Lanczos-Lovelock-Cartan Gravity from Clifford Space Geometry

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Abstract

A rigorous construction of Clifford-space Gravity is presented which is compatible with the Clifford algebraic structure and permits the derivation of the expressions for the connections with *torsion* in Clifford spaces (C-spaces). The C-space generalized gravitational field equations are derived from a variational principle based on the extension of the Einstein-Hilbert-Cartan action. We continue by arguing how Lanczos-Lovelock-Cartan higher curvature gravity with torsion can be embedded into gravity in Clifford spaces and suggest how this might also occur for extended gravitational theories based on $f(R), f(R_{\mu\nu}), ...$ actions, for polynomial-valued functions. In essence, the Lanczcos-Lovelock-Cartan curvature tensors appear as Ricci-like traces of certain components of the C-space curvatures. Torsional gravity is related to higher-order corrections of the bosonic string-effective action. In the torsionless case, black-strings and black-brane metric solutions in higher dimensions $D > 4$ play an important role in finding specific examples of solutions to Lanczos-Lovelock gravity.

1 Introduction

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. This extended relativity in Clifford spaces theory should *not* be confused with the extended relativity theory (ER) proposed by Erasmo Recami and collaborators [3] many years ago which was based on the Special Relativity theory extended to Antimatter and Superluminal motions. Since the beginning of the seventies, an "Extended special Relativity" (ER) exists, which on the basis of the ordinary postulates of Special Relativity (chosen "com grano salis") describes also superluminal motions in a rather simple way, and without any severe causality violations. Reviews of that theory of ER can be found in [3].

The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set to be equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1].

The poly-vector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate). These poly-vector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background.

The C-space poly-vector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions

$$\mathbf{X} \ = \ \sigma\,\mathbf{1} + x^\mu\ \gamma_\mu + x^{\mu\nu}\ \gamma_\mu \wedge \gamma_\nu + x^{\mu\nu\rho}\ \gamma_\mu \wedge \gamma_\mu \wedge \gamma_\rho + x^{\mu\nu\rho\tau}\ \gamma_\mu \wedge \gamma_\mu \wedge \gamma_\rho \wedge \gamma_\tau \tag{1.1}$$

σ is the Clifford scalar component of the poly-vector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in C-space

$$(d\Sigma)^2 \ = \ (d\sigma)^2 \ + \ dx_\mu \ dx^\mu \ + \ dx_{\mu\nu} \ dx^{\mu\nu} \ + \ \ldots \tag{1.2}$$

is *not* equal to the proper time Lorentz-invariant interval ds in ordinary spacetime $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$.

The main purpose of this work is to build a generalized gravitational theory in Clifford spaces and show how the Lanczos-Lovelock-Cartan higher curvature gravity with torsion can be embedded in the former. In essence, the Lanczcos-Lovelock-Cartan curvature tensors appear as Ricci-like traces of certain components of the C-space curvatures. Gravitational actions of third order in the curvature leads to a conjecture about general Palatini-Lovelock-Cartan gravity [11] where the problem of relating torsional gravity to higher-order corrections of the bosonic string-effective action was revisited. In the torsionless case, black-strings and black-brane metric solutions in higher dimensions $D > 4$ play an important role in finding specific examples of solutions to Lanczos-Lovelock gravity.

2 The construction of Clifford-space Gravity

At the beginning of this section we follow closely the work in [1] and then we depart from it by constructing Clifford space (C-space) gravity without making any a priori assumptions on the C-space connections. Let the vector fields γ_μ, $\mu = 1, 2, ..., n$ be a coordinate basis in V_n satisfying the Clifford algebra relation

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu} \tag{2.1}$$

where $g_{\mu\nu}$ is the metric of V_n. In curved space γ_μ and $g_{\mu\nu}$ cannot be constant but necessarily depend on position x^μ. An arbitrary vector is a linear superposition [4] $a = a^\mu \gamma_\mu$ where the components a^μ are *scalars* from the geometric point of view, whilst γ_μ are *vectors*.

Besides the basis $\{\gamma_\mu\}$ we can introduce the reciprocal basis[1] $\{\gamma^\mu\}$ satisfying

$$\gamma^\mu \cdot \gamma^\nu \equiv \tfrac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu} \tag{2.2}$$

where $g^{\mu\nu}$ is the covariant metric tensor such that

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu, \;\; \gamma^\mu \gamma_\nu + \gamma_\nu \gamma^\mu = 2\delta^\mu{}_\nu \text{ and } \gamma^\mu = g^{\mu\nu}\gamma_\nu$$

Following ref.[4] (see also [5]) we consider the *vector derivative* or *gradient* defined according to $\partial \equiv \gamma^\mu \partial_\mu$ where ∂_μ is an operator whose action depends on the quantity it acts on. Applying the vector derivative ∂ on a *scalar* field ϕ we have $\partial\phi = \gamma^\mu \partial_\mu \phi$ where $\partial_\mu \phi \equiv (\partial/\partial x^\mu)\phi$ coincides with the partial derivative of ϕ.

But if we apply it on a *vector* field a we have

$$\partial a = \gamma^\mu \partial_\mu (a^\nu \gamma_\nu) = \gamma^\mu (\partial_\mu a^\nu \gamma_\nu + a^\nu \partial_\mu \gamma_\nu) \tag{2.3}$$

In general γ_ν is not constant; it satisfies the relation [4], [5]

$$\partial_\mu \gamma_\nu \; = \; \Gamma_{\mu\nu}^{\;\;\alpha} \; \gamma_\alpha \tag{2.4}$$

where $\Gamma^\alpha_{\mu\nu}$ is the *connection*. Similarly, for $\gamma^\nu = g^{\nu\alpha}\gamma_\alpha$ we have

$$\partial_\mu \gamma^\nu \; = \; \Gamma_\mu{}^\nu{}_\alpha \; \gamma^\alpha \; = \; -\,\Gamma_{\mu\alpha}^{\;\;\nu} \; \gamma^\alpha \tag{2.5}$$

For further references on Clifford algebras see [6], [7].

The *non commuting* operator ∂_μ so defined determines *the parallel transport* of a basis vector γ^ν. Instead of the symbol ∂_μ Hestenes uses $\Box_\mu$, whilst Misner, Thorne and Wheeler, use ∇_μ and call it "covariant derivative". In modern, mathematically oriented literature more explicit notation such as D_{γ_μ} or ∇_{γ_μ} is used. However, such a notation, although mathematically very relevant, would not be very practical in long computations. We find it very convenient to keep

[1] In Appendix A of the Hesteness book [4] the frame $\{\gamma^\mu\}$ is called *dual* frame because the duality operation is used in constructing it.

the symbol ∂_μ for components of the geometric operator $\partial = \gamma^\mu \partial_\mu$. When acting on a scalar field the derivative ∂_μ happens to be commuting and thus behaves as the ordinary partial derivative. When acting on a vector field, ∂_μ is a *non commuting operator*. In this respect, there can be no confusion with partial derivative, because the latter normally acts on *scalar fields*, and in such a case partial derivative and ∂_μ are one and the same thing. However, when acting on a vector field, the derivative ∂_μ is non commuting. Our operator ∂_μ when acting on γ_μ or γ^μ should be distinguished from the ordinary *commuting* partial derivative, let be denoted $\gamma^\nu{}_{,\mu}$, usually used in the literature on the Dirac equation in curved spacetime. The latter derivative is not used in the present paper, so there should be no confusion.

Using (2.4), eq.(2.3) becomes

$$\partial a = \gamma^\mu \gamma_\nu (\partial_\mu a^\nu + \Gamma^\nu_{\mu\alpha} a^\alpha) \equiv \gamma^\mu \gamma_\nu \mathrm{D}_\mu a^\nu = \gamma^\mu \gamma^\nu \mathrm{D}_\mu a_\nu \tag{2.6}$$

where D_μ is the covariant derivative of tensor analysis..

Let us now consider C-space and very briefly review the procedure of [1]. A basis in C-space is given by

$$E_A = \{\gamma, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, ...\} \tag{2.7}$$

where γ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In an r-vector $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ we take the indices so that $\mu_1 < \mu_2 < ... < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X = X^A E_A = s\,\mathbf{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... \tag{2.8}$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, etc., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops, etc., onto the coordinate planes. By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter L_p in the expansion of the poly-vector X we can set set to unity to simplify matters.

In a flat C-space the basis vectors E^A are constants. In a curved C-space this is no longer true. Each E_A is a function of the C-space coordinates

$$X^A = \{s, x^\mu, x^{\mu\nu}, ...\} \tag{2.9}$$

which include scalar, vector, bivector,..., r-vector,..., coordinates. Now we define the connection $\tilde{\Gamma}^C_{AB}$ in C-space according to $\partial_A E_B = \tilde{\Gamma}^C_{AB} E_C$ where $\partial_A \equiv \partial/\partial X^A$ is the derivative in C-space. This definition is analogous to the one in ordinary space. Let us therefore define the C-space curvature as

$$\mathcal{R}_{ABC}{}^D = ([\partial_A, \partial_B] E_C) * E^D \tag{2.10}$$

which is a straightforward generalization of the ordinary relation in Riemannian geometry.

The 'star' means the *scalar product* between two polyvectors A and B, defined as

$$A * B = \langle A\,B \rangle_S \tag{2.11}$$

where 'S' means 'the scalar part' of the geometric product AB.

In [1] we explored the above relation for curvature and showed how it was related to the curvature of the ordinary space. After making several assumptions we were able to demonstrate that the derivative with respect to the bivector coordinate $x^{\mu\nu}$ is equal to the commutator of the derivatives with respect to the vector coordinates x^μ. This will not be the case in this work.

The differential of a C-space basis vector is given by

$$\mathrm{d}E_A = \frac{\partial E_A}{\partial X^B}\,\mathrm{d}X^B = \Gamma^C_{AB}\,E_C\,\mathrm{d}X^B \tag{2.12}$$

In particular, for $A = \mu$ and $E_A = \gamma_\mu$ we have

$$\mathrm{d}\gamma_\mu = \frac{\partial \gamma_\mu}{\partial X^\nu}\mathrm{d}x^\nu + \frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}}\mathrm{d}x^{\alpha\beta} + ... = \tilde{\Gamma}^A_{\nu\mu}E_A\mathrm{d}x^\nu + \tilde{\Gamma}^A_{[\alpha\beta]\mu}E_A\mathrm{d}x^{\alpha\beta} + ... =$$

$$= (\tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu}\gamma_\rho\wedge\gamma_\sigma + ...)\mathrm{d}x^\nu + (\tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu}\gamma_\rho\wedge\gamma_\sigma + ...)\mathrm{d}x^{\alpha\beta} + ... \tag{2.13}$$

We see that the differential $\mathrm{d}\gamma_\mu$ is in general a polyvector, i.e., a Clifford aggregate. In eq-(2.13) we have used

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu}\gamma_\rho\wedge\gamma_\sigma + ... \tag{2.14}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu}\gamma_\rho\wedge\gamma_\sigma + ... \tag{2.15}$$

In this work we will not assume any conditions a priori and we have now that

$$\partial_{\mu\nu} \neq [\partial_\mu, \partial_\nu],\ \ \Gamma_{[\alpha\beta]}{}^\rho{}_\mu \neq R_{\alpha\beta\mu}{}^\rho,\ \ \partial_{\mu_1\mu_2}\, g^{\rho\tau} \neq 0,\ \ \partial_{\mu_1\mu_2}R_{\alpha\beta\mu}{}^\rho \neq 0 \tag{2.16}$$

so the C-space scalar curvature $\mathbf{R}$ does *not* longer decompose as in [1]

$$\mathbf{R} = R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + ... \tag{2.17}$$

but rather it bears a closer relationship to Lanczos-Lovelock gravity as we shall see in the next section. We will derive the C-space connections from the Clifford algebraic structure, and without any a priori assumptions, as follows. In general one must include *all* poly-vector valued indices in the C-space connection which appear in the definition of the derivatives of the basis generators. However when one takes the derivatives of the geometric product of any two basis generators, using the C-space many-beins E_I^A to convert curved-base-space A indices to tangent-space indices I, one is forced to set many of the C-space connection components to zero. One needs to do that in order to have a compatible structure with the geometric product of any two basis generators. Furthermore, a C-space metric compatible connection is such that the covariant derivative of the basis generators is zero.

The use of the C-space beins allows to rewrite the geometric product of curved base-space generators, like $\gamma^\mu\gamma^\nu = e_i^\mu e_j^\nu \gamma^i\gamma^j = g^{\mu\nu} + \gamma^{\mu\nu}$, after using $g^{\mu\nu} = e^\mu_{(i} e^\nu_{j)} \eta^{ij}$ and $\gamma^{\mu\nu} = e^\mu_{[i} e^\nu_{j]} \gamma^{ij}$ so the Clifford algebraic structure is also maintained in the curved-base manifold. In this way one can decompose the C-space beins E_I^A into antisymmetrized sums of products of e_i^μ. For example, $e^{\mu_1\mu_2}_{i_1 i_2} = e^{\mu_1}_{j_1} e^{\mu_2}_{j_2} \delta^{j_1 j_2}_{i_1 i_2}$; $e^{\mu_1\mu_2\mu_3\mu_4}_{i_1 i_2 i_3 i_4} = e^{\mu_1\mu_2}_{j_1 j_2} e^{\mu_3\mu_4}_{j_3 j_4} \delta^{j_1 j_2 j_3 j_4}_{i_1 i_2 i_3 i_4}$; etc...

The Clifford scalar component $\mathbf{s}$ of the poly-vector $\mathbf{X} = X_A E^A$ will be labeled with the index $\mathbf{0}$ from now on and must not be confused with the temporal component of the vector x^μ. Based on what we wrote above, let us then begin by writing

$$\partial_{\mathbf{0}} \{\gamma^\mu,\ \gamma^\nu\} \ = \ 2\ \partial_{\mathbf{0}} g^{\mu\nu} \ \Rightarrow$$
$$\Gamma^\mu_{\mathbf{0}\ \sigma}\ \gamma^\sigma\ \gamma^\nu \ + \ \gamma^\mu\ \Gamma^\nu_{\mathbf{0}\ \sigma}\ \gamma^\sigma \ + \ \Gamma^\nu_{\mathbf{0}\ \sigma}\ \gamma^\sigma\ \gamma^\mu \ + \ \gamma^\nu\ \Gamma^\mu_{\mathbf{0}\ \sigma}\ \gamma^\sigma \ =$$
$$4\ \Gamma^{\mu\nu}_{\mathbf{0}} \ = \ 2\ \partial_{\mathbf{0}} g^{\mu\nu} \ \Rightarrow \ \Gamma^{\mu\nu}_{\mathbf{0}} \ = \ \frac{1}{2}\ \partial_{\mathbf{0}} g^{\mu\nu} \tag{2.18}$$

Eq- (2.18) is obtained after using the relations

$$\gamma^\sigma\ \gamma^\nu \ = \ \frac{1}{2}\ \{\gamma^\sigma, \gamma^\nu\} \ + \ \frac{1}{2}\ [\gamma^\sigma, \gamma^\nu] \ = \ g^{\sigma\nu} \ + \ \gamma^{\sigma\nu} \tag{2.19}$$

with symmetric $g^{\sigma\nu}$, antisymmetric $\gamma^{\sigma\nu} = -\gamma^{\nu\sigma}$ and symmetric $\Gamma^{\mu\nu}_{\mathbf{0}} = \Gamma^{\nu\mu}_{\mathbf{0}}$. Taking derivatives with respect to x^ρ gives

$$\partial_\rho \{\gamma^\mu,\ \gamma^\nu\} \ = \ 2\ \partial_\rho g^{\mu\nu} \ \Rightarrow$$
$$\Gamma^\mu_{\rho\ \sigma}\ \gamma^\sigma\ \gamma^\nu \ + \ \gamma^\mu\ \Gamma^\nu_{\rho\ \sigma}\ \gamma^\sigma \ + \ \Gamma^\nu_{\rho\ \sigma}\ \gamma^\sigma\ \gamma^\mu \ + \ \gamma^\nu\ \Gamma^\mu_{\rho\ \sigma}\ \gamma^\sigma \ =$$
$$4\ \Gamma^{\mu\nu}_{\rho} \ = \ 2\ \partial_\rho g^{\mu\nu} \ \Rightarrow \ \Gamma^{\mu\nu}_{\rho} \ = \ \frac{1}{2}\ \partial_\rho g^{\mu\nu} \tag{2.20}$$

Taking derivatives of the commutator

$$\partial_{\mathbf{0}}[\gamma^\mu,\ \gamma^\nu] \ = \ 2\ \partial_{\mathbf{0}}\gamma^{\mu\nu}, \quad \partial_\rho[\gamma^\mu,\ \gamma^\nu] \ = \ 2\ \partial_\rho\gamma^{\mu\nu} \tag{2.21}$$

gives the following relations

$$\Gamma^\mu_{\mathbf{0}\ \sigma}\ \gamma^{\sigma\nu} \ - \ \Gamma^\nu_{\mathbf{0}\ \sigma}\ \gamma^{\sigma\mu} \ = \ \Gamma^{[\mu\nu]}_{\mathbf{0}\ \tau_1\tau_2}\ \gamma^{\tau_1\tau_2} \tag{2.22}$$

$$\Gamma^\mu_{\rho\ \sigma}\ \gamma^{\sigma\nu} \ - \ \Gamma^\nu_{\rho\ \sigma}\ \gamma^{\sigma\mu} \ = \ \Gamma^{[\mu\nu]}_{\rho\ \tau_1\tau_2}\ \gamma^{\tau_1\tau_2} \tag{2.23}$$

after having

$$\partial_{\mathbf{0}}\gamma^{\mu\nu} \;=\; \Gamma^{[\mu\nu]}_{\mathbf{0}\ \tau_1\tau_2}\ \gamma^{\tau_1\tau_2}, \quad \partial_\rho\gamma^{\mu\nu} \;=\; \Gamma^{[\mu\nu]}_{\rho\ \tau_1\tau_2}\ \gamma^{\tau_1\tau_2} \tag{2.24}$$

From eqs- (2.22,2.23) one obtains, after performing contractions of the form $< \gamma^{ab}\gamma_{cd} >= (\text{constant})\cdot\ \delta^{ab}_{cd}$, the following

$$\Gamma^{\mu}_{\mathbf{0}\ \sigma}\ \delta^{\sigma\nu}_{\rho_1\rho_2} \ -\ \Gamma^{\nu}_{\mathbf{0}\ \sigma}\ \delta^{\sigma\mu}_{\rho_1\rho_2} \;=\; \Gamma^{[\mu\nu]}_{\mathbf{0}\ \tau_1\tau_2}\ \delta^{\tau_1\tau_2}_{\rho_1\rho_2} \;=\; \Gamma^{[\mu\nu]}_{\mathbf{0}\ \rho_1\rho_2} \tag{2.25}$$

$$\Gamma^{\mu}_{\alpha\ \sigma}\ \delta^{\sigma\nu}_{\rho_1\rho_2} \ -\ \Gamma^{\nu}_{\alpha\ \sigma}\ \delta^{\sigma\mu}_{\rho_1\rho_2} \;=\; \Gamma^{[\mu\nu]}_{\alpha\ \tau_1\tau_2}\ \delta^{\tau_1\tau_2}_{\rho_1\rho_2} \;=\; \Gamma^{[\mu\nu]}_{\alpha\ \rho_1\rho_2} \tag{2.26}$$

Hence from eqs-(2.25,2.26) one has an explicit form for $\Gamma^{[\mu\nu]}_{\mathbf{0}\ \rho_1\rho_2}$, $\Gamma^{[\mu\nu]}_{\alpha\ \rho_1\rho_2}$ in terms of

$$\Gamma^{\mu}_{\mathbf{0}\ \sigma} \;=\; \frac{1}{2}\ g_{\sigma\tau}\ \partial_{\mathbf{0}} g^{\mu\tau}, \quad \Gamma^{\mu}_{\alpha\ \sigma} \;=\; \frac{1}{2}\ g_{\sigma\tau}\ \partial_\alpha g^{\mu\tau} \tag{2.27}$$

respectively. From the (anti) commutators

$$[\gamma_{mn}, \gamma^{rs}] \;=\; -\,8\ \delta^{[r}_{[m}\gamma^{s]}_{n]}, \quad \{\gamma_{mn}, \gamma^{rs}\} \;=\; 2\ \gamma_{mn}^{\ \ \ rs} \ -\,4\ \delta^{rs}_{mn} \tag{2.28}$$

by taking derivatives $\partial/\partial x^\rho$ on both sides of the equations one arrives after some algebra, and by lowering indices, to the relations

$$\Gamma_{\rho\ [mn]}^{\quad\ \ [pq]}\ g_{[pq]\ [rs]} + \Gamma_{\rho\ [rs]}^{\quad\ \ [pq]}\ g_{[pq]\ [mn]} \;=\; \partial_\rho\ (g_{[mn]\ [rs]}), \quad g_{[mn]\ [rs]} \;=\; g_{[rs]\ [mn]} \tag{2.29}$$

$$\Gamma_{\rho\ [mn]}^{\quad\ \ [pq]}\ \gamma_{pqrs} \ +\ \Gamma_{\rho\ [rs]}^{\quad\ \ [pq]}\ \gamma_{pqmn} \;=\; \Gamma_{\rho\ [mnrs]}^{\qquad\ \ [abcd]}\ \gamma_{abcd} \tag{2.30}$$

$$\Gamma_{\rho\ [mn]}^{\quad\ \ [pq]}\ \delta^{[r}_{[p}\gamma^{s]}_{q]} \ +\ \Gamma^{[rs]}_{\rho\quad\ [pq]}\ \delta^{[p}_{[m}\gamma^{q]}_{n]} \;=\; \delta^{[r}_{[m}\Gamma^{s]\ \ \ \tau}_{\ \rho\ n]\ \sigma}\gamma^{\sigma}_{\ \tau} \tag{2.31}$$

and by taking derivatives with respect to $\partial/\partial x^{\rho_1\rho_2\cdots\rho_k}$, by lowering indices, one arrives at

$$\Gamma_{[\rho_1\rho_2\ldots\rho_k]\ [mn]}^{\qquad\qquad\ \ [pq]}\ g_{[pq]\ [rs]} + \Gamma_{[\rho_1\rho_2\ldots\rho_k]\ [rs]}^{\qquad\qquad\ \ [pq]}\ g_{[pq]\ [mn]} \;=\; \partial_{\rho_1\rho_2\ldots\rho_k}\ (g_{[mn]\ [rs]}) \tag{2.32}$$

$$\Gamma_{[\rho_1\rho_2\ldots\rho_k]\ [mn]}^{\qquad\qquad\ \ [pq]}\ \gamma_{pqrs} + \Gamma_{[\rho_1\rho_2\ldots\rho_k]\ [rs]}^{\qquad\qquad\ \ [pq]}\ \gamma_{pqmn} \;=\; \Gamma_{[\rho_1\rho_2\ldots\rho_k]\ [mnrs]}^{\qquad\qquad\quad\ [abcd]}\ \gamma_{abcd} \tag{2.33}$$

$$\Gamma_{[\rho_1\ldots\rho_k]\ [mn]}^{\qquad\quad\ \ [pq]}\ \delta^{[r}_{[p}\gamma^{s]}_{q]} \ +\ \Gamma_{\ [\rho_1\ldots\rho_k]\quad [pq]}^{[rs]}\ \delta^{[p}_{[m}\gamma^{q]}_{n]} \;=\; \delta^{[r}_{[m}\Gamma^{s]\ \ \tau}_{\ [\rho_1\ldots\rho_k]\ n]\ \sigma}\gamma^{\sigma}_{\ \tau} \tag{2.34}$$

In this fashion by using the remaining anti (commutators) $\{\gamma^A, \gamma^B\}, [\gamma^A, \gamma^B]$ involving the other Clifford algebra generators (poly-vector basis) , one can *recursively* obtain (define) the C-space connections in terms of derivatives of the C-space metric g_{AB}. One may notice that the expression for the C-space connections do *not* coincide with the Levi-Civita-like connections. Since the algebra is very cumbersome a computer Clifford algebra package is necessary. The commutators $[\Gamma_A, \Gamma_B]$ for $pq = odd$ one has [8]

$$[\ \gamma_{b_1 b_2 \ldots b_p},\ \gamma^{a_1 a_2 \ldots a_q}\ ] \ = \ 2\gamma^{a_1 a_2 \ldots a_q}_{b_1 b_2 \ldots b_p} \ - \\ \frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 \ldots a_q]}_{b_3 \ldots b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 \ldots a_4}_{[b_1 \ldots b_4}\ \gamma^{a_5 \ldots a_q]}_{b_5 \ldots b_p]} \ - \ \ldots \tag{2.35}$$

for $pq = even$ one has

$$[\ \gamma_{b_1 b_2 \ldots b_p},\ \gamma^{a_1 a_2 \ldots a_q}\ ] \ = \ -\frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3 \ldots a_q]}_{b_2 b_3 \ldots b_p]} \ - \\ \frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1 \ldots a_3}_{[b_1 \ldots b_3}\ \gamma^{a_4 \ldots a_q]}_{b_4 \ldots b_p]} \ + \ \ldots \tag{2.36}$$

The anti-commutators for $pq = even$ are

$$\{\ \gamma_{b_1 b_2 \ldots b_p}, \gamma^{a_1 a_2 \ldots a_q}\ \} \ = \ 2\gamma^{a_1 a_2 \ldots a_q}_{b_1 b_2 \ldots b_p} \ - \\ \frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 \ldots a_q]}_{b_3 \ldots b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 \ldots a_4}_{[b_1 \ldots b_4}\ \gamma^{a_5 \ldots a_q]}_{b_5 \ldots b_p]} \ - \ \ldots \tag{2.37}$$

and the anti-commutators for $pq = odd$ are

$$\{\ \gamma_{b_1 b_2 \ldots b_p},\ \gamma^{a_1 a_2 \ldots a_q}\ \} \ = \ -\frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3 \ldots a_q]}_{b_2 b_3 \ldots b_p]} \ - \\ \frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1 \ldots a_3}_{[b_1 \ldots b_3}\ \gamma^{a_4 \ldots a_q]}_{b_4 \ldots b_p]} \ + \ \ldots \tag{2.38}$$

For instance,

$$[\gamma_b, \gamma^a] \ = \ 2\gamma^a_b; \quad [\gamma_{b_1 b_2}, \gamma^{a_1 a_2}] \ = \ -8\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2]}_{b_2]}. \tag{2.39}$$

$$[\gamma_{b_1 b_2 b_3}, \gamma^{a_1 a_2 a_3}] \ = \ 2\ \gamma^{a_1 a_2 a_3}_{b_1 b_2 b_3} \ - 36\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3]}_{b_3]}. \tag{2.40}$$

$$[\gamma_{b_1 b_2 b_3 b_4}, \gamma^{a_1 a_2 a_3 a_4}] \ = \ -32\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3 a_4]}_{b_2 b_3 b_4]} \ + \ 192\ \delta^{[a_1 a_2 a_3}_{[b_1 b_2 b_3}\ \gamma^{a_4]}_{b_4]}. \tag{2.41}$$

etc...

To sum up, the C-space connections must be compatible with the Clifford algebra as shown in the above equations and are determined from the algebraic relations imposed by the Clifford algebra itself. In general, C-space admits torsion [1]. The C-space connections expressions are very different from the Levi-Civita-like connection

$$\{{}^{C}_{AB}\} \;=\; \frac{1}{2}\, g^{CD}\,(\partial_A g_{DB} \;+\; \partial_B g_{AD} \;-\; \partial_D g_{AB}) \qquad (2.42)$$

Furthermore, these results should be contrasted with the very restricted ansatz in [1] where one had that $R_{\mu_1\mu_2}{}^{\rho_1}{}_{\rho_2} = \Gamma_{[\mu_1\mu_2]}{}^{\rho_1}{}_{\rho_2}$ when the metric $g_{\mu\nu}$ and connection solely depended on the x^μ coordinates.

It is not difficult to verify that the metric compatibility conditions $\nabla_A g_{BC} = 0$ are equivalent to having covariantly-constant generators $\nabla_A \gamma^C = \partial_A \gamma^C - \Gamma^C_{AB}\gamma^B = 0$. Secondly, having found the equations that determine all the C-space connection components Γ^C_{AB} and which are consistent with the Clifford algebra, one can realize that in general the connection is *not* symmetric $\Gamma^C_{AB} \neq \Gamma^C_{BA}$ because C-space has *torsion*. Therefore one has a metric compatible connection with *torsion* $T^C_{AB} = \Gamma^C_{AB} - \Gamma^C_{BA}$ in C-space, assuming the anholonomy coefficients f^C_{AB} are zero, $[\partial_A, \partial_B] = f^C_{AB}\partial_C$. If the latter coefficients are not zero one must include f^C_{AB} into the definition of Torsion as follows

$$T^C_{AB} \;=\; \Gamma^C_{AB} \;-\; \Gamma^C_{BA} \;-\; f^C_{AB} \qquad (2.43)$$

In the case of nonsymmetric connections with torsion, the curvatures obey the relations under the exchange of indices

$$\mathbf{R}_{ABCD} \;=\; -\,\mathbf{R}_{BACD},\;\; \mathbf{R}_{ABDC} \;=\; -\,\mathbf{R}_{ABCD},\;\; \textit{but}\;\; \mathbf{R}_{ABCD} \;\neq\; \mathbf{R}_{CDAB} \qquad (47)$$

and are defined, when $f^C_{AB} = 0$, as

$$\mathbf{R}_{ABC}{}^{D} \;=\; \partial_A\,\Gamma_{BC}{}^{D} \;-\; \partial_B\,\Gamma_{AC}{}^{D} \;+\; \Gamma^D_{AE}\,\Gamma^E_{BC} \;-\; \Gamma^D_{BE}\,\Gamma^E_{AC} \qquad (2.44)$$

If $f^C_{AB} \neq 0$ one must also include these anholonomy coefficients into the definition of curvature (2.44) by adding terms of the form $-f^E_{AB}\Gamma^D_{EC}$.

The C-space connections are determined in terms of the C-space metric g_{AB} by the procedure described above. Some examples of the C-space curvatures are

$$\mathbf{R}_{[\mu_1\mu_2\ldots\mu_{2n}]\;\mathbf{0}\;[\rho_1\rho_2\ldots\rho_{2n}]}{}^{\mathbf{0}} \;=\; \partial_{\mu_1\mu_2\ldots\mu_{2n}}\,\Gamma^{\mathbf{0}}{}_{\mathbf{0}\;[\rho_1\rho_2\ldots\rho_{2n}]} - \partial_{\mathbf{0}}\,\Gamma^{\mathbf{0}}_{[\mu_1\mu_2\ldots\mu_{2n}]\;[\rho_1\rho_2\ldots\rho_{2n}]} +$$

$$\Gamma^{\mathbf{0}}{}_{[\mu_1\mu_2\ldots\ldots\mu_{2n}]\;A}\;\Gamma^{A}_{\mathbf{0}\;[\rho_1\rho_2\ldots\rho_{2n}]} \;-\; \Gamma^{\mathbf{0}}_{\mathbf{0}\;A}\;\Gamma^{A}_{[\mu_1\mu_2\ldots\mu_{2n}]\;[\rho_1\rho_2\ldots\rho_{2n}]} \qquad (2.45)$$

The standard Riemann-Cartan curvature tensor in ordinary spacetime is *contained* in C-space as follows

$$\mathcal{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} \;=\; \partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} \;-\; \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} \;+\; \Gamma^{\rho_2}_{\mu_1\sigma}\,\Gamma^{\sigma}_{\mu_2\rho_1} \;-\; \Gamma^{\rho_2}_{\mu_2\sigma}\,\Gamma^{\sigma}_{\mu_1\rho_1} \;\subset$$

$$\mathbf{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} \;=\partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} \;-\; \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} \;+\; \Gamma^{\rho_2}_{\mu_1\;\mathbf{A}}\,\Gamma^{\mathbf{A}}_{\mu_2\rho_1} \;-\; \Gamma^{\rho_2}_{\mu_2\;\mathbf{A}}\,\Gamma^{\mathbf{A}}_{\mu_1\rho_1} \qquad (2.46)$$

due to the contractions involving the poly-vector valued indices $\mathbf{A}$ in eq-(2.46). There is also the crucial difference that $\mathbf{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2}(s, x^\nu, x^{\nu_1\nu_2}, ...)$ has now an *additional* dependence on all the C-space poly-vector valued coordinates $s, x^{\nu_1\nu_2}, x^{\nu_1\nu_2\nu_3}, ...$ besides the x^ν coordinates.

The mixed-grade C-space metric components are *not* zero in general, there are very special cases when

$$g_{\mathbf{0}\ [\nu_1\nu_2...\rho_i]} = 0; \quad g_{[\mu_1\mu_2...\mu_i]\ [\nu_1\nu_2...\nu_j]} = 0, \ when\ i \neq j \tag{2.47}$$

occurs, but in general the mixed-grade metric components are not zero and must be included. The same-grade C-space metric components obeying $g_{AB} = g_{BA}$ are of the form

$$g_{\mathbf{00}}, \quad g_{\mu\nu}, \quad g_{\mu_1\mu_2\ \nu_1\nu_2}, \quad \ldots, \quad g_{\mu_1\mu_2...\mu_D\ \nu_1\nu_2...\nu_D} \tag{2.48}$$

In the most general case the metric *does not factorize* into antisymmetrized sums of products of the form

$$g_{[\mu_1\mu_2]\ [\nu_1\nu_2]}(x^\mu) \neq g_{\mu_1\nu_1}(x^\mu)\ g_{\mu_2\nu_2}(x^\mu) - g_{\mu_2\nu_1}(x^\mu)\ g_{\mu_1\nu_2}(x^\mu) \tag{2.49a}$$

$$g_{[\mu_1\mu_2...\mu_k]\ [\nu_1\nu_2...\nu_k]}(x^\mu) \neq det\, G_{\mu_i\nu_j} = \epsilon^{j_1 j_2 ... j_k}\ g_{\mu_1\nu_{j_1}}\ g_{\mu_2\nu_{j_2}}\ \ldots\ g_{\mu_2\nu_{j_k}}, \ k = 1, 2, 3, \ldots D \tag{2.49b}$$

The determinant of $G_{\mu_i\nu_j}$ can be written as

$$det \begin{pmatrix} g_{\mu_1\nu_1}(x^\mu) & \ldots & \ldots\ g_{\mu_1\nu_k}(x^\mu) \\ g_{\mu_2\nu_1}(x^\mu) & \ldots & \ldots\ g_{\mu_2\nu_k}(x^\mu) \\ ------------ & & --------------- \\ g_{\mu_k\nu_1}(x^\mu) & \ldots & \ldots\ g_{\mu_k\nu_k}(x^\mu) \end{pmatrix}, \tag{2.50}$$

The metric component $g_{\mathbf{00}}$ involving the scalar "directions" in C-space of the Clifford poly-vectors must also be included. It behaves like a Clifford scalar. The other component $g_{[\mu_1\mu_2...\mu_D]\ [\nu_1\nu_2...\nu_D]}$ involves the pseudo-scalar "directions". The latter scalar and pseudo-scalars might bear some connection to the dilaton and axion fields in Cosmology and particle physics.

The curvature in the presence of torsion does not satisfy the same symmetry relations when there is no torsion, therefore the Ricci-like tensor is no longer symmetric

$$\mathbf{R}_{ABC}{}^{B} = \mathbf{R}_{AC}, \quad \mathbf{R}_{AC} \neq \mathbf{R}_{CA}, \quad \mathbf{R} = g^{AC}\, \mathbf{R}_{AC} \tag{2.51}$$

For ordinary vector-valued indices one has

$$R_{abcd} = \hat{R}_{abcd} - \frac{1}{2}\left(\nabla_c T_{abd} - \nabla_d T_{abc}\right) + \frac{1}{4}\left(T_{aec}T^e_{bd} - T_{aed}T^e_{bc}\right) \tag{2.52}$$

$$R_{ab} = R_{(ab)} + R_{[ab]} = \hat{R}_{ab} + \frac{1}{2}\nabla_c T^c_{ab} - \frac{1}{4}T^d_{ca}\, T^c_{bd} \tag{2.53a}$$

$$R \;=\; \hat{R} \;-\; \frac{1}{4}\; T_{abc}\; T^{abc} \tag{2.53b}$$

where the *hatted* quantities correspond to ordinary curvatures in absence of torsion. The modified Bianchi identities include nonvanishing torsion terms in the right hand side. For ordinary vector-valued indices one has

$$R^a_{[bcd]} \;=\; \nabla_{[b} T^a_{cd]} \;+\; T^a_{m[b} T^m_{cd]} \tag{2.54a}$$

$$\nabla_{[a} R^{mn}_{bc]} \;=\; T^p_{[ab} R^{mn}_{c]p} \tag{2.54b}$$

An Einstein-Hilbert-Cartan action $S = \frac{1}{2\kappa^2}\int d^n x \sqrt{g} R$ plus matter action S_m leads to the modified Einstein equations [10]

$$\hat{R}_{ab} \;-\; \frac{1}{2} g_{ab}\; \hat{R} \;+\; \frac{1}{2}\; T^{cd}_{(a}\; T_{b)cd} \;-\; \frac{3}{8}\; g_{ab}\; T_{cde}\; T^{cde} \;=\; \kappa^2\; \mathcal{T}_{ab} \tag{2.55}$$

plus the spin energy density tensor which on-shell is given in terms of the torsion by

$$\mathcal{S}^{bc}_a \;=\; \frac{1}{\sqrt{g}}\; \frac{\delta S_m}{\delta T^a_{bc}} \;=\; \frac{1}{2}\; T^{bc}_a \tag{2.56}$$

One may write the C-space *analog* of the Einstein-Cartan's equations with a cosmological constant as

$$\hat{\mathbf{R}}_{AB} \;-\; \frac{1}{2}\; g_{AB}\; \hat{\mathbf{R}} \;+\; \Lambda\; g_{AB} \;+\; Torsion\; Terms \;=\; \mathbf{T}_{AB} \tag{2.57}$$

Below we shall derive the more complicated and different field equations from a variational principle. One may notice that nonsymmetric contributions to the stress energy tensor are possible if one has nonsymmetric metric components. Matter in C-space includes, besides ordinary bosonic and fermionic fields, spinor-valued antisymmetric tensor fields $\Psi^{\mu_1\mu_2\ldots\mu_n}_\alpha$ that contribute to the stress energy tensor $\mathbf{T}_{AB}$.

The Torsion terms in which appear in the field equations in C-space are given, up to numerical coefficients c_1, c_2 , by

$$c_1 (T^{CD}_A T_{DBC} \;+\; T^{CD}_B T_{DAC}) \;+ c_2\; g_{AB}\; T_{CDE}\; T^{CDE} \tag{2.58}$$

One could add Holst-like terms [10] to the action if one wishes, but for the moment we shall refrain from doing so. The C-space Ricci-like tensor is

$$\mathbf{R}_A^{\;B} \;=\; \sum_{j=1}^{D} \mathbf{R}_{A\;\;[\nu_1\nu_2\ldots\nu_j]}^{\;B\;[\nu_1\nu_2\ldots\nu_j]} \;+\; \mathbf{R}_{A\;\mathbf{0}}^{\;B\;\mathbf{0}} \tag{2.59}$$

and the C-space curvature scalar is

$$\mathbf{R} \;=\; \sum_{j=1}^{D}\sum_{k=1}^{D} \mathbf{R}_{[\mu_1\mu_2\ldots\mu_j]\;[\nu_1\nu_2\ldots\nu_k]}^{\qquad\qquad\qquad\;[\mu_1\mu_2\ldots\mu_j]\;[\nu_1\nu_2\ldots\nu_k]} + \sum_{j=1}^{D} \mathbf{R}_{[\mu_1\mu_2\ldots\mu_j]\;\mathbf{0}}^{\qquad\qquad\;[\mu_1\mu_2\ldots\mu_j]\;\mathbf{0}} \tag{2.60}$$

One may construct an Einstein-Hilbert-Cartan like action based on the C-space curvature scalar. This requires the use of hyper-determinants. The hyper-determinant of a hyper-matrix [15] can be recast in terms of discriminants [16]. In this fashion one can define the hyper-determinant of g_{AB} as products of the hyper-determinants corresponding to the hyper-matrices [2]

$$g_{[\mu_1\mu_2]\ [\nu_1\nu_2]},\ \ \ldots,\ g_{[\mu_1\mu_2\ldots\mu_k]\ [\nu_1\nu_2\ldots\nu_k]},\ \ for\ \ 1 < k < D \tag{2.61}$$

and construct a suitable measure of integration $\mu_{\mathbf{m}}(s, x^\mu, x^{\mu_1\mu_2}, \ldots, x^{\mu_1\mu_2\ldots\mu_D})$ in C-space which, in turn, would allow us to build the C-space version of the Einstein-Hilbert-Cartan action with a cosmological constant

$$\frac{1}{2\kappa^2}\int ds \prod dx^\mu \prod dx^{\mu_1\mu_2}\ \ldots\ dx^{\mu_1\mu_2\ldots\mu_D}\ \mu_{\mathbf{m}}(s, x^\mu, x^{\mu_1\mu_2}, \ldots)\ (\mathbf{R} - 2\Lambda) \tag{2.62}$$

κ^2 is the C-space gravitational coupling constant. In ordinary gravity it is set to $8\pi G_N$, with G_N being the Newtonian coupling constant.

The measure must obey the relation

$$[\mathbf{DX}]\ \mu_{\mathbf{m}}(\mathbf{X})\ =\ [\mathbf{DX}']\ \mu'_{\mathbf{m}}(\mathbf{X}') \tag{2.63}$$

under poly-vector valued coordinate transformations in C-space. The C-space metric transforms as

$$g'_{CD}\ =\ g_{AB}\ \frac{\partial X^A}{\partial X'^C}\ \frac{\partial X^B}{\partial X'^D} \tag{2.64}$$

but now one has that

$$\sqrt{hdet\ g'}\ \neq\ \sqrt{hdet\ g}\ hdet\left(\frac{\partial X^A}{\partial X'^B}\right) \tag{2.65}$$

due to the multiplicative "anomaly" of the product of hyper-determinants. So the measure μ_m does not coincide with the square root of the hyper-determinant. It is a more complicated function of the hyper-determinant of g_{AB} and obeying eq-(2.63). [3] One could write $hdet(X) = Z_A hdet(X) hdet(Y)$, where $Z_A \neq 1$ is the multiplicative anomaly and in this fashion rewrite eq-(2.63) leading to an *implicit* definition of the measure $\mu_m(hdet g_{AB})$.

The ordinary determinant $g = det(g_{\mu\nu})$ obeys

$$\delta\sqrt{-g} = -\ \frac{1}{2}\ \sqrt{-g}\ g^{\mu\nu}\ \delta g_{\mu\nu} \tag{2.66}$$

[2] The hyper-determinant of a product of two hyper-matrices is *not* equal to the product of their hyper-determinants. However, one is not multiplying two hyper-matrices but decomposing the hyper-matrix g_{AB} into its different blocks.

[3] There is no known generalization of the Binet-Cauchy formula $\det(AB) = \det(A)\det(B)$ for 2 arbitrary hypermatrices. However, in the case of particular types of hypermatrices, some results are known. Let X, Y be two hypermatrices. Suppose that Y is a $n \times n$ matrix. Then, a well-defined hypermatrix product XY is defined in such a way that the hyperdeterminant satisfies the rule $hdet(X \cdot Y) = hdet(X) hdet(Y)^{N/n}$. There, n is the degree of the hyperdeterminant and N is a number related to the format of the hypermatrix X.

which was fundamental in the derivation of Einstein equations from a variation of the Einstein-Hilbert action. However, when hyper-determinants of the C-space metric g_{AB} are involved it is *no* longer true that the relation (2.66) holds anymore in order to recover the C-space gravity equations (2.57) in the presence of torsion and a cosmological constant.

Using the relation $\delta \mathbf{R}_{AB} = \nabla_C \delta \Gamma^C_{AB} - \nabla_B \delta \Gamma^C_{CA}$, a variation of the action

$$\frac{1}{2\kappa^2} \int ds \prod dx^\mu \prod dx^{\mu_1 \mu_2} \ \ldots \ dx^{\mu_1 \mu_2 \ldots \mu_D} \ \mu_m(|hdet \ g_{AB}|) \ (\mathbf{R} - 2\Lambda) + S_{matter} \tag{2.66}$$

with respect to the C-space metric g_{AB} yields the correct C-space field equations

$$\mathbf{R}_{(AB)} \ + \ (\mathbf{R} - 2\Lambda) \ \frac{\delta ln(\mu_m(|hdet \ g_{AB}|))}{\delta g^{AB}} \ = \ \kappa^2 \ \mathbf{T}_{AB} \tag{2.67}$$

If, and only if,

$$\frac{\delta ln(\mu_m(|hdet \ g_{AB}|))}{\delta g^{AB}} \ = \ - \ \frac{1}{2} \ g_{AB} \tag{2.68}$$

then the field equations (2.57) would coincide with the C-space field equations (2.67) obtained from a variational principle. One should note that the field equations (2.67) *contain torsion* since $\mathbf{R}_{(AB)}, \mathbf{R}$ are defined in terms of the nonsymmetric connection $\Gamma^C_{AB} \neq \Gamma^C_{BA}$. Eqs-(2.67) are the correct C-space field equations one should use in general. Nevertheless, for practical purposes, we shall use the field equations (2.57) in the next section.

The hyper-determinant of the C-space metric g_{AB} (a hyper matrix) involving all the components of the same and different grade is defined as

$$g_{\mathbf{oo}} \ det(g_{\mu\nu}) \ hdet(g_{\mu_1\mu_2 \ \nu_1\nu_2}) \ hdet(g_{\mu \ \nu_1\nu_2}) \ \ldots \ hdet(g_{\mu_1 \ldots \mu_{D-1} \ \nu_1 \ldots \nu_{D-1}}) \ g_{\mu_1 \ldots \mu_D \ \nu_1 \ldots \nu_D} \tag{2.69}$$

where the hyper-determinant of $g_{\mu\nu}$ coincides with the ordinary determinant of $g_{\mu\nu}$. Notice once more that the hyper-determinant of a product of two hyper-matrices is *not* equal to the product of their hyper-determinants. However, in (2.69) one is not multiplying two hyper-matrices g_{AB}, g'_{AB}, but decomposing the hyper-matrix g_{AB} into different blocks.

To see how the components of g_{AB} can be realized as hyper-matrices one may choose for example the bivector-bivector metric entries $g_{12 \ 34} = g_{34 \ 12}$ such that these components are constrained to obey $g_{21 \ 34} = -g_{12 \ 34} = g_{12 \ 43}$. And $g_{11 \ 34} = g_{22 \ 34} = \ldots g_{DD \ 34} = 0$. In this fashion one can realize $g_{\mu_1\mu_2 \ \nu_1\nu_2}$ as the entries of a hyper-matrix h_{ijkl}. One may choose for example the vector-bivector metric entries $g_{1 \ 34} = g_{34 \ 1}$ such that $g_{1 \ 34} = -g_{1 \ 43}$. And $g_{1 \ 11} = g_{1 \ 22} = \ldots = g_{1 \ DD} = 0$. In this fashion one can realize $g_{\mu \ \nu_1\nu_2}$ as the entries of a hyper-matrix h_{ijk}, etc... Hence, a variation of the action (2.62) with respect to g_{AB} leads to a complicated expression (2.67) that does not necessarily coincide with the field equations (2.57). We are assuming also that the hyper-determinant exists and is non-vanishing.

Another alternative is the following. In $D = 4$, for example, one could replace the hyper-determinant of the hyper-matrix g_{AB} for the determinant of a 16×16 square matrix associated with the entries of the following square and rectangular matrices : in $D = 4$ there are 6×6 independent metric components involving the bi-vector indices $g_{\mu_1\mu_2\ \nu_1\nu_2}$. Hence one has a one-to-one correspondence of the entries of $g_{\mu_1\mu_2\ \nu_1\nu_2}$ with the entries of a 6×6 square matrix. There are 4×4 metric components involving tri-vector indices $g_{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}$, and consequently there is a 4×4 matrix associated with the latter hyper-matrix. There is one component $g_{\mu_1\mu_2\mu_3\mu_4\ \nu_1\nu_2\nu_3\nu_4}$, in addition to g_{00} and the 4×4 components of $g_{\mu\nu}$.

One must not forget also the mixed-grade components of g_{AB} that are associated with rectangular matrices, and such that the total number of hyper-matrix entries associated with all the square and rectangular matrices in $D = 4$ is then

$$16 \times 16 \ = \ (1+4+6+4+1)\ (1+4+6+4+1) \tag{2.70}$$

In D dimensions a Clifford algebra has 2^D generators, so this procedure leads to a square matrix of $2^D \times 2^D$ components. In this fashion one could trade the hyper-determinant of g_{AB} for the determinant of its associated $2^D \times 2^D$ square matrix. and which, in turn, will permit us to use the relation (2.66) in the variation of the action (2.62) leading to the *associated* 2^D-dim gravitational field equations

$$R_{\alpha\beta} \ - \frac{1}{2}\ g_{\alpha\beta}\ R \ + \ \Lambda g_{\alpha\beta} \ = \ T_{\alpha\beta}, \quad \alpha,\beta = 1,2,3,\dots\ ,2^D \tag{2.71}$$

However the associated 2^D-dim theory is physically very distinct from the C-space gravitational theory.

Concluding, the use of hyper-determinants is required to construct the analog of the Einstein-Hilbert-Cartan action in C-spaces. A variation of the action in C-space leads to the generalized field equations (2.67) (with torsion) that do *not* necessarily coincide with the field equations (2.57). In ordinary Relativity, without torsion, one can construct the Einstein tensor by performing two successive contractions of the differential Bianchi identity. It also leads to the conservation of the stress energy tensor in the right hand side. Presumably this procedure based on the *modified* Bianchi identities could apply also to C-space leading to the field equations (2.67) which *contain* torsion since $\mathbf{R}_{(AB)}, \mathbf{R}$ are defined in terms of the nonsymmetric connection $\Gamma^C_{AB} \neq \Gamma^C_{BA}$. An immediate question arises, does the Palatini formalism work also in C-spaces ? Namely, does a variation with respect to the C-space connection $(\delta S/\delta\Gamma^C_{AB}) = 0$ yield the *same* connections as those obtained from the mere structure of the Clifford algebra and depicted above in this section? We leave this difficult question for future work.

3 On C-space and Lanczos-Lovelock-Cartan Gravity

The n-th order Lanczos-Lovelock-Cartan curvature tensor is defined as

$$\mathcal{R}^{(n)}{}_{\mu_1\mu_2\ldots\mu_{2n}}^{\rho_1\rho_2\ldots\rho_{2n}} = \delta^{\rho_1\rho_2\ldots\rho_{2n}}_{\tau_1\tau_2\ldots\tau_{2n}} \delta^{\nu_1\nu_2\ldots\nu_{2n}}_{\mu_1\mu_2\ldots\mu_{2n}} \mathcal{R}_{\nu_1\nu_2}{}^{\tau_1\tau_2} \mathcal{R}_{\nu_3\nu_4}{}^{\tau_3\tau_4} \ldots \mathcal{R}_{\nu_{2n-1}\nu_{2n}}{}^{\tau_{2n-1}\tau_{2n}} \tag{3.1}$$

the n-th order Lovelock curvature scalar is

$$\mathcal{R}^{(n)} = \delta^{\nu_1\nu_2\ldots\nu_{2n}}_{\tau_1\tau_2\ldots\tau_{2n}} \mathcal{R}_{\nu_1\nu_2}{}^{\tau_1\tau_2} \mathcal{R}_{\nu_3\nu_4}{}^{\tau_3\tau_4} \ldots . \mathcal{R}_{\nu_{2n-1}\nu_{2n}}{}^{\tau_{2n-1}\tau_{2n}} \tag{3.2}$$

the above curvature tensors are antisymmetric under the exchange of any of the μ (ρ) indices and obey The Lanczos-Lovelock-Cartan Lagrangian density is

$$\mathcal{L} = \sqrt{g} \sum_{n=0}^{[\frac{D}{2}]} c_n \mathcal{L}_n, \quad \mathcal{L}_n = \frac{1}{2^n} \mathcal{R}^{(n)} \tag{3.3}$$

where c_n are arbitrary coefficients; the first term corresponds to the cosmological constant. The integer part is $[\frac{D}{2}] = \frac{D}{2}$ when $D = even$, and $\frac{D-1}{2}$ when $D = odd$. The general Lanczos-Lovelock theory in D spacetime dimensions is given by the action

$$S = \int d^D x \sqrt{|g|} \sum_{n=0}^{[\frac{D}{2}]} c_n \mathcal{L}_n, \tag{3.4}$$

One of the key properties of Lanczos-Lovelock-Cartan gravity is that the field equations do not contain higher derivatives of the metric tensor beyond the second order due to the fact that the action does not contain derivatives of the curvature, see [9], [12] and references therein.

In this section we will explore the relationship of Lanczos-Lovelock-Cartan (LLC) gravity to gravity in C-spaces in the very special case that one takes a *slice* in C-space by setting all the poly-vector coordinates to *zero* except the ordinary coordinates x^μ.

To *simplify* matters, let us take for the C-space version of the Einstein-Cartan's equations with a cosmological constant in the *vacuum* case those equations given in (2.57), instead of eqs-(2.67) derived from a variational principle of the action. So let us have

$$\hat{\mathbf{R}}_{AB} - \frac{1}{2} g_{AB} \hat{\mathbf{R}} + \Lambda g_{AB} + Torsion\ Terms = 0, \tag{3.5}$$

after evaluating the C-space curvature tensors using the connections Γ^C_{AB} with torsion. As mentioned earlier, the *hatted* quantities correspond to curvatures without torsion. The C-space curvature scalar is given by the sum of the contractions as shown in (2.60), and the C-space Ricci-like curvature is given in (2.59).

In the vacuum case $T_{AB} = 0$, the C-space version of the vacuum Einstein-Cartan's equations (66) determine the C-space metric g_{AB} when the C-space connections are given in terms of derivatives of g_{AB} as shown in section **1**.

A simple ansatz relating the LLC higher curvatures to C-space curvatures is based on the following contractions

$$\mathcal{R}^{(n)}{}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}} \sim \sum_{k=1}^{D} \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}}{}_{\rho_1\rho_2\ldots\rho_k}{}^{\rho_1\rho_2\ldots\rho_k} + \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}}{}_{\mathbf{0}}{}^{\mathbf{0}} \tag{3.6}$$

Even simpler, one may still propose for an ansatz the following

$$\mathcal{R}^{(n)}{}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}} \sim \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}}{}_{\mathbf{0}}{}^{\mathbf{0}} \tag{3.7}$$

where one must take a *slice* in C-space which requires to evaluate all the terms in the right hand side of eqs-(3.6,3.7) at the "points" $s = x^{\mu_1\mu_2} = \ldots = x^{\mu_1\mu_2\ldots\mu_D} = 0$, for all x^μ, since the left hand side of eqs-(3.6,3.7) solely depends on the vector coordinates x^μ.

Another possibility besides proposing the ansatz (3.6, 3.7) is to embed the LLC gravity equations into the C-space ones provided by eqs- (3.5). One may write the Lanczos-Lovelock-Cartan gravitational equations in the form [9], [12]

$$\sum_{n=0}^{[\frac{D}{2}]} c_n \left(n \, \hat{\mathcal{R}}^{(n)}_{\rho\sigma} - \frac{1}{2} g_{\mu\nu} \, \hat{\mathcal{R}}^{(n)} \right) + Torsion\,Terms = 0 \tag{3.8}$$

and which are more suitable to compare with the C-space gravity equations (3.8) . The embedding of the Lanczos-Lovelock-Cartan gravitational equations into the C-space gravitational equations requires

$$\mathcal{G}_{\rho\sigma} + Torsion\,Terms = 0 \leftrightarrow \hat{\mathbf{R}}_{\rho_\sigma} - \frac{1}{2} g_{\rho_\sigma} \, \hat{\mathbf{R}} + \Lambda \, g_{\rho_\sigma} + Torsion\,Terms = 0, \tag{3.9}$$

where the C-space Ricci-like curvature $\mathbf{R}_\sigma{}^\rho$ is

$$\mathbf{R}_\sigma{}^\rho = \sum_{j=1}^{D} \mathbf{R}_\sigma{}^\rho{}_{[\mu_1\mu_2\ldots\mu_j]}{}^{[\mu_1\mu_2\ldots\mu_j]} + \mathbf{R}_{\sigma\,\mathbf{0}}{}^{\rho\,\mathbf{0}} \tag{3.10}$$

and the Ricci-like scalar is given by eq-(2.60) .

The latter equations are just members of the more general C-space field equations given by eq-(3.5) involving all the poly-vector valued indices. We should emphasize that in order to match units one has to include suitable powers of the Planck length scale L_P in the summands in all of our equations. By recurring to eqs-(3.10) one finds that we can embed the Lanczos-Lovelock-Cartan gravitational equations (3.8) into the C-space gravity equations (3.5) if the following conditions on the C-space curvatures are satisfied

$$\sum_{n=1}^{[\frac{D}{2}]} c_n \; n \; \delta^\rho_{[\nu_1} \delta^{\mu_1\mu_2\ldots\mu_{2n}}_{\sigma\nu_2\nu_3\ldots\nu_{2n}]} \; \hat{\mathcal{R}}^{(n)}{}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}} =$$

$$\sum_{j=1}^{D} \hat{\mathbf{R}}_{\sigma}^{\rho} {}_{\mu_1\mu_2\ldots\mu_j}^{\mu_1\mu_2\ldots\mu_j} \; + \; \hat{\mathbf{R}}_{\mathbf{0}}{}^{\mathbf{0}}{}_{\sigma}{}^{\rho} \tag{3.11}$$

and

$$\sum_{n=1}^{[\frac{D}{2}]} c_n \; \delta^{\mu_1\mu_2\ldots\mu_{2n}}_{\nu_1\nu_2\ldots\nu_{2n}} \; \hat{\mathcal{R}}^{(n)}_{\mu_1\mu_2\ldots\mu_{2n}}{}^{\nu_1\nu_2\ldots\nu_{2n}} \; =$$

$$\sum_{j=1}^{D}\sum_{k=1}^{D} \; \hat{\mathbf{R}}_{\mu_1\mu_2\ldots\mu_j}{}^{\mu_1\mu_2\ldots\mu_j}{}_{\nu_1\nu_2\ldots\nu_k}{}^{\nu_1\nu_2\ldots\nu_k} \; + \; \sum_{j=1}^{D} \; \hat{\mathbf{R}}_{\mu_1\mu_2\ldots\mu_j}{}^{\mu_1\mu_2\ldots\mu_j}{}_{\mathbf{0}}{}^{\mathbf{0}} \tag{3.12}$$

where the *slice* in C-space requires that we evaluate all the terms in the right hand side of eqs-(3.11,3.12) at the "points" $s = x^{\mu_1\mu_2} = \ldots = x^{\mu_1\mu_2\ldots\mu_D} = 0$, for all x^μ, since the left hand side of eqs-(3.11,3.12) solely depends on x^μ. If one were to impose the condition that the C-space metric g_{AB} depends solely on x^μ this leads to very *restrictive* equations to be satisfied and which most likely lead to trivial (flat) solutions.

One should notice the key factors of nc_n in eq-(3.11) compared with the c_n factors in eq-(3.12). The $n = 0$ term in (3.11) corresponds to the cosmological constant

$$c_o \; \hat{\mathcal{R}}^{(0)} \; = \; -2\Lambda \; \Rightarrow \; -\frac{1}{2} \; c_o \; \hat{\mathcal{R}}^{(0)} \; = \; \Lambda \tag{3.13}$$

An important remark is in order. Since the C-space connection does not coincide with the Levi-Civita connection plus contorsion, one should use the appropriate connection $\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g_{\rho\nu}\partial_\mu g^{\rho\sigma}$ in the Lanczos-Lovelock-Cartan (LLC) gravity equations. However, it is still possible to use the Levi-Civita connection, with the contorsion tensor, $\Gamma^\sigma_{\mu\nu} = \{^{\sigma}_{\mu\nu}\} + K^\sigma_{\mu\nu}$ for LLC gravity and the C-space connections for C-space gravity. A bi-connection formulation of gravity based on an independent variation of *two* different connections can be found in [17], [18]. It was shown that a variation of the modified gravitational action with respect to the above independent degrees of freedom leads to the usual Einstein field equations.

The above embedding conditions (3.11,3.12) can be simplified considerable if one has the following vanishing traces

$$\mathbf{R}_{\nu_1\nu_2\ldots\nu_{2n-1}}{}^{\nu_1\nu_2\ldots\nu_{2n-1}}{}_{\mathbf{0}}{}^{\mathbf{0}} \; = \; 0, \quad \mathbf{R}_{\nu_1\nu_2\ldots\nu_{2n-1}}{}^{\nu_1\nu_2\ldots\nu_{2n-1}}{}_{\rho_1\rho_2\ldots\rho_k}{}^{\rho_1\rho_2\ldots\rho_k} \; = \; 0 \tag{3.14a}$$

$$\mathbf{R}_{\nu_1\nu_2\ldots\nu_{2n-1}}{}^{\nu_1\nu_2\ldots\nu_{2n-1}}{}_{\sigma}{}^{\rho} \; = \; 0 \tag{3.14b}$$

However the introduction of these vanishing traces will lead to an over-determined system of equations, in conjunction with the C-space field equations. As it happens with an over-determined system of equations one is hard pressed to find nontrivial solutions $R_{ABCD} \neq 0$. For this reason we shall refrain from introducing additional equations like (75).

Vanishing Torsion case

If one could choose the anholonomy coefficients f^C_{AB} in C-space such that

$$T^C_{AB} \ = \ \Gamma^C_{AB} \ - \ \Gamma^C_{BA} \ - \ f^C_{AB} \ = \ 0 \tag{3.15}$$

one would end up with a *torsionless* and metric compatible connection which would simplify matters. In this case, one must include the anholonomy coefficients into the definition of the curvature by adding the extra terms of the form $-f^E_{AB}\Gamma^D_{EC}$. In principle, a non-coordinate (non-holonomic) basis in C-space could be introduced such that it renders a zero torsion $T^C_{AB} = \Gamma^C_{AB} - \Gamma^C_{BA} - f^C_{AB} = 0$, when the Γ^C_{AB} components are constrained by the Clifford algebra itself, and by the metric compatibility condition $\nabla_A g_{BC} = 0$, whereas for the construction of the Riemmanian and LL curvature tensors in ordinary spacetimes, one must use now the torsionless Levi-Civita connection adapted to the non-holonomic (non-coordinate) basis and which requires *adding* the extra terms

$$g^{CD}(f_{ABD} + f_{BAD} - f_{DAB})$$

to the usual definition of the Levi-Civita connection.

Constant curvature vacuum solutions are much easier to study because after using the anti-symmetrized Kronecker deltas in eqs-(3.11, 3.12) one is no longer required to take a slice in C-space by evaluating the curvatures at $s = x^{\mu_1\mu_2} = \ldots = 0, \forall x^\mu$. Rather than embedding the LL gravity equations into the C-space gravity ones, the more restricted ansatz in eq-(3.7) leads to the n equations

$$\mathcal{R}^{(n)\ \nu_1\nu_2\ldots\nu_{2n}}_{\mu_1\mu_2\ldots\mu_{2n}} \ \sim \ \mathbf{R}_{\mu_1\mu_2\ldots\mu_{2n}\ \mathbf{0}}^{\nu_1\nu_2\ldots\nu_{2n}\ \mathbf{0}} \sim (\frac{2\Lambda}{(D-1)(D-2)})^n \ \delta^{\nu_1\nu_2\ldots\nu_{2n}}_{\mu_1\mu_2\ldots\mu_{2n}} \tag{3.16}$$

for all $n = 1, 2, \ldots, [D/2]$. Eq-(3.16) clearly represents and interprets the n LL curvature tensors as Ricci-like traces of certain components of the C-space curvatures. This, in a nutshell, depicts the correspondence between LL higher curvature gravity and gravity in C-spaces.

One may begin by solving the Lanczos-Lovelock equations, in the absence of torsion, which determine the ordinary metric components $g_{\mu\nu}(x^\rho)$, the connection $\Gamma^\rho_{\mu\nu}(x^\mu)$ and the LL curvature tensor $\mathcal{R}^{(n)\ \rho_1\rho_2\ldots\rho_{2n}}_{\mu_1\mu_2\ldots\mu_{2n}}(x^\mu)$. Let us look for the maximally symmetric constant curvature vacua solutions to Lovelock gravity, like de Sitter and Anti de Sitter spaces in ordinary Einstein gravity. The Riemann tensor in the latter case is

$$R_{\mu_1\mu_2\ \rho_1\rho_2} \ = \ \frac{2\Lambda}{(D-1)(D-2)} \ (g_{\mu_1\rho_1}(x^\mu) \ g_{\mu_2\rho_2}(x^\mu) \ - \ g_{\mu_2\rho_1}(x^\mu) \ g_{\mu_1\rho_2}(x^\mu)) \ \Rightarrow$$

$$R_{\mu_1\mu_2}^{\ \rho_1\rho_2} \ = \ [\frac{2\Lambda}{(D-1)(D-2)}] \ \delta^{\rho_1\rho_2}_{\mu_1\mu_2} \ \Rightarrow \ R \ = \ \frac{2D}{D-2} \ \Lambda \tag{3.17}$$

so that the n-th order constant Lanczos-Lovelock (LL) curvature tensor is

$$R^{(n)\ \rho_1\rho_2\ldots\rho_{2n}}_{\mu_1\mu_2\ldots\mu_{2n}} \ = \ [\frac{2\Lambda}{(D-1)(D-2)}]^n \ \delta^{\rho_1\rho_2\ldots\rho_{2n}}_{\mu_1\mu_2\ldots\mu_{2n}} \tag{3.18}$$

One must still check that the curvatures (3.17) are solutions to the Lanczos-Lovelock gravitational equations. In [13] the authors have shown that black-brane (black-string) solutions to Lanczos-Lovelock gravity theories in higher dimensions ($D > 4$) including higher curvature terms may, in fact, be simply constructed, but only within a certain class of Lanczos-Lovelock theories. This class of theories had the following property. Assume that L_r is the highest order term in the Lagrangian, i.e. that the coefficients c_n vanish for $n > r$. Depending on the values of the nonzero coefficients in the Lagrangian, it then turns out that the theory may have up to r distinct constant curvature vacuum solutions [13]. The different values that the constant curvature may take are the roots of a r-th order polynomial. There will, of course, generally be r roots, but only real values of the curvature are considered to be physical. The coefficients in the Lanczos-Lovelock Lagrangian may be tuned such that there are r real roots and that all these roots coincide. The theory then has a (locally) unique constant curvature vacuum solution. The authors [13] referred to these as LUV theories - standing for Lovelock-Unique-Vacuum. Those LUV theories are the ones which have simple black-brane solutions.

For the purposes of studying LUV theories, the authors [13] found it very useful to rewrite the LL equations of motion $\mathcal{G}_{ab} = 0$ in an alternative form which was very useful as we shall see below,

$$\mathcal{G}^a{}_b = \alpha_0\, \delta^{a\mu_1\mu_2\ldots\mu_{2r}}_{b\nu_1\nu_2\ldots\nu_{2r}} \left(R_{\mu_1\mu_2}{}^{\nu_1\nu_2} + \alpha_1 \delta^{\nu_1\nu_2}_{\mu_1\mu_2}\right) \cdots \left(R_{\mu_{2r-1}\mu_{2r}}{}^{\nu_{2r-1}\nu_{2r}} + \alpha_r \delta^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\nu_{2r}}\right). \tag{3.19}$$

The original form of the equations of motion can then be recovered through repeated applications of the identity

$$\delta^{a_1\ldots a_p}_{b_1\ldots b_p} \delta^{b_{p-1}b_p}_{a_{p-1}a_p} = 2(D-(p-1))(D-(p-2))\delta^{a_1\ldots a_{p-2}}_{b_1\ldots b_{p-2}} \tag{3.20}$$

The coefficients c_n are given by sums of products of the parameters α_n. The precise relation is given in reference [14]. Inverting this relation to get the α_n's in terms of the c_n's requires solving a polynomial equation of order r. Hence the α_n's are generally complex parameters.

When each one of the factors in eq-(3.19) becomes zero

$$R_{\mu_i\mu_{i+1}}{}^{\nu_i\nu_{i+1}} + \alpha_i\, \delta^{\nu_i\nu_{i+1}}_{\mu_i\mu_{i+1}} = 0 \Rightarrow R_{\mu_i\mu_{i+1}}{}^{\nu_i\nu_{i+1}} = -\alpha_i\, \delta^{\nu_i\nu_{i+1}}_{\mu_i\mu_{i+1}}, \quad i = 1,2,3,\ldots,r \tag{3.21}$$

one has then several different possible values for the constant curvature vacuum solutions. The LUV theories discussed above result from setting *all* the parameters α_n with $n = 1, \ldots r$ equal to a common value α and that is related to the cosmological constant Λ. There is then, at least locally, a unique constant curvature vacuum. If we further set $\alpha = 0$, we get a pure Lovelock theory with Lagrangian $\mathcal{L} = \alpha_0 \mathcal{L}_r$, which has flat spacetime as its unique constant curvature vacuum. For particular (black-brane) solutions for $g_{\mu\nu}$ we refer to [13] and references therein.

For the constant-curvature vacuum solutions case, the left hand side of the above embedding conditions (3.11,3.12) become

$$\sum_{n=1}^{[\frac{D}{2}]} c_n \; n \; \delta^\rho_\sigma \; [\frac{2\Lambda}{(D-1)(D-2)}]^n \; [\frac{D!}{(D-(2n-1))!}] \tag{3.22}$$

and

$$\sum_{n=1}^{[\frac{D}{2}]} c_n \; [\frac{2\Lambda}{(D-1)(D-2)}]^n \; [\frac{D!}{(D-2n)!}] \tag{3.23}$$

As usual, on must adjust units in eqs-(3.22,3.23) by taking into account that the dimensionful coefficients c_n are given in terms of powers of L_P so that the Lanczos-Lovelock action in D-dimensions is dimensionless. Having determined that the dimensions of the Lanczos-Lovelock Lagrangian density is $(length)^{-D}$ this fixes the appropriate powers of L_P which must appear in the right hand side and left hand side of eqs-(3.11,3.12). The right hand side of the embedding conditions (3.11, 3.12), when the left hand side is given by eqs-(3.22, 3.23), respectively, must contain expressions of the form

$$\sum_{j=1}^{D} \mathbf{R}_{\mu_1\mu_2\ldots\mu_j \;\; \sigma}^{\;\;\mu_1\mu_2\ldots\mu_j \;\; \rho} \;=\; \sum_{j=1}^{D} \delta^\rho_\sigma \; a_j(D) \; [\frac{2\Lambda}{(D-1)(D-2)}]^{\frac{j+1}{2}} \tag{3.24a}$$

$$\mathbf{R}_{\mathbf{0} \;\; \sigma}^{\;\;\mathbf{0} \;\; \rho} \;=\; a_0(D) \; \delta^\rho_\sigma \; [\frac{2\Lambda}{(D-1)(D-2)}]^{\frac{1}{2}} \tag{3.24b}$$

and

$$\sum_{j=1}^{D}\sum_{k=1}^{D} \; \mathbf{R}_{\mu_1\mu_2\ldots\mu_j \;\; \nu_1\nu_2\ldots\nu_k}^{\;\;\mu_1\mu_2\ldots\mu_j \;\; \nu_1\nu_2\ldots\nu_k} \;=\; \sum_{j=1}^{D}\sum_{k=1}^{D} \; b_{jk}(D) \; [\frac{2\Lambda}{(D-1)(D-2)}]^{\frac{j+k}{2}} \tag{3.25a}$$

$$\sum_{j=1}^{D} \; \mathbf{R}_{\mu_1\mu_2\ldots\mu_j \;\; \mathbf{0}}^{\;\;\mu_1\mu_2\ldots\mu_j \;\; \mathbf{0}} \;=\; \sum_{j=1}^{D} \; b_j(D) \; [\frac{2\Lambda}{(D-1)(D-2)}]^{\frac{j}{2}} \tag{3.25b}$$

where $a_j(D), a_o(D), b_{jk}(D), b_j(D)$ are suitable D-dependent dimensionful coefficients in powers of the Planck scale and which are constrained by the values of the c_n coefficients which are known for the LUV solutions. It is beyond the scope of this work to find nontrivial solutions to the embedding conditions (3.11,3.12) associated with the specific expressions in eqs-(3.24,3.25). This is a very challenging problem.

A plausible guide of how to solve such problem might be by recasting the problem in terms of generalized gauge field theories, like generalized Maxwell and Yang-Mills theories. The antisymmetry property of ordinary differential forms involving the coordinates $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$ becomes now for bivector coordinate differentials $dx^{\mu_1\mu_2} \wedge dx^{\rho_1\rho_2} = dx^{\rho_1\rho_2} \wedge dx^{\mu_1\mu_2}$. Similarly one has

$$dx^{\mu_1\mu_2\cdots\mu_{2n}} \wedge dx^{\rho_1\rho_2\cdots\rho_{2n}} \;=\; dx^{\rho_1\rho_2\cdots\rho_{2n}} \wedge dx^{\mu_1\mu_2\cdots\mu_{2n}} \tag{3.26}$$

and

$$dx^{\mu_1\mu_2\cdots\mu_{2n-1}} \wedge dx^{\rho_1\rho_2\cdots\rho_{2n-1}} \ = \ - \ dx^{\rho_1\rho_2\cdots\rho_{2n-1}} \wedge dx^{\mu_1\mu_2\cdots\mu_{2n-1}} \qquad (3.27)$$

One may rewrite our expressions in the language of poly-differential forms, for instance

$$\mathbf{F}^{(2)} \ = \ \mathbf{dA}, \quad \mathbf{d} = dx^{\mu_1\mu_2} \ \frac{\partial}{\partial x^{\mu_1\mu_2}} \ , \quad \mathbf{A} = A_{\rho_1\rho_2} \ dx^{\rho_1\rho_2} \qquad (87)$$

$$\mathbf{F}^{(2)} \ = \ (\ \partial_{\mu_1\mu_2} \ A_{\rho_1\rho_2} \ + \ \partial_{\rho_1\rho_2} \ A_{\mu_1\mu_2} \ ) \ dx^{\mu_1\mu_2} \wedge dx^{\rho_1\rho_2} \qquad (3.28)$$

one should notice the + sign in eq-(3.28) due to the properties of eq-(3.26) . And

$$\mathbf{F}^{(2n)} \ = \ \mathbf{F}^{(2)} \wedge \mathbf{F}^{(2)} \wedge \ldots \wedge \mathbf{F}^{(2)} \ (n \ factors) \qquad (3.29)$$

Therefore, from the functional form of the Lanczos-Lovelock curvatures one can infer the correspondence

$$\mathbf{F}^{(2)} \leftrightarrow \mathcal{R}_{\mu_1\mu_2}{}^{\rho_1\rho_2}, \quad \mathbf{F}^{(2n)} \leftrightarrow \mathcal{R}^{(n)}_{\mu_1\mu_2\cdots\mu_{2n}}{}^{\rho_1\rho_2\cdots\rho_{2n}} \qquad (3.30)$$

which might aid us in finding nontrivial solutions to eqs-(3.24,3.25) when we replace $\mathbf{F} = \mathbf{dA}$ for the nonabelian version $\mathbf{F} = (\mathbf{d} + \mathbf{A}) \wedge \mathbf{A}$. Furthermore, one still has to use the remaining of the C-space gravitational equations, in the absence of torsion, for the other poly-vector valued components g_{AB} of the metric. This needs to be solved before one can ascertain that *nontrivial* solutions of the Lanczos-Lovelock gravitational equations can be embedded into the C-space gravitational equations . Perhaps in this particular case the solutions for the C-space metric components admit a factorization into anti-symmetrized sums of products of $g_{\mu\nu}$. The plausible relation to extended gravitational theories based on $f(R), f(R_{\mu\nu})\ldots$ actions for polynomial-valued functions, and which obviate the need for dark matter, warrants also further investigation [19].

$$\mathbf{R} \ = \ \sum \ldots \ = \ a_1 R \ + \ a_2 R^2 \ + \ldots \ + \ a_N R^N \qquad (3.31)$$

$N = [D/2]$. where the scalar curvature with torsion in Riemann-Cartan space decomposes as $R = \hat{R} - \frac{1}{4} T_{abc} T^{abc}$. Thus in eq- (3.31) one has a special case of $f(R,T)$ for polynomial-valued functions involving curvature and torsion.

To finalize we should add that Polyvector-valued gauge field theories in noncommutative Clifford spaces were presented in [20] where we found that the study of n-ary algebras leads to interesting relationships among the **n**-ary commutators of noncommuting spacetime coordinates $[X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_n}]$ with the poly-vector valued coordinates $X^{\mu_1\mu_2\cdots\mu_n}$ in noncommutative Clifford spaces. It was given by $[X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_n}] = n! X^{\mu_1\mu_2\cdots\mu_n}$. These findings will be relevant for the quantization program.

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