Saint-Venant's Principe of the Problem of the Cylinder

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December 31, 2011

Abstract

The Statement of Modified Saint-Venant's Principle is suggested. The axisymmetrical deformation of the infinite circular cylinder loaded by an equilibrium system of forces on its near end is discussed and its formulation of Modified Saint-Venant's Principle is established. It is evident that finding solutions of boundary-value problems is a precise and pertinent approach to establish Saint-Venant type decay of elastic problems.

`AMS` Subject Classifications: 74-02, 74G50

`Keywords` : Modified Saint-Venant's Principe, statement, proof, solution, decay, axisymmetrical deformation

1 Introduction

Saint-Venant's Principle is essential and fundamental in Elasticity (See Ref.[1] and Ref.[2]). Boussinesq and Love announce statements of Saint-Venant's Principe (See Ref.[3] and Ref.[4]), but Mises and Sternberg argue, by citing counterexamples, that the statements are not clear, suggesting that Saint-Venant's Principle should be proved or given a mathematical formulation (See Ref.[5] and Ref.[6]). Truesdell asserts that if Saint-Venant's Principle of equipollent loads is true, it must be a mathematical consequence of the general equations of Linear Elasticity (See Ref.[7]).

There is no doubt that mathematical proof of Saint-Venant's Principle has become an academic attraction for contributors and much effort has been made for exploring its mysterious implications or deciphering its puzzle. Zanaboni "proved" a theorem trying to concern Saint-Venant's Principle in terms of work and energy (See Refs.[8],[9],[10]). However, Zhao argues that Zanaboni's theorem is false (See Ref.[11]). The work published by Toupin cites more counterexamples to explain that Love's statement is false, and then establishes a formulation of energy decay, which is considered as "a precise mathematical formulation and proof" of Saint-Venant's Principle for the elastic cylinder (See Refs.[12], [13]). Furthermore, Toupin's work seems to set up an example followed by a large number of papers to establish Toupin-type energy decay formulae for branches of continuum mechanics. Since 1965 the concept of energy decay suggested by Toupin has been widely accepted by authors, and various techniques have been developed to construct inequalities of Toupin-type decay of energy which are spread widely in continuum mechanics. Especially, the theorem given by Berdichevskii is considered as a generalization of Toupin's theorem (See Ref.[14]). However, Zhao points out that Toupin's theory is not a strict mathematical proof, and Toupin's Theorem is not an exact mathematical formulation, of Saint-Venant's Principle. Interestingly and significantly, Saint-Venant's Principle stated by Love is disproved mathematically from Toupin's Theorem, so Toupin's Theorem is mathematically inconsistent with Saint-Venant's Principle (See Ref.[11]).

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Boussinesq and Love and points out mathematically that Saint-Venant type decay can be proved or formulated by special formulating or adding supplementary conditions to the problems discussed (See Ref.[11]). Therefore, we suggest Modified Saint-Venant's Principle in this paper.

The cylinder discussed by Toupin is of arbitrary length and cross section and loaded with an arbitrary system of self-equilibrated forces, and Toupin's cylinder fails to be consistent with Saint-Venant type decay. The problem of proof of Saint-Venant's Principle of cylinders is still open. The question that we have to answer is : what conditions are sufficient to guarantee Saint-Venant type decay of stresses to occur in cylinders. We find in this paper that Saint-Venant type decay can be established by excluding the arbitrariness of geometry and loading of Toupin's cylinder , and suggest the problem of Saint-Venant's Principle for the axisymmetrical deformation of the circular cylinder and prove its Modified Saint-Venant Principle.

2 Modified Saint-Venant's Principle

Love's Statement of Saint-Venant's Principle: "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." (See Ref.[4])

Mises considers a body of finite dimensions and sees that " Saint-Venant's principle in its traditional form does not hold true." (See Ref.[5]) Toupin asserts: " The broader statement of the principle given by Love for bodies of arbitrary shape cannot possibly be true... " (See Ref.[12]) He also refers to counter-examples and argues that " Confronted with such a collection of counter-examples of the now traditional statement of the Saint-Venant's principle, one must agree that something is wrong with it." (See Ref.[13])

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Love, but argues by mathematical analysis that Saint-Venant type decay of strains (then stresses) described by Love's statement can be proved true by special formulating or adding supplementary conditions to the problems discussed (See Ref.[11]). Therefore, we suggest the following Modified Saint-Venant's Principle:

"In an infinite elastic body, which is loaded by an equilibrium system of forces on some small part of its surface (otherwise would be free) and supplemented with sufficient conditions , assumptions or constraints on its formation of geometry and loading, the strains and stresses tend to zero as the distances from the loaded part tend to infinite."

In the following sections we prove Modified Saint-Venant's Principle of the axisymmetrical deformation of the circular cylinder.

3 Formulating the Problem

Let us consider an axisymmetrical cylinder of length L with a constant circular cross section. The end ($z = 0$) of the cylinder is loaded by an equilibrium system of forces, otherwise the cylinder would be free. Denoting by a the radius of the circular cross section, the boundary-value problem of the cylinder is:

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2})(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2})\ \phi(r, z) = 0, \qquad (0 \le r \le a,\ 0 \le z \le L), \tag{1}$$

$$r = a\ :\quad \sigma_r = 0,\quad \tau_{rz} = 0, \tag{2}$$

$$z = L\ :\quad \sigma_z = 0,\quad \tau_{zr} = 0, \tag{3}$$

$$z = 0\ :\quad \textit{the equilibrium system of forces to be found.} \tag{4}$$

The relationship between the stresses and the stress function is:

$$\begin{aligned}
\sigma_r &= \frac{\partial}{\partial z}[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}],\\
\sigma_\theta &= \frac{\partial}{\partial z}[\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}],
\end{aligned}$$

$$
\begin{aligned}
\sigma_z &= \frac{\partial}{\partial z}[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}],\\
\tau_{rz} &= \frac{\partial}{\partial r}[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}].
\end{aligned}
\tag{5}
$$

We will establish, in the following sections, the equilibrium system of forces on the end ($z = 0$) and the conditions that bring about Saint-Venant's decay of stresses for the cylinder.

4 Eigenvalue Equation

Considering Eq.(3), let the stress function be

$$
\phi(r,z) = \sum_{n=1}^{\infty} \varphi_n(r) e^{-k_n z}, \quad (0 \le r \le a,\ 0 \le z \le L) \tag{6}
$$

Putting Eq.(6) into Eq.(1), we find the solution

$$
\phi(r,z) = \sum_{n=1}^{\infty} \varphi_n(r) e^{-k_n z} = \sum_{n=1}^{\infty} [A_n J_0(k_n r) + B_n k_n r J_1(k_n r)] e^{-k_n z}. \tag{7}
$$

From Eq.(7)

$$
\begin{aligned}
\sigma_r &= \frac{\partial}{\partial z}[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}]\\
&= \sum_{n=1}^{\infty}\{A_n[-k_n^3 J_0(k_n r) + k_n^2 \frac{1}{r} J_1(k_n r)]\\
&+ B_n[(1-2\nu)k_n^3 J_0(k_n r) - k_n^4 r J_1(k_n r)]\} e^{-k_n z},
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\tau_{rz} &= \frac{\partial}{\partial r}[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}]\\
&= \sum_{n=1}^{\infty}\{-A_n k_n^3 J_1(k_n r)\\
&- B_n[2(1-\nu)k_n^3 J_1(k_n r) + k_n^4 r J_0(k_n r)]\} e^{-k_n z}.
\end{aligned}
\tag{9}
$$

The condition (2) must be fulfilled by (8) and (9), therefore it is required that

$$
A_n[-k_n^3 J_0(k_n a) + k_n^2 \frac{1}{a} J_1(k_n a)] + B_n[(1-2\nu)k_n^3 J_0(k_n a) - k_n^4 a J_1(k_n a)] = 0, \tag{10}
$$

$$-A_n k_n^3 J_1(k_n a) - B_n[2(1-\nu)k_n^3 J_1(k_n a) + k_n^4 a J_0(k_n a)] = 0. \qquad (11)$$

The coefficients A_n and B_n have non-zero solutions from Eqs.(10) and (11) if

$$\begin{aligned}
[\quad & -k_n^3 J_0(k_n a) + k_n^2 \frac{1}{a} J_1(k_n a)][2(1-\nu)k_n^3 J_1(k_n a) + k_n^4 a J_0(k_n a) \quad] \\
- \quad & k_n^3 J_1(k_n a)[(1-2\nu)k_n^3 J_0(k_n a) - k_n^4 a J_1(k_n a)] \\
= \quad & 0. \qquad (12)
\end{aligned}$$

Excluding $k_n = 0$, Eq.(12) is changed into

$$2(1-\nu)k_n^2 a[J_0(k_n a)]^2 + (1-2\nu)k_n J_0(k_n a)J_1(k_n a) - [2(1-\nu)\frac{1}{a} + k_n^2 a][J_1(k_n a)]^2 = 0. \qquad (13)$$

Equation (13) is the eigenvalue equation of the problem, from which k_n are determined.

5 Stress Components

From Eq.(7) we find the stress components

$$\begin{aligned}
\sigma_\theta &= \frac{\partial}{\partial z}[\nu \nabla^2 \phi - \frac{1}{r}\frac{\partial \phi}{\partial r}] \\
&= \sum_{n=1}^{\infty}[-A_n \frac{1}{r} k_n^2 J_1(k_n r) + B_n(1-2\nu)k_n^3 J_0(k_n r)]e^{-k_n z}, \qquad (14)
\end{aligned}$$

$$\begin{aligned}
\sigma_z &= \frac{\partial}{\partial z}[(2-\nu)\nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2}] \\
&= \sum_{n=1}^{\infty}\{A_n k_n^3 J_0(k_n r) \\
&+ B_n[-2(2-\nu)k_n^3 J_0(k_n r) + k_n^4 r J_1(k_n r)]\}e^{-k_n z}. \qquad (15)
\end{aligned}$$

6 Saint-Venant Type Decay of Stresses and Its Requirement

To guarantee condition (3) to be satisfied and from Eq.(15) and Eq.(9), it is required that

$$L \to \infty, \qquad (16)$$

$$k_n > 0 \tag{17}$$

and

$$A_n = B_n = 0, \quad (n > N), \tag{18}$$

where N is a positive integer , that is,

$$N < \infty. \tag{19}$$

And then , from Eq.(7) Eq.(8), Eq.(9), Eq.(14) and Eq.(15),

$$\phi(r,z) = \sum_{n=1}^{N} [A_n J_0(k_n r) + B_n k_n r J_1(k_n r)] e^{-k_n z}, \tag{20}$$

$$\begin{aligned} \sigma_r &= \sum_{n=1}^{N} \{A_n[-k_n^3 J_0(k_n r) + k_n^2 \frac{1}{r} J_1(k_n r)] \\ &+ B_n[(1-2\nu)k_n^3 J_0(k_n r) - k_n^4 r J_1(k_n r)]\} e^{-k_n z}, \end{aligned} \tag{21}$$

$$\sigma_\theta = \sum_{n=1}^{N} [-A_n \frac{1}{r} k_n^2 J_1(k_n r) + B_n(1-2\nu)k_n^3 J_0(k_n r)] e^{-k_n z}, \tag{22}$$

$$\begin{aligned} \sigma_z &= \sum_{n=1}^{N} \{A_n k_n^3 J_0(k_n r) \\ &+ B_n[-2(2-\nu)k_n^3 J_0(k_n r) + k_n^4 r J_1(k_n r)]\} e^{-k_n z}, \end{aligned} \tag{23}$$

$$\begin{aligned} \tau_{rz} &= \sum_{n=1}^{N} \{-A_n k_n^3 J_1(k_n r) \\ &- B_n[2(1-\nu)k_n^3 J_1(k_n r) + k_n^4 r J_0(k_n r)]\} e^{-k_n z}, \end{aligned} \tag{24}$$

where k_n is n^{th} positive root of the eigenvalue equation Eq.(13) and N is a finite positive integer, that is,

$$k_n > 0, \quad N < \infty, \tag{25}$$

by virtue of Eq.(17) and Eq.(19).

From Eq.(23) and Eq.(24), the equilibrium system of forces loaded on the end $z = 0$ should be

$$
\begin{aligned}
z = 0: \ \sigma_z &= \sum_{n=1}^{N} \{A_n k_n^3 J_0(k_n r) \\
&+ B_n[-2(2-\nu)k_n^3 J_0(k_n r) + k_n^4 r J_1(k_n r)]\}, \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_{zr} &= \sum_{n=1}^{N} \{-A_n k_n^3 J_1(k_n r) \\
&- B_n[2(1-\nu)k_n^3 J_1(k_n r) + k_n^4 r J_0(k_n r)]\}. \qquad (27)
\end{aligned}
$$

Now we confirm that the boundary-value problem of the cylinder for Saint-Venant type decay of stresses is formulated by Equations Eq.(1)(attached by Eq.(16)), Eq.(2), Eq.(3), Eq.(26) and Eq.(27), added by Eq.(5). The solution of the problem is found to be Equations (20), (21), (22), (23) and (24), which satisfies all the fundamental equations and the boundary conditions confirmed for the problem, and is unique because of the "Uniqueness of Solution" (See Ref. [15]). The solution is the formulation of Saint-Venant type decay of the problem of the cylinder.

7 Modified Saint-Venant's Principle of the Axisymmetrical Deformation of the Circular Cylinder

From Eqs.(21), (22), (23) and (24) we come to the end equations

$$
\begin{aligned}
\lim_{z\to\infty} \sigma_r &= 0, \\
\lim_{z\to\infty} \sigma_\theta &= 0, \\
\lim_{z\to\infty} \sigma_z &= 0, \\
\lim_{z\to\infty} \tau_{rz} &= 0. \qquad (28)
\end{aligned}
$$

Thus we prove Modified Saint-Venant's Principle of the axisymmetrical deformation of the circular cylinder by the end equations in terms of Eq.(28).

8 Conclusion

1.The Statement of Modified Saint-Venant's Principle is suggested.

2.The problem of Saint-Venant's Principle of the axisymmetrical deformation of the infinite circular cylinder is suggested , and its Modified Saint-Venant's Principle is proved and the requirement for the principle to be true is found.

3.It is evident that finding solutions of boundary-value problems is a precise and pertinent approach to establish Saint-Venant type decay of elastic problems.

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