Soliton-Based Approach to Axiomatic Quantization of Classical Field Theory

Paul J. Werbos[1]
paul.werbos@gmail.com

Abstract

This paper proposes a method for the axiomatic quantization of field theories defined by a Lagrangian of the form $L(\underline{\varphi}, \nabla\underline{\varphi}, \partial_t\underline{\varphi})$, where $\underline{\varphi}$ is a mathematical vector field typically made up of scalars, covariant vectors, tensors and so on. It reviews basic challenges in the axiomatic formulation of quantum field theory, in quantizing "soliton models" which arise in grand unification and in phenomenological nuclear physics, and unresolved issues regarding mass normalization and the radius of the electron. It proposes a new fundamental theorem (conjecture) for spectra and bound states of bosonic field theories, exploiting the Glauber-Sudarshan P representation, and suggests how this might later be extended to solve the problem of choice (ill-definedness) in the S matrix in traditional formulations of quantum theory.

1. The Challenge

This paper addresses the following general problem. Suppose that we are given a classical field theory, with a Lagrangian of the form $L(\underline{\varphi}, \nabla\underline{\varphi}, \partial_t\underline{\varphi})$, where $\underline{\varphi}$ is a mathematical vector field over 3+1 dimensional space-time, a mathematical vector which may in fact be composed of objects such as covariant vector, tensors and so on. How can we "quantize" such a system? How can we map such a system into a quantum field theory (QFT), defined over distributions over the usual Fock-Hilbert space, in such a way that the key successful predictions of QFT are guaranteed to exist mathematically and be uniquely defined?

2. Review of Some Prior Work on This Challenge

A brief paper cannot give a comprehensive review of all the important prior work on this challenge. However, because this challenge has been important to many areas of physics, it is possible to cite and build upon a few key seminal sources which contain more comprehensive reviews.

2.1. Axiomatic Quantum Field Theory (AQFT)

Two of the excellent reviews of AQFT are the classical treatise of Jost [1] and the more recent critical review by Earman and Fraser [2]. These authors pose the key challenge as follows: how can we find any formulation of QFT for which the scattering matrix S is mathematically well-defined, and rich enough that the successes of quantum electrodynamics (QED) can be rederived more rigorously? Of course, they recognize that there are other key experiments in physics beyond the scope of QED, but this much would be a very substantial starting point. If we cannot even do QED, how can we expect to do more?

These reviews place great emphasis on Haag's Theorem, which many physicists now interpret as a "proof that the interaction picture does not exist." Earman and Fraser [2] explain how this interpretation is essentially just a sanitized version of a more difficult and more worrisome truth. The truth is that all predictions of scattering which rely on canonical commutation relations (CCR) and perturbation theory are affected by a "problem of choice," a problem which is unsolved to this day. In essence, the scattering matrix S is not uniquely defined by the assumptions we generally use when we make predictions via perturbation theory. Yet the great successes of QED are directly based on perturbation theory [3,4]; many other formulations of QFT have been developed [3], but the calculations which actually lead to empirical

[1] [1]The views herein do not necessarily represent the official views of the National Science Foundation.

success are al based on perturbation theory, even now. Lack of uniqueness in predicting S is a serious problem, no matter how informal one's approach to mathematics.

Haag-Ruelle scattering theory, with some recent extensions [5], could provide an answer to this problem in principle, but only if that scattering theory is applied more directly in practical calculations of S. Earman and Fraser say that people have not yet figured out how to do this. Given the normal workings of human psychology, it is not surprising that the valid aspects of this picture take some time to fully accept and face up to.

I would suggest the following adjustment of the picture as they have presented it. It is indeed essential that methods of quantization should replicate the diverse empirical successes of QED, as a kind of starting point. However, it is important to look more closely to see what these successes have actually been. The most startling initial and 16-digit successes of QED involved the prediction of *spectra* or *energy levels* [3,4], rather than the S matrix as such. These typically involved the use of time-independent perturbation theory and other methods to calculate eigenvectors and eigenvalues of the Hamiltonian operator H, without reliance on time-dependent perturbation methods. Many successful scattering calculations were also based on the concepts of "scattering states," [6], eigenvectors of H, using time-independent perturbation theory. Thus I would argue that precise prediction of spectra should be our first target in axiomatization.

In the full dynamic scattering case, as studied for example in quantum computing, the original Copenhagen formulation of QED is grossly inconsistent with experiment; a modified version of QED, cavity QED using density matrices, is necessary in order to understand many modern designs in electronics and photonics [7]. Thus we should not even attempt to replicate the Copenhagen version of the complete picture of scattering. At length scales on the order of the classical radius of the electron (3 femtometers) or more, current scattering statistics cannot be predicted well based on QED alone [8].

2.2. How to Quantize "Soliton" Models

Many branches of physics have run into the question: how can we quantize Lagrangian field theories which were developed in order to provide "soliton" models of elementary particles?

Strictly speaking, this question is not well-defined unless I specify what kind of "soliton" I am talking about. In all the examples I give here, a "soliton" is basically an equilibrium state of the fields which is "stable" because it is a strict, unconstrained local minimum of the total energy H. This concept feeds directly into the discussion of section 3.2. Russian researchers have linked the requirement for energy minimization to more formal concepts of Liapunov stability [9]. This is a special case of a more general and more rigorous concept of "chaoiton" [10], which I will mention briefly in section 3.3.

For example, in grand unification physics (e.g. [11]), people have asked how to analyze, quantize and use models such as the system which generates the famous Bogmolnyi-Prasad-Sommerfeld (BPS) monopoles [12,13]:

$$L = \tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - \tfrac{1}{2}(D_\mu Q^a)^2 - \tfrac{1}{8}\lambda(Q^2 - F^2)^2, \tag{1}$$

where the underlying (real) fields are Q^a and A^a_μ for a =1,2,3 and μ=0,1,2,3, where λ and F are parameters, and where we use the definitions:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\varepsilon_{abc} A^b_\mu A^c_\nu \tag{2}$$

$$D_\mu Q^a = \partial_\mu Q^a + e\varepsilon_{abc} A^b_\mu Q^c \tag{3}$$

Likewise, in "low energy nuclear physics" (the field which tries to confront empirical reality for phenomena which are "low energy" like fusion explosions and scattering at less than 1 GEV), people have asked how to quantize the Skyrme model and its relatives [9,13], the simplest of which may be written:

$$L = \text{a Tr}(\partial_\mu U \partial^\mu U^\dagger) + \text{b Tr}([U^\dagger \partial_\mu U, U^\dagger \partial^\sim U]^2) \quad , \tag{4}$$

where U is a tensor field whose value at any point is a matrix in SU(2).

The obvious traditional way to quantize such fields would be to associate each of the components of A, Q or even U with the usual kind of bosonic field operator, as in Weinberg [3]; for example, we would write:

$$\mathbf{Q}^{\alpha}(\underline{x}) = \mathbf{Q}^{\alpha+}(\underline{x}) + \mathbf{Q}^{\alpha-}(\underline{x}) = c\int d^3\underline{p}\frac{e^{i\underline{p}\cdot\underline{x}}a_{Q\alpha} + e^{-i\underline{p}\cdot\underline{x}}a^{+}_{Q\alpha}}{\sqrt{w_{Q\alpha}(\underline{p})}} \tag{5}$$

(Note that I changed "a" in equation 3 to "α" in equation 5, in order to reserve "a" for the usual creation and annihilation operators.) However, there are strong indications that these kinds of "solitons" may actually behave like fermions. How then, should we perform the quantization?

Starting in the 1970's or so, a large literature has grown up on "bosonization," reviewed at some length by Makhankov et al [9]. Perhaps the most important outcome of that work is a clear demonstration that fermions may emerge as the bound states of underlying bosons, in suitable field theories. Here then I propose that we should go ahead and quantize these "soliton models" as bosonic field theories, as suggested by equation 5, and treat those bosonic field theories as the fundamental laws of nature which must obey the most stringent mathematical properties (including the existence of correct spectral and scattering properties). In this picture, the "solitons" which extend over space, and have fermionic properties, are really just bound states of the underlying bosonic fields. The connection to QED, or more generally, to electroweak theory (EWT), should be done by considering the limiting case as the radius of the solitons goes to zero; only in that limit does the soliton behave like a perfect point-like fermion in the relevant sense. The key challenge to theory is to elaborate on the last three sentences, with proofs.

2.3 Explaining the Radius of the Electron

It is often said that the radius of the electron is still indistinguishable from zero. More precisely, it is said to have an upper limit of 10**-22 meters [14].

This ignores the reality that the "radius" of an elementary particle depends on how the particle is probed. It has long been well-known that the radius of a nucleon has different values depending on how it is probed [12]. The Skyrme model was developed in order to give us a better picture and understanding of how that can be. A key idea is that an elementary particle may actually be made up of several component fields (like A and Q in equations 1 and 2), and that the "radius" may not be the same for the different distributions of energy for the different components.

As an extreme example, it is well known that the source of magnetic charge in a BPS monopole is precisely a point, radius zero [15]. This is true despite the fact that the radius by other measures is not zero, and the energy in the A field is eminently finite. The work showing that the electron is "point-like" [14] is actually just probing the source of electric charge of the electron. Mathematically, it is quite reasonable to consider the possibility that the electron itself might be an extended body, like a BPS monopole, even if the source of electric charge has radius zero. Nevertheless, this would require a new type of soliton model, in which electrical charge is quantized in much the same way as magnetic charge is in the BPS system.

There are really only two other obvious ways to probe the core radius of the electron. We can probe the electron with scattering or collisions involving nucleons or other types of baryons; however, the strong nuclear force would be involved in such interactions, and no one would expect QED to predict all the details. Alternatively, we can probe the electron through high energy collisions with other electrons or leptons. These collisions clearly are not fully described by QED [8]. They provide one of the excellent empirical testbeds for more complete theories and for scattering calculations which still call out for further development.

Beyond these three types of probe, we can of course observe how far the simple Coulomb field of the electron reaches out into space, and back to the core of the electron itself. Even in classical QED, the Hamiltonian includes the energy of the electromagnetic field as part of the mass-energy of the electron. Even in QED, we know that the total energy in the electromagnetic field beyond the classical radius of the electron (about 3 femtometers) equals the total mass-energy of the electron. QED assumes that the Coulomb field extends right to the core of the electron, with no other elementary fields present, which gives the Coulomb field a total mass-energy of plus infinity. This presents a very serious and fundamental question in need of explanation: *why* does the electron have the relatively small, finite mass-energy that we actually see?

There are really only two choices of explanation here, to within some equivalence relations involving null forms. We can assume that the electron is actually made up of other fields, in addition to the electromagnetic field, which extend at least as far as 3 femtometers from the core of the electron, and result in much smaller energy densities than what QED assumes in the inner regions of the electron. (This is how the BPS example may be described.) Or else we can assume that the true energy density, the total Hamiltonian, actually does become negative in the core of the particle, somewhere inside the classical radius.

Classical QED takes the latter route. More precisely, in order to derive real predictions, it assumes a "mass renormalization," an infinite negative mass δM located precisely at the center of the electron [4]. It is conventional to say that "this is just part of the calculating machinery, not a part of the theory," but anything which massively changes the predictions should properly be understood to be part of the theory itself, especially when the raw version of the "theory" is meaningless without such additional assumptions.

Classical QED does also include "form factors" which may be interpreted as a way of giving the electron a nonzero radius; however, the form factors are not enough by themselves to make QED able to make predictions. Use of the δM mass renormalization is a core part of the very definition of classical QED.

If one is strictly conservative here, and one adheres as tightly to classical QED as possible without hiding from reality, one immediately sees a glaring gap in the discussion so far. What about gravity? If there really were a point of infinite negative mass within the electron, wouldn't that make every electron into a kind of black hole, in which gravity cannot be ignored? Preliminary calculations have been done, using valid semi-classical methods, suggesting that gravitational effects would substantially soften the problem with classical QED here [16]. Thanks to general relativity, it would be good enough to have a finite negative mass-energy concentrated in a point, in order to reproduce the mass renormalization of QED. If this were true, it would make it possible to prove that its true with experiments well within our technological and economic capability, which would allow us to bend space in remarkably interesting and useful ways. [16]. Because I truly believe in the scientific method, I would urge people to perform these experiments – not because I believe QED, but because the technology would be interesting if I am wrong, and because it would provide strong evidence for the other point of view, which I consider more likely.

The other point of view is that the true Hamiltonian is strictly nonnegative – or, more precisely, that we can rigorously replicate the successes of QED in predicting spectra with well-defined scattering predictions, consistent with experiment, by quantizing soliton models in which the Hamiltonian is positive definite. The most promising approach in my view is to start from the bosonic sector of electroweak theory (EWT), and make a small modification such that it generates solitons some of which possess electrical charge, only, unlike the previous example. I have proposed a variety of PDE systems to be explored by researchers in PDE and classical field theory [17,18]. In my view, the models which are most promising in the long-term are those which offer the possibility of *two* topological charges; the solitons with electrical charge can model the electron (and positron), while the others can offer us an alternative to the use of BPS monoples in grand unification physics [11]. For example, consider the following model:

$$L = L_W + L_B + L_{\varphi l} + L_{\varphi r} - V, \tag{6}$$

where the first two terms are the usual terms in electroweak theory (EWT) for the usual W and B fields (as on page 56 of Taylor [19]):

$$L_W = -\tfrac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu}, \; for \; F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + g W_\mu \wedge W_\nu \tag{7}$$

$$L_B = -\tfrac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial^\mu B^\nu - \partial^\nu B^\mu), \tag{8}$$

where the new "Higgs field" φ is a two-by-two complex matrix which transforms like a spinor/contravariant spinor under Lorentz transforms (i.e. a twistor [20]), and where

$$L_{\varphi l} = c_l \left| (\partial_\mu - \tfrac{i}{2} g \boldsymbol{\tau} \cdot W_\mu - \tfrac{i}{2} g' B_\mu) \varphi \right|^2 \tag{9}$$

$$L_{\varphi r} = c_r \left| (\partial_\mu - \tfrac{i}{2} g' B_\mu) \varphi^H \right|^2 \tag{10}$$

$$V = \lambda \left| I - \varphi^\dagger \varphi \right|^2 \tag{11}$$

where c_l, c_r and λ are parameters, and where I define:

$$|M|^2 = Tr(M^\dagger M), \tag{12}$$

However, such a model of the electron – like soliton models discussed in section 2.2 – requires us to go further, and study the question of how we would quantize the model. As in section 2.2, I propose that we quantize it as a bosonic field theory, derive rigorous properties of the bosonic field theory (without any requirement for renormalization), and derive QED as a limiting case as the radius of the electron goes to zero. In this view, electron-electron scattering involving lengths at about 3 femtometers would require getting past the QED approximation and more closely approximating the full bosonic theory.

3. Proposed Approach to Quantization

The discussion of section two suggests a five-step grand strategy for the development of axiomatic quantum field theory:

(1) the theory of spectra and bound states of bosonic quantum field theories, specifically to include theories which quantize "soliton" models, required to be mathematically well-defined without the need for renormalization or renormalization groups;

(2) proving that the scattering matrix S is well-defined in such field theories under reasonable suitable conditions;

(3) developing practical methods to calculate the scattering probabilities, especially for cases like deep electron-electron scattering [8] where QED is not sufficient in any case;

(4) proving that the limit of bosonic soliton models, as the radius of the soliton goes to zero, results in solitons whose behavior approaches that of a traditional point fermion;

(5) showing that the resulting scattering predictions approach those of QED, as the radius of the soliton goes to zero, in the case where the only detectable free field remaining in that limit is the electromagnetic field, or the B and W field of electroweak theory (EWT).

This paper will focus on steps one and two – the challenge of how to quantize bosonic quantum field theories. Here I propose that we continue to follow the usual canonical quantization procedure. More precisely, I propose that classical Lagrangian field theories in general should be quantized as bosonic fields, and that fermionic behavior should be derived later as an asymptotic, approximate statistical behavior of solitons. This paper focuses on the quantization part.

To begin with, I propose that we continue to follow the usual canonical quantization procedure, where we associate each classical field with the usual bosonic field operator, as in equation 5 and in [3,4]. However, as discussed in section 2.1, this is not enough to solve the problem of choice and to fully specify a quantum field theory. It does not specify a specific representation [1,2] of the system over which the canonical commutation relations (CCR) operate. Here I propose that we fill in the missing information, by deciding that the CCR operate over the subset of allowable density matrices ρ which do not contain negative probability states in the Glauber-Sudarshan P representation [21,22,23]. The next section will briefly review what that representation is, before I use it in what follows.

3.1. The Glauber-Sudarshan P Representation

The P representation is well-known in quantum optics. For example, Glauber recently received a Nobel Prize for his work using the P representation first developed by Sudarshan [22] in order to understand and use phenomena such as coherent light. Not having studied quantum optics in graduate school, I reinvented the same mapping [23] as part of my efforts to understand quantization in the more general case. The earlier work by Sudarshan mainly talked about optics, but the mathematics was quite general for any bosonic field.

Given any Lagrangian field theory with a Lagrangian of the form $L(\underline{\varphi}, \nabla\underline{\varphi}, \partial_t\underline{\varphi})$, it is well-known how to construct the corresponding Hamiltonian density $H(\underline{\varphi}, \underline{\pi})$. The state S(t) of a classical field theory at time t is specified by specifying the values of $\underline{\varphi}(t, \underline{x})$ and $\underline{\pi}(t, \underline{x})$ across all points $\underline{x}$ in three-dimensional

space. The P mapping maps any ensemble of classical states, S(t), into a corresponding density matrix ρ over bosonic Fock-Hilbert space as follows, in parallel with equations 103 through 107 of [23]:

$$\rho = \int \underline{w}(S)\underline{w}^{\dagger}(S)\Pr(S)d^{\infty}S, \tag{13}$$

where:

$$\underline{w}(S) = \frac{\underline{v}(S)}{|\underline{v}(S)|} \tag{14}$$

$$\underline{v}(S) = \exp\left(c\sum_{j=1}^{n}\int(\theta_j(\underline{p}) + i\tau_j(\underline{p}))a_j^{+}(\underline{p})d^3\underline{p}\right)|0\rangle \tag{15}$$

$$\theta_j(\underline{p}) = \sqrt{w_j(\underline{p})}\int e^{-i\underline{p}\cdot\underline{y}}\varphi_j(\underline{y})d^3\underline{y} \tag{16}$$

$$\tau_j(\underline{p}) = \frac{1}{\sqrt{w_j(\underline{p})}}\int e^{-i\underline{p}\cdot\underline{y}}\pi_j(\underline{y})d^3\underline{y} \tag{17}$$

$$w_j(\underline{p}) = \sqrt{m_j^2 + |\underline{p}|^2} \tag{18}$$

Note that a state of the corresponding quantum system is specified by specifying the density matrix ρ. Wave functions or vectors in Fock-Hilbert space like $\underline{\mathbf{w}}$ are not enough to specify the state of the system. The specific wave functions $\underline{\mathbf{w}}$ given by equation 14 are not the same as "pure states" in QFT; in quantum optics, they are called "coherent states."

The main practical value of the P mapping comes from the following remarkable property, which does not depend on the specific choice of Lagrangian. Suppose that we consider any function $g(\underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\pi}})$ of the classical fields $\underline{\boldsymbol{\varphi}}$ and $\underline{\boldsymbol{\pi}}$, where g is expressed as a polynomial in the components of $\underline{\boldsymbol{\varphi}}$ and $\underline{\boldsymbol{\pi}}$ or as the limit of a series of such polynomials. Suppose that we define the quantized version of that observable as $G_n(\underline{\boldsymbol{\Phi}}, \underline{\boldsymbol{\Pi}})$, which is derived from g by performing two substitutions: (1) we substitute the operators $\underline{\boldsymbol{\Phi}}$ and $\underline{\boldsymbol{\Pi}}$ for $\underline{\boldsymbol{\varphi}}$ and $\underline{\boldsymbol{\pi}}$, where $\underline{\boldsymbol{\Phi}}$ and $\underline{\boldsymbol{\Pi}}$ are defined exactly as in Weinberg [3]; we replace classical multiplication with the usual operator normal product. In that case it has been proven that:

$$\mathrm{Tr}(\rho G_n(\underline{\Phi},\underline{\Pi})) = \int g(\underline{\varphi},\underline{\pi})\Pr(S)d^{\infty}S \tag{19}$$

Of course, P maps from ensembles of classical states Pr(S) to density matrices ρ in Fock-Hilbert space. It is interesting to ask whether this mapping is invertible. Can we map back from any matrix ρ over Fock-Hilbert space to a corresponding ensemble of classical field states? Sudarshan [22] has actually shown that we can do so, but only if we allow for the possibility that "Pr(S)" may at times be negative for some states S.

The topic of negative probability states has spawned a great diversity of reactions, just as Haag's Theorem has. Perhaps it would do less violence to logic to simply admit negative probabilities than it does to admit probability amplitudes in the complex plane. Yet most physical situations in quantum optics do not seem to require such states. The rare exceptions (e.g. [24]) seem to involve intermediate states of very special dynamic scattering experiments. Thus I conjecture that we can capture the relevant physics by simply excluding such states in the analysis of allowable equilibrium bound states or spectra, and in the representation of incoming or outgoing particles (simple or composite) to scattering experiments. More precisely – in the approaches to scattering to be described in section 3.3, phenomena like Bell's Theorem

experiments do not require invoking negative probability states at intermediate times; "preselection" takes care of those unusual cases.

Recent work in quantum optics and quantum computing has shown that negative probability states are often not necessary, even in situations where they appear necessary for the usual P representation, if one uses a modified version of the P mapping [25,26]. Both the original P mapping and the modified mapping are equally valid ways of analyzing the original classical field theory; thus if one line of analysis should encounter difficulties, the other may provide a way to make things work.

It should be noted that the beautiful classical-quantum equivalence of equation 19 does *not* imply that we can rederive the usual Schrodinger equation from the dynamics of classical fields, in the general case. That appears to work in the case of electromagnetism, but not for general nonlinear bosonic field theories [23]. This is part of why I conjecture that this kind of bosonic field theory converges to QED only in the special limit, where the free bosonic field equations fall within that kind of special case. In that case, special systems like Bell's Theorem experiments or [24] can be explained as the result of what Aharonov calls "preselection phenomena," which appear paradoxical only if one does not account for such phenomena [7].

3.2 Spectra and Bound States for Bosonic Field Theory (BQFT)

The basic principle which needs to be proven here is that we can consistently derive the stable or bound states of any BQFT simply by looking at the eigenvectors and eigenvalues of $H = H_n(\underline{\mathbf{\Phi}}, \underline{\mathbf{\Pi}})$, the Hamiltonian operator integrated over all points $\underline{\mathbf{x}}$ in space. For empirical work with soliton models, we also need a way to calculate these states, hopefully without the need for perturbation theory or renormalization.

For the method of quantization proposed here, I conjecture that there is a kind of exact correspondence between the stable or bound states of the original classical field model and the corresponding eigenvectors of the Hamiltonian. Taking a different approach to quantization, based on functional integral formulations of QFT, the energy of solitons is typically higher than that of the corresponding classical versions [12]; however, in our approach, based on more traditional canonical quantization, the energy levels would be exactly the same, because of equation 19. Thus for example, according to this kind of quantization, the energy levels computed by Manton [13] using elaborate but classical PDE simulations yield exact quantum predictions of the energy levels for the models he assumes.

Of course, the coherent states $\underline{\mathbf{w}}(S)$ in equation 14 do not correspond exactly to the eigenvectors of any Hermitian operator, let alone to those of H. They are not orthogonal to each other. In a system which possesses soliton solutions, the wave function $\underline{\mathbf{w}}(T(\underline{\Delta \mathbf{x}})S)$, for the state $T(\underline{\Delta \mathbf{x}})S$ derived by translating the state S by some small displacement $\underline{\Delta \mathbf{x}}$, will certainly not be orthogonal to $\underline{\mathbf{w}}(S)$, is S is a localized "soliton" solution.

Thus to establish an exact correspondence, I propose that we augment section 3.1 by defining:

$$\underline{w}^*(S) = \alpha \int e^{-i\underline{p}\cdot\underline{\chi}} \underline{w}(\mathrm{T}(\underline{\chi})S) d^3\underline{\chi} \tag{20}$$

where $\underline{\mathbf{p}}$ is the usual total classical momentum of the state S and α is some kind of normalization factor, *and that* we perform a similar smearing over the rotated version of S (based on angular momentum), and that we smear even further for the special case of field theories which have additional local continuous conserved quantities, based on those quantities.

Of course, this kind of $\underline{\mathbf{w}}^*$ vector or wave function is not a coherent state. Yet $\underline{\mathbf{w}}^*(S)$ is well-defined for any state S, and it provides one way of referring to the ensemble of states generated by translating and rotating S. My conjecture is that this kind of smearing results in orthogonalizing the representation of different stable states.

I would then conjecture the validity of the following fundamental theorem for bosonic scattering. The theorem would be that there is an exact correspondence between the $\underline{\mathbf{w}}^*(S)$ vectors for states S which are variational in the classical energy H and the eigenvectors and eigenvalues of the quantum Hamiltonian H, to within whatever discrete constraints also apply. By "variational" I mean that the energy H of the classical state has a variation of zero for small perturbations of $\underline{\boldsymbol{\varphi}}$ and $\underline{\boldsymbol{\pi}}$, for perturbations which do not change discrete constraints such as the topological charge of the soliton. (In actuality, this mention of constraints is probably not necessary, since small perturbations cannot change topological charge [9,13];

however, the related Higgs boundary conditions almost certainly require that all of this must be formulated over tempered distributions rather than vectors in Hilbert space, to be proven in the relevant general case.)

Of course, we all know that eigenvectors and eigenvalues already have variational properties, which, combined with the equivalence in equation 19 and the orthogonality we derive from smearing, suggests that this really ought to work. Likewise, because of that equivalence, the eigenvectors should be well-defined with finite eigenvalues, whenever the corresponding classical field theory itself is well-behaved in that way.

Of course, the set of states which are variational in energy is a superset in general of the states which are local minima of energy. In field theories which do not support stable solitons, the variational states may represent saddle points and other states which do not possess the kind of stability we see for the electron. But for a model which is suitable for representing atomic energy levels, for example, as bound states, we need to consider a set of stable states which may be greater than the set of traditional solitons. For example, what if bound states actually involve some kind of stable motion or fluctuation? In that case, the entire set of states which the system fluctuates through may constitute a set of states which are lower in energy than all neighboring states which are not in that set. This is the underlying concept for a "chaoiton," as defined in [10].

In this view, the usual "metastable" excited states of atoms are actually strong local minima of energy, which would be strongly stable except for the possibility of emitting a large discrete bundle of light propagated to receivers at future time, receivers which are usually plentiful in our universe except when specially designed cavities (like Vertical Cavity Surface-Emitting Lasers) block the path, as described in Cavity QED and in [7].

Two classical states S and S' which have norms less than epsilon apart should result in density matrices ρ and ρ' which are also close, if the classical states meet basic conditions of boundedness. The "infraparticle problem"[5] seems to mirror a lack of match of norms in the direction back from ρ to S, which suggests that the use of different norms in Fock space might solve some of the problems in more traditional AQFT; however, that should not be necessary for the approach here.

3.3. Alternative Options for Specifying the Scattering Matrix

Theoretical papers should not be overly dogmatic in specifying specific choices, when more than one plausible choice exists and there is a need for computational and empirical work to do a comparative evaluation of them (or, in some cases, to understand unexpected equivalences).

One obvious approach to specifying the scattering matrix here is to go back to the original approach of Dyson and Heisenberg, and use the concept of "scattering states" which are a generalization of the eigenvector/eigenvalue approach of section 3.2. Many of the practical calculations of scattering are actually based on some use of time-independent perturbation theory to calculate states in which there is a steady stream of inputs and outputs, effectively smeared over time; these states are simply eigenstates of H.

After the fundamental theorem/conjecture of section 3.2 is proved, it would make sense to try to prove a generalization for ensemble states like what $\underline{\mathbf{w}}^*$ represents, but with multiple channels of inputs and outputs. In a variation of this, it is straightforward to modify equations 13 to 18 to define a new $P^{[4]}$ mapping which goes from possible states of the 4-dimensional Minkowski continuum to the four-dimensional kind of transparently relativistic Fock-Hilbert space discussed by Streater and Wightman [1]. (Some details of the latter approach are in my notes, but would add too much to the length of this letter.)

Because I believe in Occam's Razor, I would now tend to prefer an alternative specification of the scattering matrix, which is far simpler and far more radical. Following [7], I would propose that we encode the initial state information into an initial density matrix (basically by convolving the smeared states $\underline{\mathbf{w}}^*$ for the input channels); then use the reverse P mapping to derive an initial classical ensemble probability Pr(S); and then run that ensemble in forwards time, to generate an ensemble for the time of final measurement; and then throw out all possible classical states S at time t_+ which do not meet the final boundary conditions implied by our choice of what to measure at time t_+. Aharanov has aptly referred to this kind of situation as "preselection," and talked about experimental evidence for it above and beyond the discussion in [7]. As in section 3.2, the scattering matrix should be well-defined and unique so long as the related properties of the corresponding classical field theory are also well-defined and unique.

As I revise this paper, on June 4, 2012, I see a tentative indication that these two approaches might possibly be equivalent. See the Appendix.

4. Summary and Conclusions

A method has been proposed for quantizing any classical Lagrangian field theory into a bosonic quantum field theory. Considerable new work is needed to proved the key conjecture of section 3.2, and to better understand the properties of classical field theories like those discussed in section 2. Nevertheless, it now seems as if there is substantial hope to reconcile the key goals of underlying simplicity, mathematical clarity and the requirements of physical and empirical realism. I would like to express deepest thanks to the spirit of Von Neumann, whose prior work on these issues (e.g. [27]) was the ultimate source on inspiration for this work and who, of course, could do it all much better. And likewise to my wife, whose spirit and mathematical abilities are also beyond the scope of this letter.

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Appendix. A Very Tentative Hypothesis About Scattering

Canonical quantum field theory [3,4] in the usual Schrodinger picture proposes that we encode our knowledge of any physical system at any time t into a wave function ψ or density matrix ρ defined over a Fock-Hilbert space representing particle coordinates over three-dimensional space. The usual Schrodinger equation over Fock-Hilbert space does not fulfill the spirit of special relativity, because the treatment of space appears radically different from the treatment of time.

Streater and Wightman (reviewed in [1]) propose a different formulation, in which wave functions are defined over four-dimensional space-time, treating time and space on a transparently equal footing. This requires the use of creation and annihilation operators which still follow the usual algebra, except that they range over 3+1-dimensional Minkowski space instead of three-dimensional space.

Following that formulation, we may define an extended mapping:

$$\rho^{[4]} = \int \underline{w}'(S')\underline{w}'^{\dagger}(S')\Pr(S')d^{\infty}S', \tag{21}$$

where S' refers to a state of the fields $\underline{\varphi}$ and $\underline{\pi}$ across all space-time, and where:

$$\underline{w}'(S') = \frac{\underline{v}'(S')}{\left|\underline{v}'(S')\right|} \tag{22}$$

$$\underline{v}'(S') = \exp\left(c\sum_{j=1}^{n}\int(\varphi_j(x_\mu) + i\pi_j(x_\mu))a_j^{+}(x_\mu)d^4x_\mu\right)|0\rangle \tag{23}$$

This results in the same relation as in equation 19, but extended to functions of $\underline{\varphi}$ and $\underline{\pi}$ across different points in space-time. One may also defined a smeared state, w'*(S'):

$$\underline{w}'^{*}(S') = \alpha\int e^{-ip_\mu\chi^\mu}\,\underline{w}'(\mathrm{T}(\chi^\mu)S')d^4\chi^\mu \tag{24}$$

For the semiclassical version of scattering, we seek to calculate an ensemble of states S' which obey the classical dynamics and meet the boundary conditions. Obeying the classical dynamics simply means that every state of the ensemble is a variational state of the total Lagrangian over space-time. Meeting the usual boundary conditions means that each of the n input channels is an ensemble of classical states represented

by a smeared state, which corresponds to a eigenstate of H in our analysis of spectra, and that we discard all states which are inconsistent with the final choice of measurement operator (as in "preselection" [7]).

This hints (but does not at all prove as yet) that the "scattering states" in 3+1 dimensions, as discussed by Streater and Wightman, correspond to the smeared states w'. More precisely, the usual eigenstates of H in 3-dimensional Fock Hilbert space obeying the usual boundary conditions may correspond to eigenstates of the total Lagrangian operator over 4-dimensional Fock-Hilbert space, after smearing. Requiring that an ensemble of space be variational in the classical Lagrangian over 3+1-dimensional space time may be equivalent to making the corresponding w'* an eigenfunction of the total Lagrangian quantized in the style of Streater and Wightman over 3+1-dimensional space-time, at least for states which are localized in an interaction zone as in standard scattering analysis. Nevertheless [22], this does not imply equivalence of either of these two to the usual Schrodinger dynamics.