A simple and compact approach to hydrodynamic using geometric algebra

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Abstract

A new simple and compact approach to hydrodynamic is presented using the formalism of geometric algebra (GA).

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I. INTRODUCTION

A. Maxwell's equations

Maxwell's equations were first published in 1865 [1] . In the manner of vector calculus, as now seen in most modern textbooks [2] and revert to natural units, these equations read ($c = \epsilon_0 = \mu_0 = 1$)

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{B} = 0, & \text{(Gauss' law of magnetism)}, \\ \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} + \partial_t \mathbf{E}, & \text{(Ampère's law)}; \\ \boldsymbol{\nabla} \cdot \mathbf{E} = \rho, & \text{(Gauss' law)}; \\ \boldsymbol{\nabla} \times \mathbf{E} = -\partial_t \mathbf{B}, & \text{(Faraday's law)}; \end{cases} \tag{1}$$

where $\mathbf{E}, \mathbf{B}, \mathbf{J}$ are conventional vector fields, with $\mathbf{E}$ the electric field strength and $\mathbf{B}$ the magnetic field strength and $\boldsymbol{\nabla}$ is the three-vector gradient (identified in bold) given by

$$\boldsymbol{\nabla} = \boldsymbol{\sigma}_i \frac{\partial}{\partial x^i} = \boldsymbol{\sigma}_i \partial_i$$

and $\boldsymbol{\sigma}_i = \gamma_i \gamma_0$ denote a right-handed orthonormal frame for relative space defined by the timelike vector γ_0. Maxwell's four equations given in Eq. (1) along with the Lorentz force law

$$\mathbf{K} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{2}$$

completely summarize classical electrodynamics [2].

B. Navier-Stokes equation

Generally, a fluid flow is described by the Navier-Stokes (NS) equation [3]:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho} \boldsymbol{\nabla} P - \nu \boldsymbol{\nabla}^2 \mathbf{v} \ , \tag{3}$$

where $\mathbf{v}$ is fluid velocity, P is pressure, ρ is density and ν is the coefficient of viscosity which is assumed constant. Note that only the convective terms are nonlinear for incompressible Newtonian flow. The convective acceleration is an acceleration caused by a (possibly steady) change in velocity over position. In principle, the study of fluid dynamics is focused on solving the Navier-Stokes equation with particular boundary conditions depend on the phenomenon under consideration. Mathematically it has been known as the boundary value problem. The most interesting problem in fluid dynamics is turbulence phenomenon.

II. SPACETIME IN GEOMETRIC ALGEBRA

Nature's law is believed to be written in the language of mathematics. The mathematical formalism of geometric algebra[4–6, 8, 11], developed by Clifford in 1878, as convincingly argued by Hestenes [9], provides a single unified mathematical language for physics. To model the Minkowski spacetime, we introduce a set of four basis vectors $\{\gamma_\mu,\ \mu = 0\ldots 3,$ satisfying

$$\gamma_\mu\cdot\gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+\,-\,-\,-). \tag{4}$$

The vectors $\{\gamma_\mu\}$ satisfy the same algebraic relations as Dirac's γ-matrices, but they now form a set of four independent basis vectors for spacetime, not four components of a single vector in an internal 'spin space'. When manipulating (geometric) products of these vectors, one simply uses the rule that parallel vectors commute and orthogonal vectors anticommute.

From the four vectors $\{\gamma_\mu\}$ we can construct a set of six basis elements for the space of bivectors:

$$\{\gamma_1\gamma_0, \gamma_2\gamma_0, \gamma_3\gamma_0, \gamma_3\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_1\}. \tag{5}$$

After the bivectors comes the space of grade-3 objects or trivectors. This space is four-dimensional and is spanned by the basis

$$\{\gamma_3\gamma_2\gamma_1, \gamma_0\gamma_3\gamma_2, \gamma_0\gamma_1\gamma_3, \gamma_0\gamma_2\gamma_1\}. \tag{6}$$

Finally, there is a single grade-4 element. This is called the *pseudoscalar* and is given the symbol I, so that

$$i = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{7}$$

The symbol I is used because the square of I is -1, but the pseudoscalar must not be confused with the unit scalar imaginary employed in quantum mechanics. The pseudoscalar I is a geometrically-significant entity and is responsible for the duality operation in the algebra. Furthermore, I *anti*commutes with odd-grade elements (vectors and trivectors), and commutes only with even-grade elements.

The full STA is spanned by the basis

$$1, \qquad \{\gamma_\mu\}, \qquad \{\boldsymbol{\sigma}_k,\ I\boldsymbol{\sigma}_k\}, \qquad \{I\gamma_\mu\}, \qquad I, \tag{8}$$

where

$$\boldsymbol{\sigma}_k = \gamma_k\gamma_0, \qquad k = 1, 2, 3. \tag{9}$$

III. FORMAL SIMILARITY BETWEEN ELECTROMAGNETISM AND HYDRODYNAMIC

In order to see directly the formal similarity between mathematical structure of electromagnetism and hydrodynamic, we start our discussion first by simply considering the ideal (with coefficient of viscosity $\nu = 0$) and incompressible fluid, the Navier-Stokes equation Eq. (3) then becomes

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} \right) = -\boldsymbol{\nabla} P \ , \tag{10}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0 \ . \tag{11}$$

Taking advantage of the equation in vector calculus $\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla}(\frac{1}{2}\mathbf{v}^2) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$, it can be rewritten as follows,

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \left(\frac{1}{2}\mathbf{v}^2 \right) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = -\frac{1}{\rho} \boldsymbol{\nabla} P \ , \tag{12}$$

or,

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) - \boldsymbol{\nabla} \left(\frac{1}{2}\mathbf{v}^2 + \frac{P}{\rho} \right) \ . \tag{13}$$

Here we define the *scalar potential or the Bernoulli energy function* as $\Phi \equiv \frac{1}{2}\mathbf{v}^2 + \frac{P}{\rho}$, the *vorticity* as $\mathbf{W} \equiv \boldsymbol{\nabla} \times \mathbf{v}$ and the *Lamb's vector* as $\mathbf{L} \equiv \mathbf{W} \times \mathbf{v}$, then the equation becomes,

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{W} \times \mathbf{v} - \boldsymbol{\nabla}\Phi \\ &= -\mathbf{L} - \boldsymbol{\nabla}\Phi \ . \end{aligned} \tag{14}$$

By definition, we get immediately

$$\boldsymbol{\nabla} \cdot \mathbf{W} = 0. \tag{15}$$

Imposing curl operation in Eq. (14) we obtain the vorticity equation as follow,

$$\frac{\partial \mathbf{W}}{\partial t} = -\boldsymbol{\nabla} \times (\mathbf{W} \times \mathbf{v}) = -\boldsymbol{\nabla} \times \mathbf{L} \ , \tag{16}$$

which gives the curl of $\mathbf{L}$ field.

Using the incompressible condition $0 = \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{v}) = -\boldsymbol{\nabla} \cdot \mathbf{L} - \boldsymbol{\nabla}^2\Phi$, we can obtain the information about the divergence of $\mathbf{L}$ field:

$$\boldsymbol{\nabla} \cdot \mathbf{L} = -\boldsymbol{\nabla}^2\Phi \equiv j \ . \tag{17}$$

We still need the curl of $\mathbf{W}$ to form a Maxwell-like equation. Consider the definition of the Lamb's vector $\mathbf{L} = \mathbf{W} \times \mathbf{v}$. Taking the derivative $\partial/\partial t$ in the definition we obtain,

$$\frac{\partial \mathbf{L}}{\partial t} = \frac{\partial \mathbf{W}}{\partial t} \times \mathbf{v} + \mathbf{W} \times \frac{\partial \mathbf{v}}{\partial t} . \tag{18}$$

Substituting Eq. (14) and (16), we get,

$$\boldsymbol{\nabla} \times \mathbf{W} = \mathbf{J} + \frac{1}{\mathbf{v}^2} \frac{\partial \mathbf{L}}{\partial t} , \tag{19}$$

where,

$$\mathbf{J} \equiv \frac{1}{\mathbf{v}^2} \{ -\mathbf{v} \boldsymbol{\nabla}^2 \Phi + [\boldsymbol{\nabla} \times (\mathbf{v} \cdot \mathbf{W})]\, \mathbf{v} + \mathbf{W} \times \boldsymbol{\nabla}(\Phi + \mathbf{v}^2) + 2\, [(\boldsymbol{\nabla} \times \mathbf{v}) \cdot \boldsymbol{\nabla}]\, \mathbf{v} \} . \tag{20}$$

Then we get the Maxwell-like equations for ideal and incompressible fluid,

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{W} = 0 \, , \\ \boldsymbol{\nabla} \times \mathbf{W} = \mathbf{J} + \frac{1}{\mathbf{v}^2} \partial_t \mathbf{L} \, , \\ \boldsymbol{\nabla} \cdot \mathbf{L} = j \, , \\ \boldsymbol{\nabla} \times \mathbf{L} = -\partial_t \mathbf{W} \, , \end{cases} , \tag{21}$$

IV. GEOMETRIC ALGEBRA APPROACH TO HYDRODYNAMIC EQUATION

We assume $\frac{1}{\mathbf{v}^2} = 1$. In this section, we derive a compact hydrodynamic equation using the powerful geometric algebra approach. First we replace the cross product with the *exterior product* $\boldsymbol{\nabla} \wedge \mathbf{W} = I \boldsymbol{\nabla} \times \mathbf{W}$, so now the equations read:

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{W} = 0 \, , \\ \boldsymbol{\nabla} \wedge \mathbf{W} = I(\mathbf{J} + \partial_t \mathbf{L}) \, , \\ \boldsymbol{\nabla} \cdot \mathbf{L} = j \, , \\ \boldsymbol{\nabla} \wedge \mathbf{L} = -\partial_t I \mathbf{W} \, , \end{cases} , \tag{22}$$

Then we assemble equations for the separate divergence and curl parts of *vector derivative* $\boldsymbol{\nabla} \mathbf{W} = \boldsymbol{\nabla} \cdot \mathbf{W} + \boldsymbol{\nabla} \wedge \mathbf{W}$.

$$\boldsymbol{\nabla}(I\mathbf{W}) = -\mathbf{J} - \partial_t \mathbf{L} \, , \boldsymbol{\nabla} \mathbf{L} = j - \partial_t (I\mathbf{W}) \, ,$$

It follows that we can introduce the term *hydrodynamic field strength* as $F = \mathbf{L} + I\mathbf{W}$. Then we can combine all these equations into a single multivector equation:

$$\boldsymbol{\nabla}(\mathbf{L} + I\mathbf{W}) + \partial_t (\mathbf{L} + I\mathbf{W}) = j - \mathbf{J}$$

Each of the separate equations can be recovered by picking out terms of a given grade. We can still introduce the *spacetime current for the flow* J, which has

$$j = J \cdot \gamma_0, \qquad \mathbf{J} = J \wedge \gamma_0.$$

It then follows that

$$j - \mathbf{J} = \gamma_0 \cdot J + \gamma_0 \wedge J = \gamma_0 J.$$

Using the *spacetime vector derivative* $\nabla = \gamma^\mu \partial_\mu$ and $\nabla\gamma_0 = \partial_t - \boldsymbol{\nabla}$, we can get the most compact version of convariant equation

$$\boxed{\nabla F = J} \tag{23}$$

If we multiply $\nabla F = J$ by ∇, we get

$$\nabla^2 F = \nabla J = \nabla \cdot J + \nabla \wedge J.$$

Here the

$$\nabla^2 = \nabla\gamma_0\gamma_0\nabla = (\partial_t - \boldsymbol{\nabla})(\partial_t + \boldsymbol{\nabla}) = \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$$

is a scalar operator, so the left-hand side can only contain bivector terms. It follows directly that the scalar part of the right-hand side vanishes

$$\nabla \cdot J = \boldsymbol{\nabla} \cdot \mathbf{J} + \tfrac{\partial j}{\partial t} = 0$$

which simply recoveres the total charge conservation.

The above discussion can be summarized by the following table 1.

V. CONCLUSION

A simple and compact approach to hydrodynamic using geometric algebra is proposed. The great formal similarity between the mathematical structures of electromagnetism and hydrodynamic is very worthy further study. It may help gain more in-depth understanding of both fields.

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[1] J. C. Maxwell, "A dynamical theory of the electromagnetic field", Royal Society Transactions **155**, 459-512 (1865).

Unified singe compact GA equation for both Electromagnetism and Hydrodynamic $\nabla F = J$			
Electromagnetism Theory		Hydrodynamic Theory	
Scalar potential	V	Bernoulli energy function	$\Phi \equiv \frac{1}{2}\mathbf{v}^2 + \frac{P}{\rho}$
Vector potential	A	Velocity field	$\mathbf{v}$
Electric field	$\mathbf{E}$	Lamb's vector	$\mathbf{L} \equiv \mathbf{W} \times \mathbf{v}$
Magnetic field	$\mathbf{B}$	Vorticity	$\mathbf{W} \equiv \boldsymbol{\nabla} \times \mathbf{v}$
Charge density	ρ	Hydro-charge	$j \equiv -\boldsymbol{\nabla}^2 \Phi$
Current density	$\mathbf{J}$	Hydro-current	$\mathbf{J} \equiv \mathbf{J}(\Phi, \mathbf{v})$
Gauss's law for magnetism	$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$		$\boldsymbol{\nabla} \cdot \mathbf{W} = 0$
Ampère's law	$\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} + \partial_t \mathbf{E}$		$\boldsymbol{\nabla} \times \mathbf{W} = \mathbf{J} + \frac{1}{\mathbf{v}^2}\partial_t \mathbf{L}$
Gauss's law	$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho$		$\boldsymbol{\nabla} \cdot \mathbf{L} = j$
Faraday's law	$\boldsymbol{\nabla} \times \mathbf{E} = -\partial_t \mathbf{B}$		$\boldsymbol{\nabla} \times \mathbf{L} = -\partial_t \mathbf{W}$
Electromagnetism field strength	$F = \mathbf{E} + I\mathbf{B}$	Hydrodynamic field strength	$F = \mathbf{L} + I\mathbf{W}$
Potential function A	$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t},\ \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$		$\mathbf{L} \equiv \mathbf{W} \times \mathbf{v}, \mathbf{W} \equiv \boldsymbol{\nabla} \times \mathbf{v}$
Lorenz gauge	$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\partial V}{\partial t} = 0$		
Coulomb gauge	$\boldsymbol{\nabla} \cdot \mathbf{A} = 0$	Incompressible condition	$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$
Charge conservation	$\nabla \cdot J = \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$	Hydro-charge conservation	$\nabla \cdot J = \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial j}{\partial t} = 0$
EM Invariants F^2	$\mathbf{B}^2 - \mathbf{E}^2$	Hydro-Invariants	$\mathbf{W}^2 - \mathbf{L}^2$
	$2\mathbf{B} \cdot \mathbf{E}$		$2\mathbf{W} \cdot \mathbf{L}$
Force	$\mathbf{K} = \gamma q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$		

TABLE I: Analogy between electromagnetism and hydrodynamics

[2] D. J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, 1999).

[3] P. Kundu (1996), *Fluids Mechanics*, Addison-Wesley, New York.

[4] D. Hestenes, *Spacetime Algebra*, (Gordon and Breach, New York, 1966).

[5] D. Hestenes, "Spacetime physics with geometric algebra", Am. J. Phys., **71**, 691-714 (2003).

[6] T. G. Vold, "An introduction to geometric calculus and its application to electrodynamics", Am. J. Phys., **61** Issue 6, 505-513 (1993).

[7] W. E. Baylis, J. Bonenfant, J. Derbyshire and J. Huschilt, "Motion of charged particles in crossed and equal E and B fields", Am. J. Phys, **62**, 899 (1994).

[8] P. Simons, "Vectors and beyond: geometric algebra and its philosophical significance", Dialectica, **61**, 381-395 (2009).

[9] D. Hestenes, "Oersted Medal Lecture 2002: Reforming the mathematical language of physics" Am. J. Phys, **71**, 104-121 (2003).

[10] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, (Macmillan and Co., London, 1873).

[11] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003).

[12] Feynman, R. The Feynman Lectures on Physics. 2. p. 15-5.