**Precession of perihelion of the planet with the critical photon orbit**

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**Abstract** The solution of geodesic equation for Schwarzschild's metric is used to estimate the precession of perihelion of the planets with aid of the critical photon orbit's radius and Schwarzschild's radius. We introduced new equation of outer planets.

**Key words**: celestial mechanics: Precession-perihelion: Schwarzschild's metric

## 1 Introduction

The disagreement of precession of the perihelion of the planets, Mercury, between measurements precession and calculations of the precession, had been known and unexplained by the basic laws of Newtonian mechanics. This problem was solved by Einstein's theory of relativity. (Sean M. Caroll.2004),( R.Wald.1984),( C.Misner ,K .Thorne, and J.Wheeler.1973) and ( S.Weinberg.1972).

Complexity in dealing with Celestial Mechanics, and the needing of accurate prediction of the dynamics of the objects of the solar system, laid many methods to eases the solutions. Such as perturbation theories, (A. Celletti, L. Chierchia.2007). and canonical perturbation theory, ( S. Ferraz-Mello.2007).

## 2 Solution of geodesic equation

In this work, we solved the geodesic equation for Schwarzschild's metric, to find out the relation of the precision of perihelion with the circular orbit of the photon around the Schwarzschild's black hole.

Schwarzschild's spherically symmetric Vacuum solution has a line element

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\emptyset^2)$$

Where: $r_s = \frac{2GM}{c^2}$, Schwarzschild's radius .

The geodesics equation with the substitution $u = \frac{1}{r}$ gives:

$$\frac{d^2u}{d\emptyset^2} + u - \frac{\alpha}{2} = \frac{3}{2}\beta u^2 \qquad (1)$$

The solution in the form of:

$$u = A + Ae\cos(\emptyset - \varphi) \qquad (2)$$

Where $\varphi$ varies slowly as a function of $\lambda\emptyset$ , $\frac{d\varphi}{d\emptyset} = \lambda\acute{\varphi}$ , ($\emptyset$ is the true anomaly associated with the orbit, $\lambda$ is gravitational parameter which will be determined); $\alpha = \frac{2}{a(1-e^2)}$ ; $\beta = \frac{2GM}{c^2}$ and e is the eccentricity.

$$\frac{du}{d\emptyset} = -Ae\sin(\emptyset - \varphi) + Ae\sin(\emptyset - \varphi)\,\lambda\acute{\varphi} \tag{3}$$

$$\frac{d^2u}{d\emptyset^2} = -Ae\cos(\emptyset - \varphi) + 2Ae\cos(\emptyset - \varphi)\,\lambda\acute{\varphi} - Ae\cos(\emptyset - \varphi)\,\acute{\varphi}^2 + Ae\sin(\emptyset - \varphi)\,\acute{\acute{\varphi}} \tag{4}$$

Substituting equations, (2) and (4) into equation (1). By equating the constant terms and the coefficients of the cosine and sine terms, we obtained the following relations:

1. $A - \frac{\alpha}{2} = \frac{3}{2}\beta A^2$
2. $\lambda^2\acute{\varphi}^2 - 2\lambda\acute{\varphi} + \left[\frac{3}{2}\beta Ae\cos(\emptyset - \varphi) + 3\beta A\right] = 0$

The first solution gives:

$$A = \frac{1}{3\beta} \pm \left(\frac{1}{3\beta} - \frac{\alpha}{2}\right) \tag{5}$$

Where there are two roots of A

$$A_+ = \frac{2}{3\beta} - \frac{\alpha}{2} \quad ; \quad A_- = \frac{\alpha}{2} \tag{6}$$

Dimensionally, $A = L^{-1}$ is potential energy. $A_-$ is Newtonian limit, $A_+$ is the contribution of general relativity, represented in $\frac{2}{3\beta}$ term. Authors used to ignore this term.

The second solution produces:

$$\acute{\varphi} = \frac{1}{\lambda} \pm \frac{1}{\lambda}\sqrt{1 - \left[\frac{3}{2}\beta Ae\cos(\emptyset - \varphi) + 3\beta A\right]}$$

$$\acute{\varphi} = \frac{1}{\lambda} \pm \frac{1}{\lambda}\left[1 - \frac{1}{2}\left(\frac{3}{2}\beta Ae\cos(\emptyset - \varphi) + 3\beta A\right)\right] \tag{7}$$

$$\acute{\varphi}_+ = \frac{2}{\lambda} - \frac{1}{2\lambda}\left(\frac{3}{2}\beta Ae\cos(\emptyset - \varphi) + 3\beta A\right) \tag{8}$$

$$\acute{\varphi}_- = \frac{3}{4\lambda}\beta Ae\cos(\emptyset - \varphi) + \frac{3}{2\lambda}\beta A \tag{9}$$

For simplicity, we assume that, $\cos(\emptyset - \varphi) = 1$

$$\acute{\varphi}_+ = \frac{1}{\lambda} - \frac{e}{2\lambda} + \frac{3}{8\lambda}\beta\alpha(e + 2) \tag{10}$$

$$\acute{\varphi}_- = \frac{3}{8\lambda}\beta\alpha(e + 2) \tag{11}$$

## 3 Results and discussions

The relation between $\acute{\varphi}_+$ and $\acute{\varphi}_-$ is:

$$\acute{\varphi}_+ - \acute{\varphi}_- = \frac{1}{\lambda}\left(1 - \frac{e}{2}\right) \tag{12}$$

For eccentricity, less than 1, the elliptical motion is affected by this change in $\varphi$. It causes the semi-major axis of the ellipse to slowly rotate.

$$\frac{d\varphi}{dt} = \frac{1}{\lambda}\frac{d\varphi}{d\emptyset}\frac{d\emptyset}{dt}$$

Where $\frac{d\emptyset}{dt} = \sqrt{\frac{GM}{a^3}}$ Kepler's third law (Herbert Goldstein.1980).

$$\frac{d\varphi_-}{dt} = \frac{3}{8}\sqrt{\frac{GM}{a^3}}\,\frac{\beta\alpha}{\lambda}(e+2) \qquad (13)$$

Equation (13) calculates the rotation of the semi – major axis in planet orbit. It has an external term, $\frac{1}{\lambda}(e+2)$, which differs from that in many context books. To explain the significant of this term, in figure (1), we plotted the ratio of the eccentricity $\frac{e+2}{1-e^2}$ of solar planets we found:

$$\frac{e+2}{1-e^2} \approx 2$$

This occurs when $e \ll 1$. Otherwise, the parameter $\lambda$ demands that, the ratio to be steady in two. Therefore, $\lambda$ is orbit's fine-tuning, or a representative factor for the gravitational system, it varies according to the gravitational force to keep the ratio of eccentricity constant

$$\frac{e+2}{1-e^2} = 2\lambda$$

Substitution of the above result in (11), $\acute{\varphi}_-$ becomes:

$$\acute{\varphi}_- = \frac{r_c}{a}$$

Where $r_c = \frac{3GM}{c^2}$ ,$r_c$ is a radius where the photon can orbit forever in a circle orbit around Schwarzschild's black hole.

$$\frac{\omega_\varphi}{\omega_\emptyset} = \frac{r_c}{a} \qquad (14)$$

$\omega_\varphi \equiv \frac{d\varphi_-}{dt}$ , is the precession angular rate,$\omega_\emptyset \equiv \frac{d\emptyset}{dt}$, is the angular velocity. Equation (14) represents a condition of perihelion precession of the planet.

Table (1) shows precession's values of the solar system planets according to equation (14).In Mercury it shows $41.13''$ and equation (13) gives $43.52''$ .

Integrating equation (11), and for complete revolution $\emptyset = 2\pi$ ,one finds:

$$\varphi_- = \frac{3\pi GM}{c^2 a} \qquad (15)$$

Rewrite equation (6) with + sign as:

$$A_+ = \frac{2}{3r_s} - \frac{2}{a(1-e^2)} \qquad (16)$$

Where $r_s$ is Schwarzschild's radius.When $a \gg r_s$ equation (10) becomes

$$\acute{\varphi}_{+} = \frac{1}{\lambda}\left(1 - \frac{e}{2}\ \right) = \frac{1-e^2}{1-\left(e/_2\right)^2}\left(1 - \frac{e}{2}\right)^2 \tag{17}$$

Comparing equation (12) and (17), we find $\acute{\varphi}_{-} \rightarrow 0$ for large orbits.

Therefore, $\acute{\varphi}_{+}$ is suitable to be used in case of outer planets with high eccentricity, approaches one, $e \sim 1$ .

**4 Conclusion**

We found equation (14) which gives lower limit of the precession of perihelion and represents a condition to precession. Equation (17) gives the precession of perihelion in case of outer planets, where $a \gg r_s$ .

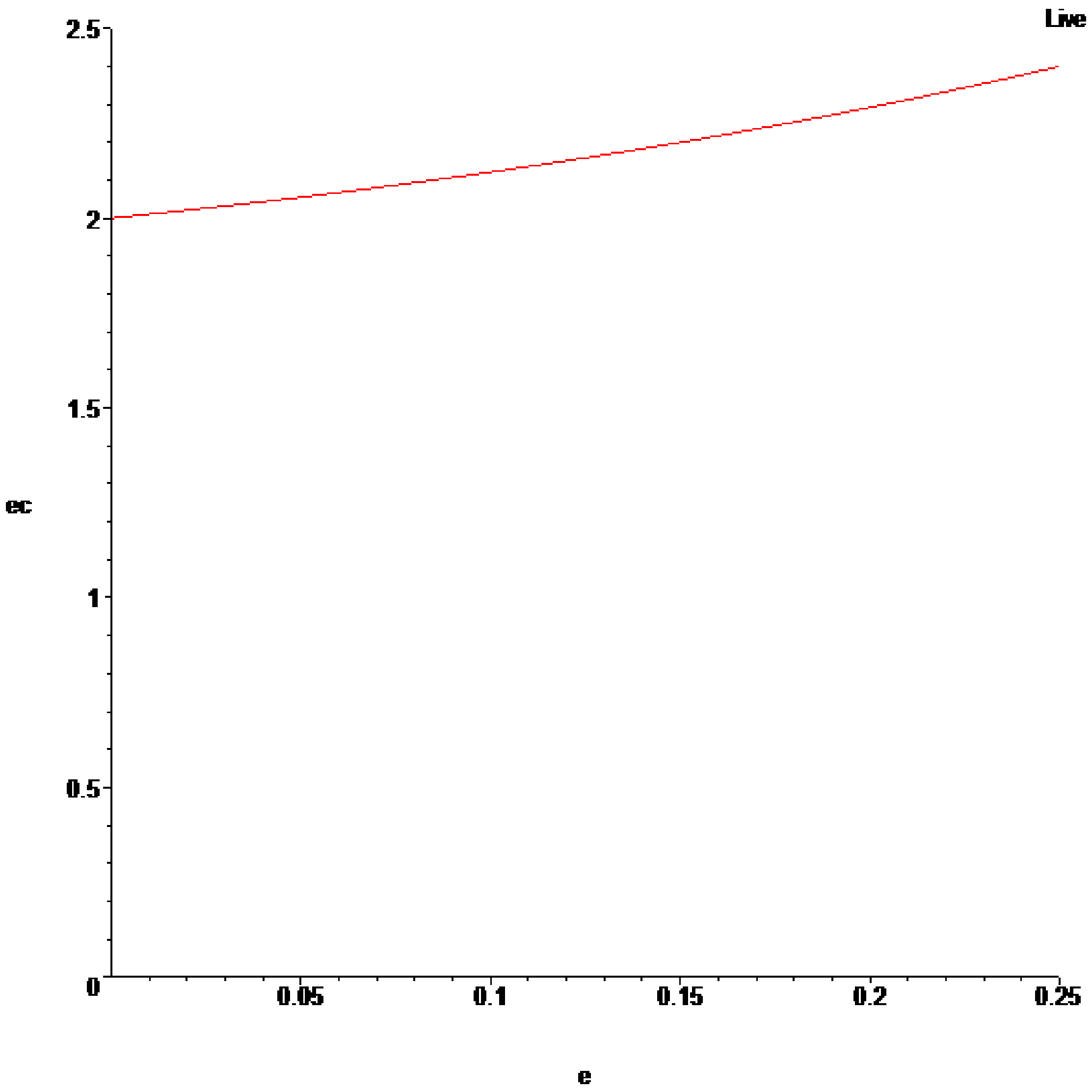


**Figure (1): shows the** $\mathbf{ec = \frac{e+2}{1-e^2}}$

**Table 1** Shows the precession of perihelion of the planets with the critical photon orbit

| Planet | Semi-major axis $10^{12}$m | Precession/century (arc sec) |
|---|---|---|
| Mercury | 0.0579 | 41.13 |
| Venus | 0.1082 | 8.615 |
| Earth | 0.1496 | 3.832 |
| Mars | 0.2279 | 1.338 |
| Jupiter | 0.7783 | 0.062 |
| Saturn | 1.427 | 0.019 |
| Uranus | 2.8696 | 0.002 |
| Neptune | 4.4966 | 0.001 |

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