants, since the expected value of the Hamiltonian is positive $\langle \psi_n | H | \psi_n \rangle \geq 0$ and Hermitian ,with $f(x) \geq 0$ then all the Energies are positive $E_n = s(1-s) \in R^+$ Riemann Hypothesis should hold. If we set $s = \frac{1}{2} + i\sqrt{E}$ then is it clear that the roots of the functional determinant $\det(E - H)$ are the roots of the function $\xi\left(\frac{1}{2} + i\sqrt{E}\right)$

In the limit $x \to \infty$,the smooth part of the Eigenvalue staircase is given by $N(E) \approx \frac{\sqrt{E}}{2\pi} \log\left(\frac{\sqrt{E}}{2\pi e}\right)$, $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$, if we use the expression for the logarithm $\log(x) \approx \frac{x^\epsilon - 1}{\epsilon}$ as $\epsilon \to 0$ and apply the half derivative expression, then the following holds $\epsilon \to 0$

$$f_{smooth}(x) \approx 4\pi^2 e^2 \left(\frac{\epsilon\sqrt{\pi}x + B}{A(\epsilon)}\right)^{\frac{2}{\epsilon}} \qquad f_{smooth}^{-1}(x) \approx \frac{\left(4\pi^2 e^2\right)^{-\epsilon/2} A(\epsilon) x^{\epsilon/2} - B}{\sqrt{\pi}\epsilon} \tag{24}$$

$A(\epsilon) = \frac{\Gamma\left(\frac{3+\epsilon}{2}\right)}{\Gamma\left(1+\frac{\epsilon}{2}\right)}$ and $B = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, the second expression inside (24) is the asymptotic of $f(x)$ as $x \to \infty$, for this potential , the energies are

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n \qquad E_n^{smooth} = f(n) = N_{smooth}^{-1}(E) \approx \frac{4\pi^2 n^2}{W^2\left(ne^{-1}\right)} \tag{25}$$

We can also test our formula $f^{-1}(x) = 2\sqrt{\pi}\frac{d^{\frac{1}{2}} n(x)}{dx^{\frac{1}{2}}}$ (and compare it with (24)) with the potentials x^n $n = 1, 2, \infty$ (an infinite potential well is assumed at the point $x = 0$) these are the cases of the linear potential (bouncer) , Harmonic oscillator and infinite potential well

$$f^{-1}(x) = \frac{2\sqrt{x}}{\omega} \qquad f(x) = \frac{(\omega x)^2}{4} \qquad N(E) = \frac{E}{2\omega} \tag{26}$$

$$f^{-1}(x) = x^{\frac{1}{n}} \qquad f(x) = x^n \qquad n \to \infty \qquad N(E) = \frac{1}{\sqrt{4\pi}} \cdot \frac{\Gamma\left(\frac{1}{m}+1\right)}{\Gamma\left(\frac{1}{m}+\frac{3}{2}\right)} E^{\frac{1}{m}+\frac{1}{2}} \tag{27}$$

$$f^{-1}(x) = \frac{x}{k} \qquad f(x) = kx \qquad N(E) = \frac{2E^{3/2}}{3\pi k} \tag{28}$$

From expression (23) we can also compute the density of states of our Hamiltonian with $p = \sqrt{E}$ and $\delta\left(E - \gamma^2\right) = \frac{\delta(p-\gamma)+\delta(p+\gamma)}{2p}$ we will also use the Shokhotsky's formula for the delta function $-\frac{1}{\pi}\Im m\left(\frac{1}{x+i\epsilon}\right) = \delta(x)$ $\epsilon \to 0$

$$\begin{aligned} &-\tfrac{1}{2\pi}\tfrac{d}{dE}\arg\xi\left(\tfrac{1}{2}+i\epsilon+i\sqrt{E}\right) = \sum_{\gamma}\delta\left(E-\gamma^2\right) = \sum_{\gamma}\delta\left(p^2-\gamma^2\right) = \\ &\tfrac{1}{\pi}\tfrac{\zeta}{\zeta}\left(\tfrac{1}{2}+ip\right)\tfrac{1}{2p} + \tfrac{1}{\pi}\tfrac{\zeta'}{\zeta}\left(\tfrac{1}{2}-ip\right)\tfrac{1}{2p} - \tfrac{\ln\pi}{2\pi p} + \tfrac{\Gamma'}{\Gamma}\left(\tfrac{1}{4}+i\tfrac{p}{2}\right)\tfrac{1}{4\pi p} + \\ &\tfrac{\Gamma'}{\Gamma}\left(\tfrac{1}{4}-i\tfrac{p}{2}\right)\tfrac{1}{4\pi p} + \tfrac{\delta\left(p-\frac{i}{2}\right)+\pi\delta\left(p+\frac{i}{2}\right)}{2p} = \rho(E) = \tfrac{1}{2\pi p}\tfrac{d}{dp}\mathrm{argdet}\left(H+i\epsilon-p\right) \end{aligned} \tag{29}$$

Here $\frac{1}{-\pi}\lim_{\epsilon\to 0}\Im m\left(\frac{2}{2x\pm i+2i\epsilon}\right) = \delta\left(x \pm \frac{i}{2}\right)$, this factor comes from the logarithmic derivative of $s(s-1)$ along the critical line $s = \frac{1}{2} + ip$, equation (29) is a distributional version of the Riemann-Weil trace formula , taking formally the logarithm of the Euler product for the Riemann Zeta function on the critical line yields to $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} e^{-ip\ln n} =_{reg} -\frac{\zeta'}{\zeta}\left(\frac{1}{2}+ip\right)$, using two test functions h(x) and g(x) $g(x) = \frac{1}{\pi}\int_0^{\infty} dr\cos(rx)h(r)$ we recover the oscillatory part of the Riemann-Weil trace formula $-2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\ln n)$.

Unlike the model of Wu and Sprung, we have considered also the oscillatory part of the Riemann Eigenvalue Staircase $\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+i\sqrt{E}\right)$, which satisfy Bolte's semiclassical law , Wu and Sprung [17] considered only the smooth part of teh Eigenvalue staircase in the limit $T >> 1$ $\frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) \approx N(T)$ in order to get a Hamiltonian whose Energies are the positive imaginary part of the Riemann Zeros, their starting point is the Harmonic oscillator [15] , but unlike the normal quantum mechanical oscillator whose functional determinant gives the Gamma function $\frac{\sqrt{2\pi}}{\Gamma(s)} = \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)$ the product taken ONLY over the positive imaginary part of the zeros (even if it converges) $\prod_{n=0}^{\infty}\left(1+\frac{s}{\gamma_n}\right)$ has no meaning, also the Wu-Sprung model doesn't obey Weyl's law in one dimension mainly $N_{smooth}(E) = O\left(E^{d/2}\right)$, in our case , the Hamiltonian (8) with the Smooth part of the Eigenvalue staircase $N(E) \approx \frac{\sqrt{E}}{2\pi}\log\left(\frac{\sqrt{E}}{2\pi e}\right)$, satisfies a Weyl's law with $d = 1 + \frac{\epsilon}{2}$ and the spectral determinant (quotient) $\frac{\Delta(E)}{\Delta(0)} = \prod_{n=0}^{\infty}\left(1-\frac{E}{E_n}\right)$ $E_n = \gamma_n^2$ is proportional to the Riemann xi function on the critical line $\xi\left(\frac{1}{2}+i\sqrt{E}\right)$

By analogy with the zeros of the Selberg Zeta function, is better to consider the case with the Energies $E_n = \gamma_n^2$, in this case the Trace of the Resolvent of the Hamiltonian $(E+i\epsilon-H)^{-1}$ is the Riemann-Weil trace for the Riemann zeros.

o *Analytic expressions for the potential from Riemann-Weil trace formula*:

From the expression for the fractional derivative of powers $\frac{d^k x^\lambda}{dx^k} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-k+1)}x^{\lambda-k}$, we can obtain for the inverse function

$$f^{-1}(x) = \frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right) = 2\sum_{\gamma>0}\frac{H(x-\gamma^2)}{\sqrt{x-\gamma^2}} \tag{30}$$

Using the Riemann-Weil formula we can rewrite (28) as

$$f^{-1}(x) = \frac{4}{\sqrt{4x+1}}+\frac{1}{2\pi}\int_{-\sqrt{x}}^{\sqrt{x}}\frac{dr}{\sqrt{x-r^2}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)-\ln\pi\right)-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right) \tag{31}$$

Here $g(u = \ln n, x) = \frac{1}{\pi}\int_0^{\sqrt{x}}\frac{\cos(ut)}{\sqrt{x-t^2}}dt = \frac{J_0\left(u\sqrt{x}\right)}{2}$, here the integral can be expressed in terms of the zeroeth order Bessel function.

o *Numerical evaluation of the functional determinant* :

We need to evaluate the half-derivative inside the inverse of the potential $f^{-1}(x) = \frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$, to do so we can use the Grunwald-Letnikov formula [13] with an step $\epsilon = 0.01$ and $q = \frac{1}{2}$

$$\frac{\Delta^q g(x)}{\epsilon^q} \approx \frac{d^{\frac{1}{2}} g(x)}{dx^{\frac{1}{2}}} \approx \frac{1}{\epsilon^q}\sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(q+1)}{\Gamma(n+1)\Gamma(q-n+1)} g\left(x+(q-n)\epsilon\right) \quad (32)$$

In order to evaluate $\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+i\sqrt{x}\right)$ for big 'x' , we have used the Riemann-Siegel formula [10] $k = \sqrt{x}$

$$Z(k) = \zeta\left(\frac{1}{2}+ik\right)e^{i\vartheta(k)} = 2\sum_{n=1}^{U(k)}\frac{\cos\left(\vartheta(k) - k\ln n\right)}{\sqrt{n}} + O\left(\frac{1}{k^{1/4}}\right) \quad k\to\infty \quad (33)$$

The functions inside (A.3) are $u(k) = \left[\sqrt{\frac{k}{2\pi}}\right]$, $[x]$ is the floor function and $\vartheta(T) = \arg\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right) - \frac{T}{2}\ln\pi \approx \frac{T}{2}\ln\left(\frac{T}{2\pi e}\right) - \frac{\pi}{8} + \frac{1}{48T} + ...$ (34)

For the case of the functional determinant of our Hamiltonian operator with the potential $f^{-1}(x) = \frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ defined as

$$L\to\infty \quad (35)$$
$$\left(-\frac{d^2}{dx^2}+\frac{1}{4}+f(x)-\lambda\right)y(x,\lambda) = 0 \;\; y(0,\lambda) = 0 = y(L,\lambda)$$
$$\lambda = s(1-s),$$

e can evaluate the functional determinant of the operator (35) by the Gelfand-Yaglom method [18] , in this case we need to solve the initial value problem

$$\left(-\frac{d^2}{dx^2}+\frac{1}{4}+f(x)-\lambda\right)y(x,\lambda) = 0 \;\; y(0,\lambda) = 0 \;\; \frac{dy(0,\lambda)}{dx} = 1 \quad (36)$$

Unfortunately exact solutions can not be found , in the WKB approximation (36) has the solution

$$\Pi(x) = \sqrt{f(x)+\frac{1}{4}-\lambda} \quad (37)$$
$$y(x,\lambda) \approx \frac{1}{\Pi(x)^{1/2}}\left(C_+\exp\int_0^x \Pi(t)dt + C_-\exp - \int_0^x \Pi(t)dt\right)$$

The 2 constants $C_\pm$ are chosen so (37) solves the initial value problem (36)

The Gelfand-Yaglom theorem, [18] tells us that the functional determinant is related to the solution of the initial value problem (36) and the boundary value problem (35) in the form

$$\lambda = s(1-s) \quad (38)$$

$$\lim_{L\to\infty}\frac{y(L,\lambda)}{y(L,0)}=\frac{\det\left(H+\frac{1}{4}-s(1-s)\right)}{\det\left(H+\frac{1}{4}\right)}=\frac{\xi(s)}{\xi(0)}=\prod_{\rho}\left(1-\frac{s}{\rho}\right)$$

The main advantage of the Gelfand-Yaglom method , is that we do not need to evaluate any single eigenvalue in order to obtain the functiona determinant $\det\left(H+\frac{1}{4}-\lambda\right)$, unfortunately this method is only valid for ordinary differential equations

TABLE1 : comparison between the Riemann Zeros (square) from the tables of Odlyzko and the Numerical values of the energies for our Hamiltonian operator (8) with $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$, to obtain numerically the potential we have used formula (34) to evaluate the fractional derivative and the Riemann-Siegel formula (32) to evaluate $S(T)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right)$

n	Zeros (square)	Energies
0	199.7897	198.7886
1	441.9244	441.9240
2	625.5401	625.5406
3	925.6684	925.6683
4	1084.7142	1084.7139
5	1412.7149	1412.7146
6	1674.3400	1674.3398
7	1877.2289	1877.2287
8	2304.4896	2304.4893
9	6363.8591	6363.8589

o Bessel function $J_0\left(a\sqrt{x}\right)$ and the density of states for our Hamiltonian and the Riemann zeros $\rho(x)=\sum_{n=0}^{\infty}\delta\left(x-\gamma_n^2\right)$:

Let us compare the Riemann-Weyl explicit formula and the definition for the inverse of our potencial function $f^{-1}(x)$

$$\begin{aligned}&\sum_{\gamma}\delta\left(x-\gamma^2\right)=\frac{1}{\pi}\frac{\zeta}{\zeta}\left(\frac{1}{2}+i\sqrt{x}\right)\frac{1}{2\sqrt{x}}+\frac{1}{\pi}\frac{\zeta'}{\zeta}\left(\frac{1}{2}-i\sqrt{x}\right)\frac{1}{2\sqrt{x}}-\frac{\ln\pi}{2\pi p}\\&+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{\sqrt{x}}{2}\right)\frac{1}{4\pi\sqrt{x}}+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-i\frac{\sqrt{x}}{2}\right)\frac{1}{4\pi\sqrt{x}}+\frac{\delta\left(\sqrt{x}-\frac{i}{2}\right)+\delta\left(\sqrt{x}+\frac{i}{2}\right)}{2\sqrt{x}}=\rho(x)\end{aligned}\tag{39}$$

$$f^{-1}(x)=\frac{4}{\sqrt{4x+1}}+\frac{1}{2\pi}\int_{-\sqrt{x}}^{\sqrt{x}}\frac{dr}{\sqrt{x-r^2}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)-\ln\pi\right)-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right)\tag{40}$$

From the equation for our potential $f^{-1}(x)=2\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}N(x)$ taking again the half-derivative operator we get $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}f^{-1}(x)=2\sqrt{\pi}\frac{dN(x)}{dx}=\rho(x)$, this means that perhaps the equations (39) and (40) should be related by a fractional operator $\frac{d^{\pm\frac{1}{2}}}{dx^{\pm\frac{1}{2}}}$, for the smooth part this is almost trivial to find from the definition of the half-integral $\frac{1}{\sqrt{\pi}}\int_0^x \frac{dt}{\sqrt{x-t}}f(t)$ and the fact that $\left(\frac{d^{\pm\frac{1}{2}}}{dx^{\pm\frac{1}{2}}}\right)^2 g(x)=\frac{dg(x)}{dx}$, for teh oscillating part of the potential we must recall the identity

$$\frac{\cos(x)}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n-1} \qquad J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} \qquad \sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}J_0\left(a\sqrt{x}\right)=\frac{\cos\left(\sqrt{ax}\right)}{\sqrt{x}} \tag{41}$$

quation (41) is a bit harder to prove, we can prove (41) by using the Taylor expansion of the 2 functions and then applying the property of the half derivative operator for power series $\frac{d^{\frac{1}{2}}x^n}{dx^{\frac{1}{2}}}=\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}x^{n-\frac{1}{2}}$, so we choose the potential from the WKB formalism and we also have proved that the density of states of our Hamiltonian defined in (8) is just the Riemann-Weil explicit formula in a distributional framework so $\frac{f^{-1}(x)}{2\sqrt{\pi}}=\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\rho(x)$

APPENDIX A: An implicit equation for the potential f(x) from the Bohr-Sommerfeld quantization conditions

The expression $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ can also be obtained from the Bohr-Sommerfeld quantization conditions [7]

$$E=f(a) \qquad 2\int_0^a dx\sqrt{E-f(x)}=p(x) \qquad \oint_C pdq=2\pi\left(n+\frac{1}{2}\right) \tag{A.1}$$

' is the classical turning point $n=N(E)$ is the Eigenvalue staircase, the first integral inside (A.1) is a line integral taken over the closed orbit of the classical system, equation (A.1) can be understood as an integral equation for the inverse of the potential in the form

$$2\pi\left(\frac{1}{2}+n(E)\right)=2\int_0^{a=a(E)}\sqrt{E-V(x)}dx=2\int_0^E\sqrt{E-x}\frac{df^{-1}}{dx}=\sqrt{\pi}D_x^{-\frac{1}{2}}f(x) \tag{A.2}$$

If we take the half derivative on both sides of (A.2) we would get $f^{-1}(x)=2\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{1}{2}+\frac{1}{\pi}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)\right)$ in this case this result is completely equivalent to the one we got by Zeta regularization and by the WKB approximation of the Theta function $\frac{1}{2\sqrt{\pi t}}\int_0^\infty dxe^{-tf(x)}=\Theta_{WKB}(t)$.

From equation $f^{-1}(x)=2\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{1}{2}+\frac{1}{\pi}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)\right)$ the density of states could be evaluated approximately as $\frac{1}{2\sqrt{\pi}}\frac{d^{\frac{1}{2}}f^{-1}(x)}{dx^{\frac{1}{2}}}=\rho(x)=\sum_n\delta\left(x-\gamma_n^2\right)$

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