RLC circuit derived from particle and field electromagnetic equations

Valery P. Dmitriyev
*Ushakov Maritime University, Novorossiysk, Russia**
(Dated: February 24, 2012)

The RLC circuit equation is derived step-by-step from basic equations of classical electrodynamics. The system is shown to be oscillating even if elements of the linear current would not interact with each other. Their mutual electromagnetic interaction due to acceleration of charges results in the phenomenon that looks like the increase of the effective mass of the charged particles. The increase of the mass makes the oscillations more persistent with respect to the damping caused by the friction.

PACS numbers: 84.30Bv
Keywords: oscillatory circuit, electromagnetic force, Maxwell equations, vector potential

1. INTRODUCTION

Any proposition of a regular theory can be deduced from the first principles of this theory. It would be reasonable to follow this rule in electrodynamics and its applications. Maxwell equations together with the Coulomb and Lorentz laws of interaction of electrically charged particles are sufficient in order to solve any problem of classical electromagnetism. The description of oscillations in the electric circuit is the key topic in applications. However, until now the equation of the RLC circuit was not deduced from the basic equations of electrodynamics. In common textbooks on electrodynamics or electrical engineering the equation of oscillatory circuit is derived from the Ohm's law plus the Kirchhoff's voltage law, value of the capacitor and Faraday's law of induction (see e.g. [1]-[6]). In this event no relation of the Ohm's law with the motion equation of the electric charge is indicated. Moreover, the steady-state equation is used in order to describe the transient process. The Kirchhoff's law is served without reference to Maxwell equations. At last, the Faraday law of induction is presented as a prerequisite for Maxwell equations but not vice versa. Such a perverted order of presentation of the educational material is commonly justified by that it was the historical sequence. Below I try to improve these shortcomings presenting fundamentals of electrical engineering not in historical but in logical sequence. The problem is stated as follows. Given are Maxwell equations in terms of electromagnetic potentials

$$\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} + \boldsymbol{\nabla}\varphi = 0, \tag{1}$$

$$\frac{\partial \mathbf{E}}{\partial t} - c\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\mathbf{A}) + 4\pi\mathbf{j} = 0, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho \tag{4}$$

and Coulomb and Lorentz forces for motion equation of an electrically charged particle

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}). \tag{5}$$

We need to derive from them the equation for oscillations in RLC circuit. In the course of derivation I will not make allowances for effects of the medium as minor details which do not change the general shape of the phenomenon.

2. DISCHARGING OF THE CAPACITOR THROUGH THE ELECTRIC CIRCUIT

For the collection of electric charges with the density $\rho = \mathsf{const}$ Eq.(5) is modified to

$$\frac{m}{q}\frac{d\mathbf{j}}{dt} = \rho\mathbf{E} + \frac{\mathbf{j}}{c}\times(\boldsymbol{\nabla}\times\mathbf{A}) \tag{6}$$

*Electronic address: dmitriyev@nsma.ru, aether@yandex.ru

where $\mathbf{j} = \rho\mathbf{v}$ is the electric current density. We will consider a capacitor which stores the charge Q. And describe the discharging of the capacitor via the conducting wire which connects two plates of the capacitor. With this end let us integrate Eq.(6) over the wire path from one plate of the capacitor to another

$$-\frac{m}{q}\int_1^2 \frac{d\mathbf{j}}{dt}\cdot d\mathbf{l} + \rho\int_1^2 \mathbf{E}\cdot d\mathbf{l} + \int_1^2 \left[\frac{\mathbf{j}}{c}\times(\boldsymbol{\nabla}\times\mathbf{A})\right]\cdot d\mathbf{l} = 0. \tag{7}$$

The third integral in Eq.(7) vanishes because $\mathbf{j}\|d\mathbf{l}$:

$$\int_1^2 \left[\frac{\mathbf{j}}{c}\times(\boldsymbol{\nabla}\times\mathbf{A})\right]\cdot d\mathbf{l} = 0. \tag{8}$$

The uniform electric current equals to the decrease rate of the electric charge Q in the capacitor

$$jS = -\frac{dQ}{dt} \tag{9}$$

where $S = \mathsf{const}$ is the cross section of the wire. Using Eq.(9) in the first integral of Eq.(7):

$$\frac{m}{q}\int_1^2 \frac{d\mathbf{j}}{dt}\cdot d\mathbf{l} = -\frac{ml}{qS}\frac{d^2Q}{dt^2}. \tag{10}$$

Using the first Maxwell equation (1) in the second term of Eq.(7), we obtain

$$\int_1^2 \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{c}\int_1^2 \frac{\partial\mathbf{A}}{\partial t}\cdot d\mathbf{l} - \int_1^2 \boldsymbol{\nabla}\varphi\cdot d\mathbf{l}. \tag{11}$$

Insofar as the geometry of the circuit is invariable, in the first integral of the right-hand part of (11) the differential $d\mathbf{l}$ can be introduced under the derivative and partial derivative changed to the total one:

$$\int_1^2 \frac{\partial\mathbf{A}}{\partial t}\cdot d\mathbf{l} = \int_1^2 \frac{\partial(\mathbf{A}\cdot d\mathbf{l})}{\partial t} = \frac{d}{dt}\int_1^2 \mathbf{A}\cdot d\mathbf{l}, \tag{12}$$

and for the second integral in the right-hand part of (11) easily:

$$\int_1^2 \boldsymbol{\nabla}\varphi\cdot d\mathbf{l} = \varphi_2 - \varphi_1. \tag{13}$$

Next we evaluate the integral in the right-hand part of Eq.(12). From the second Maxwell equation (2) with $\partial\mathbf{E}/\partial t = 0$ and the Coulomb gauge (3) the vector potential $d\mathbf{A}$ produced by the element $jd\mathbf{l}'$ of the current at the distance $|\mathbf{r}-\mathbf{r}'|$:

$$d\mathbf{A} = \frac{jSd\mathbf{l}'}{c|\mathbf{r}-\mathbf{r}'|}. \tag{14}$$

In the result we have for the circulation of the vector potential around the circuit

$$\frac{1}{c}\int_1^2 \mathbf{A}d\mathbf{l} = \frac{jS}{c^2}\int_1^2\int_1^2 \frac{d\mathbf{l}'\cdot d\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|} = LSj = -L\frac{dQ}{dt} \tag{15}$$

where $L = \mathsf{const}$ and (9) was used. Here we may expect that the inductance L of the circuit has a positive sign, $L > 0$, since the predominant contribution to the integral (15) are given by adjacent portions of the circuit where $d\mathbf{l}'$ approximately parallel to $d\mathbf{l}$. At last, for a capacitor with the invariable configuration, integrating the third Maxwell equation (4), we may find the relation between the potential difference (13) and charge Q at the plates of the capacitor:

$$\varphi_2 - \varphi_1 = -\frac{Q}{C} \tag{16}$$

where $C = \mathsf{const}$. Using (15) and (16) through (12) and (13) in Eq.(11) gives

$$\int_1^2 \mathbf{E}\cdot d\mathbf{l} = L\frac{d^2Q}{dt^2} + \frac{Q}{C}. \tag{17}$$

Substituting (17), (8) and (10) into Eq.(7) we come to the equation sought for

$$\left(\frac{ml}{q\lambda} + L\right)\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \tag{18}$$

where λ is the linear density of the moving electric charges in the wire.

3. OSCILLATIONS IN LC CIRCUIT

Solving equation (18) we obtain with the account of the above definition the following time profiles

$$Q = Q_0 \cos(\omega t), \tag{19}$$

$$\varphi_1 - \varphi_2 = \frac{Q_0}{C} \cos(\omega t), \tag{20}$$

$$j = \omega Q_0 \sin(\omega t), \tag{21}$$

$$\frac{1}{c} \int_1^2 \mathbf{A} \cdot d\mathbf{l} = L\omega Q_0 \sin(\omega t), \tag{22}$$

$$-\frac{1}{c} \int_1^2 \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} = -L\omega^2 Q_0 \cos(\omega t), \tag{23}$$

$$\omega = \frac{1}{\sqrt{(L + \frac{ml}{q\lambda})C}}. \tag{24}$$

The process can bee seen as vibration of the "mass" $\frac{ml}{q\lambda} > 0$. It is initiated by the action of the electric potential (20) and then hindered by the kinetic factor (23) so that according to (11) the whole electromotive force is

$$\int_1^2 \mathbf{E} \cdot d\mathbf{l} = \frac{Q_0}{C} \left(\frac{\frac{ml}{q\lambda}}{L + \frac{ml}{q\lambda}} \right) \cos(\omega t). \tag{25}$$

The system can be alternatively viewed as the vibration of the effective "mass" $L + \frac{ml}{q\lambda}$ under the action of the electric potential (20), i.e. the corresponding to (23) kinetic force plays the role of the inertial factor.

Take notice that the oscillations also take place when $L = 0$, i.e. in the hypothetical case of the absence of the kinetic factor. So, it is not the Faraday law of the electromagnetic induction that should be considered as the cause of oscillations in the LC circuit. Here the influence of the inductance L is the same as the increase of the mass in the mechanical pendulum: it makes the process more stable with respect to the inevitable in reality dissipation.

4. MECHANICAL ANALOGY

We consider a mechanical oscillator with the mass m:

$$m\frac{dv}{dt} = -kx, \tag{26}$$

$$v = \frac{dx}{dt} \tag{27}$$

where $k = \mathsf{const} > 0$.

The physical analog of the LC circuit is the mechanical oscillator with the additional force f that depends on the acceleration:

$$m\frac{dv}{dt} = f - kx, \tag{28}$$

$$f = -\mu\frac{dv}{dt}, \tag{29}$$

$$v = \frac{dx}{dt} \tag{30}$$

where $\mu = \mathsf{const} > 0$. The model system described by the latter equations behaves itself exactly as the collection of positively charged carriers of the electric current in the LC circuit. When the system is accelerated the additional force f hinders it. And vice versa, when the system is decelerated the additional force f promotes its motion. Alternatively, the system can be viewed as a particle with the effective mass $m + \mu$ which moves under the action of the force $-kx$ of linear elasticity.

5. MOTION WITH FRICTION

To describe the motion with friction there can be added to the force in Eq.(5) the empirical term $-\kappa\mathbf{v}$, where $\kappa > 0$:

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \kappa\mathbf{v}. \tag{31}$$

In the case of the collection of electric charges with the density $\rho = \mathsf{const}$ Eq.(31) is modified to

$$\frac{m}{q}\frac{d\mathbf{j}}{dt} = \rho\mathbf{E} + \frac{\mathbf{j}}{c} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \frac{\kappa\mathbf{j}}{q} \tag{32}$$

where $\mathbf{j} = \rho\mathbf{v}$ is the electric current density. Further, making the same operations as in the above derivation we obtain the generalization of Eq.(18) to the case of the circuit with the resistance $R = \frac{\kappa l}{q\lambda}$ to the electric current

$$\left(\frac{ml}{q\lambda} + L\right)\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0. \tag{33}$$

Eq.(33) describes oscillations with damping:

$$Q = Q_0 e^{-\alpha t}\cos(\omega t), \tag{34}$$

$$\alpha = \frac{R}{2\tilde{L}}, \tag{35}$$

$$\omega^2 = \frac{1}{\tilde{L}C} - \alpha^2, \tag{36}$$

$$\tilde{L} = L + \frac{ml}{q\lambda}. \tag{37}$$

As we can see from Eq.(35), the increase in the inductance L decreases the damping factor α. So, designers of real oscillatory circuits, where the resistance $R \neq 0$, strive for increasing L. With this end they roll up the wire into a coil, where there are many closely arranged segments of the electric current which are parallel to each other, $d\mathbf{l} \| d\mathbf{l}'$.

6. CONCLUSION

In the system of two oppositely charged bodies connected by the conducting wire the oscillations are possible even in the absence of mutual electromagnetic influence of adjacent parts of the electric current on each other. This is a kind of oscillation that is common to any mechanical system possessing the inertial mass. The influence on the moving charge of the adjacent accelerated charges is executed via the kinetic part

$$-\frac{q}{c}\frac{\partial\mathbf{A}}{\partial t} \tag{38}$$

of the electromagnetic interaction (5) according to Maxwell equation (1). The mutual interaction by (38) retards the motion of the collection of charged particles in the same way as the growth of the inertial mass does. We may say even about a special case of the effect of attached mass of a charged particle moving in the environs of accelerated charges. The increase of the inertial mass leads to that the oscillations become more stable with respect to the friction.

---

[1] I.E.Irodov, Basic laws of electromagnetism, chapter 11.1 Equation of an oscillatory circuit, p.277, Mir Publishers Moscow, reprint 2001.

[2] D.J.Griffiths, Introduction to electrodynamics, chapter 7.2 Electromagnetic induction, p.310, 3d edition, Prentice Hall, Upper Saddle River, New Jersey, 1999.

[3] J.A.Edminister, Theory and problems of electromagnetics, chapter 11 Inductance and magnetic circuits, p.169, chapter 15 Transmiison lines (by Milton Kult), p.237, 2nd Edition, Shaum's outline series, McGraw Hill, 1995.

[4] H.P.Neff, Introductory electromagnetics, Section 3.10 Circuit theory from field theory, p.162, John Wiley & Sons, 1991.

[5] P.Lorrain, D.L.Corson, F.Lorrain, Electromagnetic fields and waves, including electric circuits, section 24.2 Self-inductance, p.441, Third Edition, W.H.Freeman and Company, New-York, 1988.

[6] W.R.Smythe, Static and dynamic electricity, chapter 9 Transient phenomena in networks, p.327, McGraw Hill Book Company, 1950.