Saint-Venant's Principe of the " Cavity in Cylinder " Problem

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Abstract

The problem of a cylinder with a small spherical cavity loaded by an equilibrium system of forces is suggested and discussed and its formulation of Saint-Venant's Principle is established. It is evident that finding solutions of boundary-value problems is a precise and pertinent approach to establish Saint-Venant type decay of elastic problems.

Keywords : Saint-Venant's Principe, proof, provability, solution, decay, formulation, cavity

AMS Subject Classifications: 74-02, 74G50

1 Introduction

Saint-Venant's Principle is essential and fundamental in Elasticity (See Ref.[1] and Ref.[2]). Boussinesq and Love announce statements of Saint-Venant's Principe (See Ref.[3] and Ref.[4]), but Mises and Sternberg argue, by citing counterexamples, that the statements are not clear, suggesting that Saint-Venant's Principle should be proved or given a mathematical formulation (See Ref.[5] and Ref.[6]). Truesdell asserts that if Saint-Venant's Principle of equipollent loads is true, it must be a mathematical consequence of the general equations of Linear Elasticity (See Ref.[7]).

There is no doubt that mathematical proof of Saint-Venant's Principle has become an academic attraction for contributors and much effort has been made for exploring its mysterious implications or deciphering its puzzle. Zanaboni "proved" a theorem trying to concern Saint-Venant's Principle in terms of work and energy (See Refs.[8],[9],[10]). However, Zhao argues that Zanaboni's theorem is false (See Ref.[11]). The work published by Toupin cites more counterexamples to explain that Love's statement is false, and then establishes a formulation of energy decay, which is considered as "a precise mathematical formulation and proof" of Saint-Venant's Principle for the elastic cylinder (See Refs.[12], [13]). Furthermore, Toupin's work seems to set up an example followed by a large number of papers to establish Toupin-type energy decay formulae for branches of continuum mechanics. Since 1965 the concept of energy decay suggested by Toupin has been widely accepted by authors, and various techniques have been developed to construct inequalities of Toupin-type decay of energy which are spread widely in continuum mechanics. Especially, the theorem given by Berdichevskii is considered as a generalization of Toupin's theorem (See Ref.[14]). However, Zhao points out that Toupin's theory is not a strict mathematical proof, and Toupin's Theorem is not an exact mathematical formulation, of Saint-Venant's Principle. Interestingly and significantly, Saint-Venant's Principle stated by Love is disproved mathematically from Toupin's Theorem, so Toupin's Theorem is mathematically inconsistent with Saint-Venant's Principle (See Ref.[11]).

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Boussinesq and Love and points out mathematically that Special Saint-Venant's Principle or Modified Saint-Venant's Principle can be proved or formulated though Saint-Venant's Principle in its general form stated by Boussinesq or Love is not true (See Ref.[11]).

In this paper we suggest the problem of a cylinder with a small spherical cavity loaded by an axisymmetric equilibrium system of forces and prove its Special Saint-Venant's Principle.

2 Love's Statement of Saint-Venant's Principle and Its Provability

Love's Statement : "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." (See Ref.[4])

Mises considers a body of finite dimensions and sees that " Saint-Venant's principle in its traditional form does not hold true." (See Ref.[5]) Toupin asserts: " The broader statement of the principle given by Love for bodies of arbitrary shape cannot possibly be true... " (See Ref.[12]) He also refers to counter-examples and argues that " Confronted with such a collection of counter-examples of the now traditional statement of the Saint-Venant's principle, one must agree that something is wrong with it." (See Ref.[13])

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Love, but argues by mathematical analysis that Saint-Venant's decay of strains (then stresses) described by Love's statement can be proved true by special formulating or adding supplementary conditions to the problems discussed

(See Ref.[11]).

3 “ Cavity in Cylinder ” Problem

Now we suggest the “ Cavity in Cylinder ” Problem. Suppose that a small cavity in an infinite cylinder is loaded by an axisymmetric equilibrium system of forces, otherwise the cylinder would be free. Thus the stress function should satisfy the biharmonic equation

$$(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\cot\psi\frac{\partial}{\partial\psi}+\frac{1}{R^2}\frac{\partial^2}{\partial\psi^2})$$
$$(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\cot\psi\frac{\partial}{\partial\psi}+\frac{1}{R^2}\frac{\partial^2}{\partial\psi^2})\phi(R,\psi)=0. \tag{1}$$

The boundary condition on $R = b$ is

$$R=b: \quad \lim_{b\to\infty}\sigma_R=0, \quad \lim_{b\to\infty}\tau_{R\psi}=0. \tag{2}$$

The boundary condition on the cavity $R = a$ is described by

$$R=a: \quad the\ axisymmetric\ equilibrium\ system\ of\ forces. \tag{3}$$

The relationship between the stresses and the stress function is:

$$\begin{aligned}
\sigma_r &= \frac{\partial}{\partial z}[\nu\nabla^2\phi-\frac{\partial^2\phi}{\partial r^2}],\\
\sigma_\theta &= \frac{\partial}{\partial z}[\nu\nabla^2\phi-\frac{1}{r}\frac{\partial\phi}{\partial r}],\\
\sigma_z &= \frac{\partial}{\partial z}[(2-\nu)\nabla^2\phi-\frac{\partial^2\phi}{\partial z^2}],\\
\tau_{rz} &= \frac{\partial}{\partial r}[(1-\nu)\nabla^2\phi-\frac{\partial^2\phi}{\partial z^2}].
\end{aligned} \tag{4}$$

4 Stress Function of “ Cavity in Cylinder ” Problem

Considering Eq.(2), the boundary condition on $R = b$, the stress function is formulated as

$$\phi(R,\psi)=\sum_{n=0}^{\infty}A_nR^{-(n+1)}P_n(\cos\psi)+\sum_{n=-2}^{\infty}B_nR^{-(n+1)}P_{n+2}(\cos\psi) \tag{5}$$

where $P_n(\cos\psi)$ and $P_{n+2}(\cos\psi)$ are Legendre functions.

As is well known, Eq. (5) satisfies the biharmonic equation Eq.(1).

5 Stresses of "Cavity in Cylinder" Problem

By virtue of recursion formula

$$(x^2-1)P_n'(x) = nxP_n(x) - nP_{n-1}(x), \tag{6}$$

we find, from Eq.(5), that

$$\begin{aligned}
\sigma_r(R,\psi) &= \frac{\partial}{\partial z}[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}] \\
&= \frac{3A_0}{R^4}[-\cos\psi + 5\sin^2\psi\cos\psi] \\
&+ \frac{3A_1}{R^5}[-4 + 35\sin^2\psi\cos^2\psi] \\
&+ \frac{5A_2}{R^6}[-4\cos\psi - 14\sin^2\psi\cos\psi + 63\sin^2\psi\cos^3\psi] \\
&+ \frac{B_{-2}}{R^2}[(1-2\nu)\cos\psi - 3\sin^2\psi\cos\psi] \\
&+ \frac{B_{-1}}{R^3}[-2(1+\nu) + 6\nu\cos^2\psi + 15\sin^2\psi\cos^2\psi] \\
&+ \frac{B_0}{R^4}[(7-18\nu)\cos\psi - 15(1-2\nu)\cos^3\psi - 35\sin^2\psi\cos\psi \\
&+ 105\sin^2\psi\cos^3\psi] \\
&+ \sum_{n=3}^{\infty} A_n R^{-(n+4)}\{[2n\cot\psi\csc\psi + (2n^2+3n-3)\cos\psi \\
&- 3n(2n+1)\cos\psi\cot^2\psi \\
&+ (n+1)(n+3)(2n+5)\sin^2\psi\cos\psi - 2n(n+2)(2n+5)\cos^3\psi \\
&+ n^2(2n+5)\cos^3\psi\cot^2\psi]P_n(\cos\psi) - [n(n+2) + n\csc^2\psi \\
&- n(9n-2)\cot^2\psi \\
&+ n(n+1)(n+3)\sin^2\psi - n(6n^2+16n+9)\cos^2\psi \\
&+ n(5n^2+4n-3)\cos^2\psi\cot^2\psi]P_{n-1}(\cos\psi) \\
&- [3n(n-1)\cot\psi\csc\psi + (n-1)n(2n+3)\cos\psi \\
&- 4(n-1)n^2\cos\psi\cot^2\psi]P_{n-2}(\cos\psi) \\
&- (n-2)(n-1)n\cot^2\psi P_{n-3}(\cos\psi)\} \\
&+ \sum_{n=1}^{\infty} B_n R^{-(n+4)}\{[(2n^2+5n-1)\cos\psi \\
&+ 2(2n+3)(2n+5)\nu\cos\psi + 2n\cot\psi\csc\psi
\end{aligned}$$

$$
\begin{aligned}
&- && n(6n+13)\cos\psi\cot^2\psi + (n+1)(n+3)(2n+7)\sin^2\psi\cos\psi \\
&- && 2(2n+7)(n^2+3n+1)\cos^3\psi \\
&+ && n(n+2)(2n+7)\cos^3\psi\cot^2\psi]P_{n+2}(\cos\psi) \\
&- && [n(n+2)+2(n+2)(2n+3)\nu + n\csc^2\psi - n(9n+14)\cot^2\psi \\
&+ && (n+1)(n+2)(n+3)\sin^2\psi \\
&- && (n+2)(6n^2+18n+7)\cos^2\psi + n(5n^2 \\
&+ && 2n+19)\cos^2\psi\cot^2\psi]P_{n+1}(\cos\psi) \\
&- && [3n(n+1)\cot\psi\csc\psi + (n+1)(n+2)(2n+1)\cos\psi \\
&- && 2n(n+1)(2n+3)\cos\psi\cot^2\psi]P_n(\cos\psi) \\
&- && n^2(n+1)\cot^2\psi P_{n-1}(\cos\psi)\},
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\sigma_\theta(R,\psi) &= \frac{\partial}{\partial z}[\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}] \\
&= -\frac{3A_0}{R^4}\cos\psi \\
&+ \frac{3A_1}{R^5}[1-5\cos^2\psi] \\
&+ \frac{5A_2}{R^6}[3\cos\psi - 7\cos^3\psi] \\
&+ \frac{B_{-2}}{R^2}(1-2\nu)\cos\psi \\
&+ \frac{B_{-1}}{R^3}[(1-2\nu) - 3(1-2\nu)cos^2\psi] \\
&+ \frac{B_0}{R^4}[(7-18\nu)\cos\psi - 15(1-2\nu)\cos^3\psi] \\
&+ \sum_{n=3}^{\infty} A_n R^{-(n+4)}\{[-(n+1)(2n+3)\cos\psi \\
&- 2n\cot\psi\csc\psi + n(2n+3)\cos\psi\cot^2\psi]P_n(\cos\psi) \\
&+ [n(n+1) + n\csc^2\psi - n(3n+1)\cot^2\psi]P_{n-1}(\cos\psi) \\
&+ n(n-1)\cot\psi\csc\psi P_{n-2}(\cos\psi)\} \\
&+ \sum_{n=1}^{\infty} B_n R^{-(n+4)}\{[-(n+1)(2n+5)\cos\psi \\
&+ 2(2n+3)(2n+5)\nu\cos\psi + n(2n+5)\cos\psi\cot^2\psi \\
&- 2n\cot\psi\csc\psi]P_{n+2}(\cos\psi) + [(n+1)(n+2) \\
&- 2(n+2)(2n+3)\nu + n\csc^2\psi \\
&- n(3n+5)\cot^2\psi]P_{n+1}(\cos\psi) \\
&+ n(n+1)\cot\psi\csc\psi P_n(\cos\psi)\},
\end{aligned}
$$

(8)

$$
\begin{aligned}
\sigma_z(R,\psi) &= \frac{\partial}{\partial z}[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}] \\
&= \frac{3A_0}{R^4}[-3\cos\psi + 5\cos^3\psi] \\
&+ \frac{3A_1}{R^5}[3 - 30\cos^2\psi + 35\cos^4\psi] \\
&+ \frac{5A_2}{R^6}[15\cos\psi - 70\cos^3\psi + 63\cos^5\psi] \\
&+ \frac{B_{-2}}{R^2}[-(1-2\nu)\cos\psi - 3\cos^3\psi] \\
&+ \frac{B_{-1}}{R^3}[-(1-2\nu) - 6(1+\nu)cos^2\psi + 15\cos^4\psi] \\
&+ \frac{B_0}{R^4}[(3+18\nu)\cos\psi - (80+30\nu)\cos^3\psi + 105\cos^5\psi] \\
&+ \sum_{n=3}^{\infty} A_n R^{-(n+4)}\{[(8n+9)(2n+5)\cos^3\psi \\
&- (4n^2+24n+21)\cos\psi]P_n(\cos\psi) \\
&+ [3n(2n+1) - n(8n^2+24n+15)\cos^2\psi]P_{n-1}(\cos\psi) \\
&+ 3n(n-1)(2n+1)\cos\psi P_{n-2}(\cos\psi) \\
&- n(n-1)(n-2)P_{n-3}(\cos\psi)\} \\
&+ \sum_{n=1}^{\infty} B_n R^{-(n+4)}\{[(2n+3)(2n+5)(2n+7)\cos^3\psi \\
&+ (2n+3)(2n+5)(1-2\nu)\cos\psi]P_{n+2}(\cos\psi) \\
&+ [-(n+2)(2n+3)(1-2\nu) - (n+2)(2n+3)(6n+15)\cos^2\psi]P_{n+1}(\cos\psi) \\
&+ 3(n+1)(n+2)(2n+3)\cos\psi P_n(\cos\psi) - n(n+1)(n+2)P_{n-1}(\cos\psi)\},
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\tau_{rz}(R,\psi) &= \frac{\partial}{\partial r}[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}] \\
&= \frac{3A_0}{R^4}[-\sin\psi + 5\sin\psi\cos^2\psi] \\
&+ \frac{15A_1}{R^5}[-3\sin\psi\cos\psi + 7\sin\psi\cos^3\psi] \\
&+ \frac{5A_2}{R^6}[3\sin\psi - 42\sin\psi\cos^2\psi + 63\sin\psi\cos^4\psi] \\
&+ \frac{B_{-2}}{R^2}[-(1-2\nu)\sin\psi - 3\sin\psi\cos^2\psi]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{B_{-1}}{R^3}[-3(1+2\nu)\sin\psi\cos\psi + 15\sin\psi\cos^3\psi] \\
&+ \quad \frac{B_0}{R^4}[(1+6\nu)\sin\psi - (50+30\nu)\sin\psi\cos^2\psi \\
&+ \quad 105\sin\psi\cos^4\psi] \\
&+ \quad \sum_{n=3}^{\infty} A_n R^{-(n+4)}\{[n(2n+1)\cos\psi\cot\psi \\
&- \quad (n+3)(2n+1)\sin\psi \\
&+ \quad (n+5)(8n+9)\sin\psi\cos^2\psi - n(8n+9)\cos^3\psi\cot\psi]P_n(\cos\psi) \\
&- \quad [n(2n+1)\cot\psi \\
&+ \quad 2n(n+4)(2n+1)\sin\psi\cos\psi - n(4n^2 \\
&+ \quad 6n+7)\cos^2\psi\cot\psi]P_{n-1}(\cos\psi) \\
&+ \quad [n(n-1)(n+3)\sin\psi - 5n^2(n-1)\cos\psi\cot\psi]P_{n-2}(\cos\psi) \\
&+ \quad (n-2)(n-1)n\cot\psi P_{n-3}(\cos\psi)\} \\
&+ \quad \sum_{n=1}^{\infty} B_n R^{-(n+4)}\{[(n+3)(2n+3)(1-2\nu)\sin\psi \\
&- \quad (n+2)(2n+3)(1-2\nu)\cos\psi\cot\psi \\
&+ \quad (n+5)(2n+3)(2n+5)\sin\psi\cos^2\psi \\
&- \quad (n+2)(2n+3)(2n+5)\cos^3\psi\cot\psi]P_{n+2}(\cos\psi) \\
&- \quad [-(n+2)(2n+3)(1-2\nu)\cot\psi + 2(n+2)(n+4)(2n+3)\sin\psi\cos\psi \\
&- \quad (n+2)(2n+3)(4n+7)\cos^2\psi\cot\psi]P_{n+1}(\cos\psi) \\
&+ \quad [(n+1)(n+2)(n+3)\sin\psi \\
&- \quad (n+1)(n+2)(5n+6)\cos\psi\cot\psi]P_n(\cos\psi) \\
&+ \quad n(n+1)(n+2)\cot\psi P_{n-1}(\cos\psi)\}.
\end{aligned} \tag{10}
$$

Then stresses in spherical coordinates , $\sigma_R(R,\psi)$ and $\tau_{R\psi}(R,\psi)$, are determined by

$$
\begin{aligned}
\sigma_R(R,\psi) &= \sigma_r(R,\psi)\sin^2\psi + \sigma_z(R,\psi)\cos^2\psi + 2\tau_{rz}(R,\psi)\sin\psi\cos\psi, \\
\tau_{R\psi}(R,\psi) &= [\sigma_r(R,\psi) - \sigma_z(R,\psi)]\sin\psi\cos\psi - \tau_{rz}(R,\psi)(\sin^2\psi - \cos^2\psi)
\end{aligned} \tag{11}
$$

where $\sigma_r(R,\psi)$, $\sigma_z(R,\psi)$ and $\tau_{rz}(R,\psi)$ are given by Eq.(7), Eq.(9) and Eq.(10).

Stresses $\sigma_R(R,\psi)$ and $\tau_{R\psi}(R,\psi)$ in Eq.(11) must satisfy Eq.(2) so that Eq.(7), Eq.(9) and Eq.(10) are modified into

$$
\sigma_r(R,\psi) = \frac{3A_0}{R^4}[-\cos\psi + 5\sin^2\psi\cos\psi]
$$

$$
\begin{aligned}
&+ \frac{3A_1}{R^5}[-4 + 35\sin^2\psi\cos^2\psi] \\
&+ \frac{5A_2}{R^6}[-4\cos\psi - 14\sin^2\psi\cos\psi + 63\sin^2\psi\cos^3\psi] \\
&+ \frac{B_{-2}}{R^2}[(1-2\nu)\cos\psi - 3\sin^2\psi\cos\psi] \\
&+ \frac{B_{-1}}{R^3}[-2(1+\nu) + 6\nu\cos^2\psi + 15\sin^2\psi\cos^2\psi] \\
&+ \frac{B_0}{R^4}[(7-18\nu)\cos\psi - 15(1-2\nu)\cos^3\psi - 35\sin^2\psi\cos\psi \\
&+ 105\sin^2\psi\cos^3\psi] \\
&+ \sum_{n=3}^{N_1} A_n R^{-(n+4)}\{[2n\cot\psi\csc\psi + (2n^2+3n-3)\cos\psi \\
&- 3n(2n+1)\cos\psi\cot^2\psi \\
&+ (n+1)(n+3)(2n+5)\sin^2\psi\cos\psi - 2n(n+2)(2n+5)\cos^3\psi \\
&+ n^2(2n+5)\cos^3\psi\cot^2\psi]P_n(\cos\psi) - [n(n+2) + n\csc^2\psi \\
&- n(9n-2)\cot^2\psi \\
&+ n(n+1)(n+3)\sin^2\psi - n(6n^2+16n+9)\cos^2\psi \\
&+ n(5n^2+4n-3)\cos^2\psi\cot^2\psi]P_{n-1}(\cos\psi) \\
&- [3n(n-1)\cot\psi\csc\psi + (n-1)n(2n+3)\cos\psi \\
&- 4(n-1)n^2\cos\psi\cot^2\psi]P_{n-2}(\cos\psi) \\
&- (n-2)(n-1)n\cot^2\psi P_{n-3}(\cos\psi)\} \\
&+ \sum_{n=1}^{N_2} B_n R^{-(n+4)}\{[(2n^2+5n-1)\cos\psi \\
&+ 2(2n+3)(2n+5)\nu\cos\psi + 2n\cot\psi\csc\psi \\
&- n(6n+13)\cos\psi\cot^2\psi + (n+1)(n+3)(2n+7)\sin^2\psi\cos\psi \\
&- 2(2n+7)(n^2+3n+1)\cos^3\psi \\
&+ n(n+2)(2n+7)\cos^3\psi\cot^2\psi]P_{n+2}(\cos\psi) \\
&- [n(n+2) + 2(n+2)(2n+3)\nu + n\csc^2\psi - n(9n+14)\cot^2\psi \\
&+ (n+1)(n+2)(n+3)\sin^2\psi \\
&- (n+2)(6n^2+18n+7)\cos^2\psi + n(5n^2 \\
&+ 2n+19)\cos^2\psi\cot^2\psi]P_{n+1}(\cos\psi) \\
&- [3n(n+1)\cot\psi\csc\psi + (n+1)(n+2)(2n+1)\cos\psi \\
&- 2n(n+1)(2n+3)\cos\psi\cot^2\psi]P_n(\cos\psi) \\
&- n^2(n+1)\cot^2\psi P_{n-1}(\cos\psi)\},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\sigma_z(R,\psi) &= \frac{3A_0}{R^4}[-3\cos\psi + 5\cos^3\psi] \\
&+ \frac{3A_1}{R^5}[3 - 30\cos^2\psi + 35\cos^4\psi] \\
&+ \frac{5A_2}{R^6}[15\cos\psi - 70\cos^3\psi + 63\cos^5\psi] \\
&+ \frac{B_{-2}}{R^2}[-(1-2\nu)\cos\psi - 3\cos^3\psi] \\
&+ \frac{B_{-1}}{R^3}[-(1-2\nu) - 6(1+\nu)cos^2\psi + 15\cos^4\psi] \\
&+ \frac{B_0}{R^4}[(3+18\nu)\cos\psi - (80+30\nu)\cos^3\psi + 105\cos^5\psi] \\
&+ \sum_{n=3}^{N_1} A_n R^{-(n+4)}\{[(8n+9)(2n+5)\cos^3\psi \\
&- (4n^2+24n+21)\cos\psi]P_n(\cos\psi) \\
&+ [3n(2n+1) - n(8n^2+24n+15)\cos^2\psi]P_{n-1}(\cos\psi) \\
&+ 3n(n-1)(2n+1)\cos\psi P_{n-2}(\cos\psi) \\
&- n(n-1)(n-2)P_{n-3}(\cos\psi)\} \\
&+ \sum_{n=1}^{N_2} B_n R^{-(n+4)}\{[(2n+3)(2n+5)(2n+7)\cos^3\psi \\
&+ (2n+3)(2n+5)(1-2\nu)\cos\psi]P_{n+2}(\cos\psi) \\
&+ [-(n+2)(2n+3)(1-2\nu) - (n+2)(2n+3)(6n+15)\cos^2\psi]P_{n+1}(\cos\psi) \\
&+ 3(n+1)(n+2)(2n+3)\cos\psi P_n(\cos\psi) - n(n+1)(n+2)P_{n-1}(\cos\psi)\},
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\tau_{rz}(R,\psi) &= \frac{3A_0}{R^4}[-\sin\psi + 5\sin\psi\cos^2\psi] \\
&+ \frac{15A_1}{R^5}[-3\sin\psi\cos\psi + 7\sin\psi\cos^3\psi] \\
&+ \frac{5A_2}{R^6}[3\sin\psi - 42\sin\psi\cos^2\psi + 63\sin\psi\cos^4\psi] \\
&+ \frac{B_{-2}}{R^2}[-(1-2\nu)\sin\psi - 3\sin\psi\cos^2\psi] \\
&+ \frac{B_{-1}}{R^3}[-3(1+2\nu)\sin\psi\cos\psi + 15\sin\psi\cos^3\psi] \\
&+ \frac{B_0}{R^4}[(1+6\nu)\sin\psi - (50+30\nu)\sin\psi\cos^2\psi \\
&+ 105\sin\psi\cos^4\psi]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=3}^{N_1} A_n R^{-(n+4)}\{[n(2n+1)\cos\psi\cot\psi \\
&- (n+3)(2n+1)\sin\psi \\
&+ (n+5)(8n+9)\sin\psi\cos^2\psi - n(8n+9)\cos^3\psi\cot\psi]P_n(\cos\psi) \\
&- [n(2n+1)\cot\psi \\
&+ 2n(n+4)(2n+1)\sin\psi\cos\psi - n(4n^2 \\
&+ 6n+7)\cos^2\psi\cot\psi]P_{n-1}(\cos\psi) \\
&+ [n(n-1)(n+3)\sin\psi - 5n^2(n-1)\cos\psi\cot\psi]P_{n-2}(\cos\psi) \\
&+ (n-2)(n-1)n\cot\psi P_{n-3}(\cos\psi)\} \\
&+ \sum_{n=1}^{N_2} B_n R^{-(n+4)}\{[(n+3)(2n+3)(1-2\nu)\sin\psi \\
&- (n+2)(2n+3)(1-2\nu)\cos\psi\cot\psi \\
&+ (n+5)(2n+3)(2n+5)\sin\psi\cos^2\psi \\
&- (n+2)(2n+3)(2n+5)\cos^3\psi\cot\psi]P_{n+2}(\cos\psi) \\
&- [-(n+2)(2n+3)(1-2\nu)\cot\psi + 2(n+2)(n+4)(2n+3)\sin\psi\cos\psi \\
&- (n+2)(2n+3)(4n+7)\cos^2\psi\cot\psi]P_{n+1}(\cos\psi) \\
&+ [(n+1)(n+2)(n+3)\sin\psi \\
&- (n+1)(n+2)(5n+6)\cos\psi\cot\psi]P_n(\cos\psi) \\
&+ n(n+1)(n+2)\cot\psi P_{n-1}(\cos\psi)\},
\end{aligned} \tag{14}
$$

where N_1 and N_2 are finite positive integers , that is,

$$
3 \le N_1 < \infty, \quad 1 \le N_2 < \infty. \tag{15}
$$

Then the stress $\sigma_\theta(R,\psi)$ should be

$$
\begin{aligned}
\sigma_\theta(R,\psi) &= -\frac{3A_0}{R^4}\cos\psi \\
&+ \frac{3A_1}{R^5}[1-5\cos^2\psi] \\
&+ \frac{5A_2}{R^6}[3\cos\psi - 7\cos^3\psi] \\
&+ \frac{B_{-2}}{R^2}(1-2\nu)\cos\psi \\
&+ \frac{B_{-1}}{R^3}[(1-2\nu) - 3(1-2\nu)cos^2\psi] \\
&+ \frac{B_0}{R^4}[(7-18\nu)\cos\psi - 15(1-2\nu)\cos^3\psi]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=3}^{N_1} A_n R^{-(n+4)}\{[-(n+1)(2n+3)\cos\psi \\
&- 2n\cot\psi\csc\psi + n(2n+3)\cos\psi\cot^2\psi]P_n(\cos\psi) \\
&+ [n(n+1) + n\csc^2\psi - n(3n+1)\cot^2\psi]P_{n-1}(\cos\psi) \\
&+ n(n-1)\cot\psi\csc\psi P_{n-2}(\cos\psi)\} \\
&+ \sum_{n=1}^{N_2} B_n R^{-(n+4)}\{[-(n+1)(2n+5)\cos\psi \\
&+ 2(2n+3)(2n+5)\nu\cos\psi + n(2n+5)\cos\psi\cot^2\psi \\
&- 2n\cot\psi\csc\psi]P_{n+2}(\cos\psi) + [(n+1)(n+2) \\
&- 2(n+2)(2n+3)\nu + n\csc^2\psi \\
&- n(3n+5)\cot^2\psi]P_{n+1}(\cos\psi) \\
&+ n(n+1)\cot\psi\csc\psi P_n(\cos\psi)\},
\end{aligned}
\tag{16}
$$

from Eq.(8).

6 The Axisymmetric Equilibrium System of Forces on the Cavity

From Eq.(12), Eq.(13), Eq.(14) and Eq.(11), the axisymmetric equilibrium system of forces on the cavity ($R = a$) is expressed by

$$
\begin{aligned}
&R = a : \\
\sigma_R(a,\psi) &= \sigma_r(a,\psi)\sin^2\psi + \sigma_z(a,\psi)\cos^2\psi + 2\tau_{rz}(a,\psi)\sin\psi\cos\psi, \\
\tau_{R\psi}(a,\psi) &= [\sigma_r(a,\psi) - \sigma_z(a,\psi)]\sin\psi\cos\psi - \tau_{rz}(a,\psi)(\sin^2\psi - \cos^2\psi)
\end{aligned}
\tag{17}
$$

where $\sigma_r(a,\psi)$, $\sigma_z(a,\psi)$ and $\tau_{rz}(a,\psi)$ are given by

$$
\begin{aligned}
\sigma_r(a,\psi) &= \frac{3A_0}{a^4}[-\cos\psi + 5\sin^2\psi\cos\psi] \\
&+ \frac{3A_1}{a^5}[-4 + 35\sin^2\psi\cos^2\psi] \\
&+ \frac{5A_2}{a^6}[-4\cos\psi - 14\sin^2\psi\cos\psi + 63\sin^2\psi\cos^3\psi] \\
&+ \frac{B_{-2}}{a^2}[(1-2\nu)\cos\psi - 3\sin^2\psi\cos\psi] \\
&+ \frac{B_{-1}}{a^3}[-2(1+\nu) + 6\nu\cos^2\psi + 15\sin^2\psi\cos^2\psi]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{B_0}{a^4}[(7-18\nu)\cos\psi - 15(1-2\nu)\cos^3\psi - 35\sin^2\psi\cos\psi \\
&+ \quad 105\sin^2\psi\cos^3\psi] \\
&+ \quad \sum_{n=3}^{N_1} A_n a^{-(n+4)}\{[2n\cot\psi\csc\psi + (2n^2+3n-3)\cos\psi \\
&- \quad 3n(2n+1)\cos\psi\cot^2\psi \\
&+ \quad (n+1)(n+3)(2n+5)\sin^2\psi\cos\psi - 2n(n+2)(2n+5)\cos^3\psi \\
&+ \quad n^2(2n+5)\cos^3\psi\cot^2\psi]P_n(\cos\psi) - [n(n+2) + n\csc^2\psi \\
&- \quad n(9n-2)\cot^2\psi \\
&+ \quad n(n+1)(n+3)\sin^2\psi - n(6n^2+16n+9)\cos^2\psi \\
&+ \quad n(5n^2+4n-3)\cos^2\psi\cot^2\psi]P_{n-1}(\cos\psi) \\
&- \quad [3n(n-1)\cot\psi\csc\psi + (n-1)n(2n+3)\cos\psi \\
&- \quad 4(n-1)n^2\cos\psi\cot^2\psi]P_{n-2}(\cos\psi) \\
&- \quad (n-2)(n-1)n\cot^2\psi P_{n-3}(\cos\psi)\} \\
&+ \quad \sum_{n=1}^{N_2} B_n a^{-(n+4)}\{[(2n^2+5n-1)\cos\psi \\
&+ \quad 2(2n+3)(2n+5)\nu\cos\psi + 2n\cot\psi\csc\psi \\
&- \quad n(6n+13)\cos\psi\cot^2\psi + (n+1)(n+3)(2n+7)\sin^2\psi\cos\psi \\
&- \quad 2(2n+7)(n^2+3n+1)\cos^3\psi \\
&+ \quad n(n+2)(2n+7)\cos^3\psi\cot^2\psi]P_{n+2}(\cos\psi) \\
&- \quad [n(n+2) + 2(n+2)(2n+3)\nu + n\csc^2\psi - n(9n+14)\cot^2\psi \\
&+ \quad (n+1)(n+2)(n+3)\sin^2\psi \\
&- \quad (n+2)(6n^2+18n+7)\cos^2\psi + n(5n^2 \\
&+ \quad 2n+19)\cos^2\psi\cot^2\psi]P_{n+1}(\cos\psi) \\
&- \quad [3n(n+1)\cot\psi\csc\psi + (n+1)(n+2)(2n+1)\cos\psi \\
&- \quad 2n(n+1)(2n+3)\cos\psi\cot^2\psi]P_n(\cos\psi) \\
&- \quad n^2(n+1)\cot^2\psi P_{n-1}(\cos\psi)\},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\sigma_z(a,\psi) \quad &= \quad \frac{3A_0}{a^4}[-3\cos\psi + 5\cos^3\psi] \\
&+ \quad \frac{3A_1}{a^5}[3 - 30\cos^2\psi + 35\cos^4\psi] \\
&+ \quad \frac{5A_2}{a^6}[15\cos\psi - 70\cos^3\psi + 63\cos^5\psi] \\
&+ \quad \frac{B_{-2}}{a^2}[-(1-2\nu)\cos\psi - 3\cos^3\psi]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{B_{-1}}{a^3}[-(1-2\nu) - 6(1+\nu)cos^2\psi + 15\cos^4\psi] \\
&+ \quad \frac{B_0}{a^4}[(3+18\nu)\cos\psi - (80+30\nu)\cos^3\psi + 105\cos^5\psi] \\
&+ \quad \sum_{n=3}^{N_1} A_n a^{-(n+4)}\{[(8n+9)(2n+5)\cos^3\psi \\
&- \quad (4n^2+24n+21)\cos\psi]P_n(\cos\psi) \\
&+ \quad [3n(2n+1) - n(8n^2+24n+15)\cos^2\psi]P_{n-1}(\cos\psi) \\
&+ \quad 3n(n-1)(2n+1)\cos\psi P_{n-2}(\cos\psi) \\
&- \quad n(n-1)(n-2)P_{n-3}(\cos\psi)\} \\
&+ \quad \sum_{n=1}^{N_2} B_n a^{-(n+4)}\{[(2n+3)(2n+5)(2n+7)\cos^3\psi \\
&+ \quad (2n+3)(2n+5)(1-2\nu)\cos\psi]P_{n+2}(\cos\psi) \\
&+ \quad [-(n+2)(2n+3)(1-2\nu) - (n+2)(2n+3)(6n+15)\cos^2\psi]P_{n+1}(\cos\psi) \\
&+ \quad 3(n+1)(n+2)(2n+3)\cos\psi P_n(\cos\psi) - n(n+1)(n+2)P_{n-1}(\cos\psi)\},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\tau_{rz}(a,\psi) \quad &= \quad \frac{3A_0}{a^4}[-\sin\psi + 5\sin\psi\cos^2\psi] \\
&+ \quad \frac{15A_1}{a^5}[-3\sin\psi\cos\psi + 7\sin\psi\cos^3\psi] \\
&+ \quad \frac{5A_2}{a^6}[3\sin\psi - 42\sin\psi\cos^2\psi + 63\sin\psi\cos^4\psi] \\
&+ \quad \frac{B_{-2}}{a^2}[-(1-2\nu)\sin\psi - 3\sin\psi\cos^2\psi] \\
&+ \quad \frac{B_{-1}}{a^3}[-3(1+2\nu)\sin\psi\cos\psi + 15\sin\psi\cos^3\psi] \\
&+ \quad \frac{B_0}{a^4}[(1+6\nu)\sin\psi - (50+30\nu)\sin\psi\cos^2\psi \\
&+ \quad 105\sin\psi\cos^4\psi] \\
&+ \quad \sum_{n=3}^{N_1} A_n a^{-(n+4)}\{[n(2n+1)\cos\psi\cot\psi \\
&- \quad (n+3)(2n+1)\sin\psi \\
&+ \quad (n+5)(8n+9)\sin\psi\cos^2\psi - n(8n+9)\cos^3\psi\cot\psi]P_n(\cos\psi) \\
&- \quad [n(2n+1)\cot\psi \\
&+ \quad 2n(n+4)(2n+1)\sin\psi\cos\psi - n(4n^2 \\
&+ \quad 6n+7)\cos^2\psi\cot\psi]P_{n-1}(\cos\psi) \\
&+ \quad [n(n-1)(n+3)\sin\psi - 5n^2(n-1)\cos\psi\cot\psi]P_{n-2}(\cos\psi)
\end{aligned}
$$

$$
\begin{aligned}
+ & \quad (n-2)(n-1)n \cot\psi P_{n-3}(\cos\psi)\} \\
+ & \quad \sum_{n=1}^{N_2} B_n a^{-(n+4)}\{[(n+3)(2n+3)(1-2\nu)\sin\psi \\
- & \quad (n+2)(2n+3)(1-2\nu)\cos\psi\cot\psi \\
+ & \quad (n+5)(2n+3)(2n+5)\sin\psi\cos^2\psi \\
- & \quad (n+2)(2n+3)(2n+5)\cos^3\psi\cot\psi]P_{n+2}(\cos\psi) \\
- & \quad [-(n+2)(2n+3)(1-2\nu)\cot\psi + 2(n+2)(n+4)(2n+3)\sin\psi\cos\psi \\
- & \quad (n+2)(2n+3)(4n+7)\cos^2\psi\cot\psi]P_{n+1}(\cos\psi) \\
+ & \quad [(n+1)(n+2)(n+3)\sin\psi \\
- & \quad (n+1)(n+2)(5n+6)\cos\psi\cot\psi]P_n(\cos\psi) \\
+ & \quad n(n+1)(n+2)\cot\psi P_{n-1}(\cos\psi)\}.
\end{aligned}
\tag{20}
$$

7 Confirming the Boundary-value Problem and Its Solution

Now we confirm that the boundary-value problem of the “Cavity in Cylinder” Problem is formulated by Eq.(1), Eq.(2) and Eqs.(17), added by Eq.(4).The solution of the problem is found to be Equations (12), (13), (14), (16) and

$$
\begin{aligned}
\sigma_R(R,\psi) &= \sigma_r(R,\psi)\sin^2\psi + \sigma_z(R,\psi)\cos^2\psi + 2\tau_{rz}(R,\psi)\sin\psi\cos\psi, \\
\tau_{R\psi}(R,\psi) &= [\sigma_r(R,\psi) - \sigma_z(R,\psi)]\sin\psi\cos\psi - \tau_{rz}(R,\psi)(\sin^2\psi - \cos^2\psi)
\end{aligned}
\tag{21}
$$

where $\sigma_r(R,\psi)$, $\sigma_z(R,\psi)$ and $\tau_{rz}(R,\psi)$ are given by Eq.(12), Eq.(13) and Eq.(14).

The solution satisfies all the fundamental equations and the boundary conditions confirmed for the problem, and is unique because of the ”Uniqueness of Solution” (See Ref. [15]). The solution is the formulation of Saint-Venant type decay of the “Cavity in Cylinder” Problem.

8 Saint-Venant’s Principle of “Cavity in Cylinder” Problem

From Eq.(12), Eq.(13), Eq.(14), Eq.(16) and Eqs.(21) we have

$$
\begin{aligned}
\lim_{R\to\infty}\sigma_r(R,\psi) &= 0, \\
\lim_{R\to\infty}\sigma_z(R,\psi) &= 0, \\
\lim_{R\to\infty}\tau_{rz}(R,\psi) &= 0,
\end{aligned}
$$

$$
\begin{aligned}
\lim_{R\to\infty} \sigma_\theta(R,\psi) &= 0, \\
\lim_{R\to\infty} \sigma_R(R,\psi) &= 0, \\
\textit{and} \quad \lim_{R\to\infty} \tau_{R\psi}(R,\psi) &= 0.
\end{aligned}
\tag{22}
$$

Now we prove Saint-Venant's Principle of the "Cavity in Cylinder" Problem by the end equations in terms of Eqs.(22). Equations (12), (13), (14), (16) and (21) are the formulation of Saint-Venant's decay of the problem of the small cavity loaded by the axisymmetric equilibrium system of forces , Eqs.(17).

9 Conclusion

The Special Saint-Venant's Principle of the effect of the axisymmetric equilibrium system of forces loaded on the small cavity in the infinite cylinder is proved in this paper. It is evident that finding solutions of boundary-value problems is a precise and pertinent approach to establish Saint-Venant type decay of elastic problems.

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