SOLUTION TO ONE OF LANDAU'S PROBLEMS AND INFINITELY MANY PRIME NUMBERS OF THE FORM $ap \pm b$

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ABSTRACT. In this paper it is proved that for every positive integer k there are infinitely many prime numbers of the form $n^2 + k$, which means that there are infinitely many prime numbers of the form $n^2 + 1$. In addition to this, in this document it is proved that if a and b are two positive integers which are coprime and also have different parity, then there are infinitely many prime numbers of the form $ap + b$, where p is a prime number. Moreover, it is also proved that there are infinitely many prime numbers of the form $ap - b$. In other words, it is proved that the progressions $ap+b$ and $ap-b$ generate infinitely many prime numbers. In particular, all this implies that there are infinitely many prime numbers of the form $2p + 1$ (since the numbers 2 and 1 are coprime and have different parity), which means that there are infinitely many *Sophie Germain Prime Numbers*. This paper also proposes an important new conjecture about prime numbers called *Conjecture C*. If this conjecture is true, then *Legendre's Conjecture*, *Brocard's Conjecture* and *Andrica's Conjecture* are all true, and also some other important results will be true.

1. INTRODUCTION

FOREWORD

When I wrote my original document **Números Primos de Sophie Germain, Demostración de su Infinitud** (*There Are Infinitely Many Sophie Germain Prime Numbers*), I used the ***'Breusch Interval'*** in order to prove theorems and to prove that if i is any positive odd integer, then there are infinitely many prime numbers of the form $2p + i$ (where p is a prime number) and infinitely many prime numbers of the form $2p - i$.

I took the Breusch Interval from the following websites created by **Carlos Giraldo Ospina (Lic. Matemáticas, USC, Cali, Colombia)**:

- `http://www.matematicainsolita.8m.com`
- `http://matematicainsolita.8m.com/Archivos.htm`
- `http://numerosprimos.8m.com/Documentos.htm`

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C. G. Ospina utilized the Breusch Interval in some of his documents to prove some specific conjectures. This author makes reference to the article *Números Primos, Balance de Nuestra Ignorancia* written a long time ago by Joaquín Navarro, where this person mentions the Breusch Interval and that it was proved that there is always a prime number in this interval.

One of the people who read my document told me that they had never heard about a so-called 'Breusch Interval' before, and that they were unable to find any information on this interval in any Mathematics book.

For this reason, I wrote another document where I proved that for every positive integer n there is always a prime number in the Breusch Interval. To do this, I used another interval which I call ***'Iwaniec-Pintz Interval'***.

Rather than rewriting my document *Números Primos de Sophie Germain, Demostración de su Infinitud* using the Iwaniec-Pintz Interval, it was easier for me to prove that there is always a prime number in the Breusch Interval for every positive integer n.

The present document is much more general, because here it is proved that there are infinitely many prime numbers of the forms $n^2 + k$, $ap + b$ and $ap - b$. In order to prove this, the Breusch Interval is used. This is the reason why this document starts with giving a proof that the Breusch Interval contains at least one prime number for every positive integer n.

2. Breusch Interval

Definition 1. The interval $\left[n, \frac{9(n+3)}{8}\right]$, where $\boldsymbol{n}$ is a positive integer, is what we call ***'Breusch Interval'***.

We are going to prove that for every positive integer n there is always a prime number in the interval mentioned above. Probably, this was already proved a long time ago, but in this document we are going to give a clear proof that this is true.

In order to achieve the goal, we are going to use another interval, which we will call ***'Iwaniec-Pintz Interval'***.

3. Iwaniec-Pintz Interval

Definition 2. The **Iwaniec-Pintz Interval** is the interval $\left[n - n^{\theta}, n\right]$, where $\theta = 23/42$.

Iwaniec and Pintz proved that there is always a prime number in the interval $\left[n - n^{\frac{23}{42}}, n\right]$. The proof that this is true was given in their document ***Primes in Short Intervals*** [1].

In the Iwaniec-Pintz Interval, n must be considered as a positive integer greater than 1. This is because in the interval $\left[1 - 1^{\frac{23}{42}}, 1\right]$ there is no prime number.

To sum up, there is always a prime number in the interval $\left[n - n^{\frac{23}{42}}, n\right]$ for every positive integer $n > 1$.

4. Modified Iwaniec-Pintz Interval

Definition 3. Working with an exponent like $\frac{23}{42}$ may be very difficult. Consequently, we define the ***'Modified Iwaniec-Pintz Interval'*** as the interval $\left[n - n^{\frac{2}{3}}, n\right]$.

We know that $\frac{2}{3} > \frac{23}{42}$, because $0.\bar{6} > 0.5476190...$ Now, if $n = 1$, then $n^{\frac{2}{3}} = n^{\frac{23}{42}} = 1$, but if $n > 1$ then $n^{\frac{2}{3}} > n^{\frac{23}{42}}$, as we will prove now:

Proof.

$$n^{\frac{2}{3}} > n^{\frac{23}{42}} \quad \Rightarrow \quad \frac{n^{\frac{2}{3}}}{n^{\frac{23}{42}}} > \frac{n^{\frac{23}{42}}}{n^{\frac{23}{42}}} \quad \Rightarrow \quad n^{\frac{5}{42}} > 1 \quad \Rightarrow \quad n > \sqrt[\frac{5}{42}]{1} \quad \Rightarrow \quad n > 1$$

The reciprocal is also correct:

$$n^{\frac{2}{3}} > n^{\frac{23}{42}} \quad \Leftarrow \quad \frac{n^{\frac{2}{3}}}{n^{\frac{23}{42}}} > \frac{n^{\frac{23}{42}}}{n^{\frac{23}{42}}} \quad \Leftarrow \quad n^{\frac{5}{42}} > 1 \quad \Leftarrow \quad n > \sqrt[\frac{5}{42}]{1} \quad \Leftarrow \quad n > 1$$

For this reason, it is better to say that

$$n^{\frac{2}{3}} > n^{\frac{23}{42}} \quad \Leftrightarrow \quad \frac{n^{\frac{2}{3}}}{n^{\frac{23}{42}}} > \frac{n^{\frac{23}{42}}}{n^{\frac{23}{42}}} \quad \Leftrightarrow \quad n^{\frac{5}{42}} > 1 \quad \Leftrightarrow \quad n > \sqrt[\frac{5}{42}]{1} \quad \Leftrightarrow \quad n > 1$$

□

Remark 1. In general, to prove that an inequality is correct, we can solve that inequality step by step. If we get a result which is obviously correct, then we can start with that correct result, 'work backwards from there' and prove that the initial statement is true.

If $n^{\frac{2}{3}} > n^{\frac{23}{42}}$, then $n - n^{\frac{2}{3}} < n - n^{\frac{23}{42}}$.

Proof.

$$n^{\frac{2}{3}} > n^{\frac{23}{42}} \quad \Rightarrow \quad n^{\frac{2}{3}}/(-1) < n^{\frac{23}{42}}/(-1) \quad \Rightarrow \quad -n^{\frac{2}{3}} < -n^{\frac{23}{42}} \quad \Rightarrow$$
$$\Rightarrow \quad n - n^{\frac{2}{3}} < n - n^{\frac{23}{42}}$$

□

Now, let us make a graphic of both the Iwaniec-Pintz Interval and the Modified Iwaniec-Pintz Interval for every positive integer $n > 1$:

$$\underbrace{n - n^{\frac{2}{3}} \quad < \quad \underbrace{n - n^{\frac{23}{42}} \quad < \quad n}_{1.}}_{2.}$$

- **1. Iwaniec-Pintz Interval**
- **2. Modified Iwaniec-Pintz Interval**

For every positive integer n greater than 1, the interval $\left[n - n^{\frac{23}{42}}, n\right]$ contains at least one prime number. Therefore, the interval $\left[n - n^{\frac{2}{3}}, n\right]$ also contains at least one prime number, according to the graphic we just saw.

Theorem 1. *If n is any positive integer greater than 1, then the Modified Iwaniec-Pintz Interval $\left[n - n^{\frac{2}{3}}, n\right]$ contains at least one prime number.*

5. Value of n

Now, we need to know what value n needs to have so that a Breusch Interval can contain at least one Modified Iwaniec-Pintz Interval. We are going to use Theorem 1, which we have just proved.

We do not have any way of knowing if $\frac{9(n+3)}{8}$ (upper endpoint of the Breusch Interval) is an integer or not if we do not know the value of n first. The problem is that we use the interval $\left[n - n^{\frac{2}{3}}, n\right]$ with positive **integers** only.

Let us assume y is the integer immediately preceding the number $\frac{9(n+3)}{8}$. In other words, y will be the largest integer that is less than $\frac{9(n+3)}{8}$:

- If $\frac{9(n+3)}{8}$ is an integer, then $\frac{9(n+3)}{8} - d = y$, where $d = 1$.
- If $\frac{9(n+3)}{8}$ is not an integer, then $\frac{9(n+3)}{8} - d = y$, and in this case we know for sure that $0 < d < 1$, but we do not have any way of knowing the exact value of d if we do not know the value of $\frac{9(n+3)}{8}$ first.

All things considered, we have $\frac{9(n+3)}{8} - 1 \le y$ and $y < \frac{9(n+3)}{8}$. This means that

$$\frac{9(n+3)}{8} - 1 \le y < \frac{9(n+3)}{8}$$

It is correct to say that if $y - y^{\frac{2}{3}} \ge n$, then a Breusch Interval will contain at least one Modified Iwaniec-Pintz Interval. Let us see the graphic:

$$\underbrace{n \quad \le \quad \underbrace{y - y^{\frac{2}{3}} \qquad\qquad \frac{9(n+3)}{8} - 1 \quad \le \quad y}_{2.} \quad < \quad \frac{9(n+3)}{8}}_{3.}$$

- **2. Modified Iwaniec-Pintz Interval**
- **3. Breusch Interval**

Let us calculate the value n needs to have so that $y - y^{\frac{2}{3}} \geq n$:

$$y - y^{\frac{2}{3}} \geq n$$

$$\left(\frac{9(n+3)}{8} - d\right) - \left(\frac{9(n+3)}{8} - d\right)^{\frac{2}{3}} \geq n \text{ (we know that } y = \frac{9\,(n+3)}{8} - d)$$

$$\left(\frac{9n}{8} + \frac{9}{8} \times 3 - d\right) - \left(\frac{9n}{8} + \frac{9}{8} \times 3 - d\right)^{\frac{2}{3}} \quad \geq \quad n$$

$$\left(\frac{9n}{8} + \underbrace{\frac{27}{8} - d}_{g}\right) - \left(\frac{9n}{8} + \underbrace{\frac{27}{8} - d}_{g}\right)^{\frac{2}{3}} \quad \geq \quad n$$

In order to make the process easier, the number $\frac{27}{8} - d$ will be called g. This number will always be positive, because $\frac{27}{8} = 3.375$, and d can not be greater than 1.

$$\begin{aligned}
\left(\frac{9n}{8} + g\right) - \left(\frac{9n}{8} + g\right)^{\frac{2}{3}} &\geq n \\
\frac{9n}{8} + g - \left(\frac{9n}{8} + g\right)^{\frac{2}{3}} &\geq n \\
-\left(\frac{9n}{8} + g\right)^{\frac{2}{3}} &\geq n - \frac{9n}{8} - g \\
-\left(\frac{9n}{8} + g\right)^{\frac{2}{3}} &\geq -\frac{n}{8} - g \\
\left[-\left(\frac{9n}{8} + g\right)^{\frac{2}{3}}\right]/(-1) &\leq \left[-\frac{n}{8} - g\right]/(-1) \\
\left(\frac{9n}{8} + g\right)^{\frac{2}{3}} &\leq \frac{n}{8} + g \\
\sqrt[3]{\left(\frac{9n}{8} + g\right)^{2}} &\leq \frac{n}{8} + g \\
\left(\frac{9n}{8} + g\right)^{2} &\leq \left(\frac{n}{8} + g\right)^{3} \\
\left(\frac{9n}{8}\right)^{2} + 2 \times \frac{9n}{8} \times g + g^{2} &\leq \left(\frac{n}{8}\right)^{3} + 3\left(\frac{n}{8}\right)^{2} \times g + 3\left(\frac{n}{8}\right) g^{2} + g^{3}
\end{aligned}$$

$$\begin{aligned}
\frac{81n^2}{64} + \frac{9ng}{4} + g^2 &\leq \frac{n^3}{512} + \frac{3n^2g}{64} + \frac{3ng^2}{8} + g^3 \\
\frac{81n^2 + 144ng + 64g^2}{64} &\leq \frac{n^3 + 24n^2g + 192ng^2 + 512g^3}{512} \\
512 \times \frac{81n^2 + 144ng + 64g^2}{64} &\leq n^3 + 24n^2g + 192ng^2 + 512g^3 \\
8\left(81n^2 + 144ng + 64g^2\right) &\leq n^3 + 24n^2g + 192ng^2 + 512g^3 \\
648n^2 + 1152ng + 512g^2 &\leq n^3 + 24n^2g + 192ng^2 + 512g^3
\end{aligned}$$

Now we need to know what value n needs to have so that the following inequalities are all true:

$$n^3 \geq 648n^2 \tag{5.1}$$

$$24n^2g \geq 1152ng \tag{5.2}$$

$$192ng^2 \geq 512g^2 \tag{5.3}$$

If these three inequalities are all true at the same time, then

$$n^3 + 24n^2g + 192ng^2 + 512g^3 \quad \geq \quad 648n^2 + 1152ng + 512g^2$$

Let us calculate the values of n:

1)

$$n^3 \geq 648n^2 \quad \Rightarrow \quad \frac{n^3}{n^2} \geq 648 \quad \Rightarrow \quad n \geq 648$$

2)

$$24n^2g \geq 1152ng \quad \Rightarrow \quad \frac{24n^2g}{ng} \geq 1152 \quad \Rightarrow \quad 24n \geq 1152 \quad \Rightarrow$$
$$\Rightarrow \quad n \geq \frac{1152}{24} \quad \Rightarrow \quad n \geq 48$$

3)

$$192ng^2 \geq 512g^2 \quad \Rightarrow \quad \frac{192ng^2}{g^2} \geq 512 \quad \Rightarrow \quad 192n \geq 512 \quad \Rightarrow$$
$$\Rightarrow \quad n \geq \frac{512}{192} \quad \Rightarrow \quad n \geq 2.\bar{6}$$

Result 1: $n \geq 648$ **Result 2:** $n \geq 48$ **Result 3:** $n \geq 2.\bar{6}$

If $n \geq 648$, then $n > 48$ and $n > 2.\bar{6}$. Therefore, if $n \geq 648$ then the inequalities 5.1, 5.2 and 5.3 are all true at the same time. This means that if $n \geq 648$, then

$$n^3 + 24n^2 g + 192ng^2 + 512g^3 \quad \geq \quad 648n^2 + 1152ng + 512g^2$$

If we start with this true statement and we 'work backwards from here', we prove that if $n \geq 648$ is an integer, then $y - y^{\frac{2}{3}} \geq n$. To sum up,

$$n \quad \leq \quad y - y^{\frac{2}{3}} \quad < \quad y \quad < \quad \frac{9\,(n+3)}{8} \qquad \forall n \in \mathbb{Z}^+ \text{ such that } n \geq 648$$

Since $n \geq 648$ and $y > n$, then $y > 1$. This is the reason why in the interval $\left[y - y^{\frac{2}{3}}, y\right]$ there is always at least one prime number, according to ***Theorem 1***.

In other words, for every positive integer $n \geq 648$ the Breusch Interval will contain at least one Modified Iwaniec-Pintz Interval, which will contain at least one prime number. This means that for every positive integer $n \geq 648$ the Breusch Interval contains at least one prime number.

We will also prove that there is at least one prime number in the Breusch Interval for every positive integer n such that $1 \leq n \leq 647$:

n	Example of prime number in the interval:	$\frac{9(n+3)}{8}$
1	2	**4.5**
2	2	**5.625**
3	3	**6.75**
4	5	**7.875**
5	5	**9**
6	7	**10.125**
7	7	**11.25**
8	11	**12.375**
9	11	**13.5**
10	11	**14.625**
11	11	**15.75**
12	13	**16.875**
13	13	**18**
14	17	**19.125**
15	17	**20.25**
16	17	**21.375**
17	17	**22.5**
18	19	**23.625**
19	19	**24.75**

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20	23	**25.875**
21	23	**27**
22	23	**28.125**
23	23	**29.25**
24	29	**30.375**
25	29	**31.5**
26	29	**32.625**
27	29	**33.75**
28	29	**34.875**
29	29	**36**
30	31	**37.125**
31	31	**38.25**
32	37	**39.375**
33	37	**40.5**
34	37	**41.625**
35	37	**42.75**
36	37	**43.875**
37	37	**45**
38	41	**46.125**
39	41	**47.25**
40	41	**48.375**
41	41	**49.5**
42	43	**50.625**
43	43	**51.75**
44	47	**52.875**
45	47	**54**
46	47	**55.125**
47	47	**56.25**
48	53	**57.375**
49	53	**58.5**
50	53	**59.625**
51	53	**60.75**
52	53	**61.875**
53	53	**63**
54	59	**64.125**
55	59	**65.25**
56	59	**66.375**
57	59	**67.5**
58	59	**68.625**
59	59	**69.75**
60	61	**70.875**
61	61	**72**

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62	67	**73.125**
63	67	**74.25**
64	67	**75.375**
65	67	**76.5**
66	67	**77.625**
67	67	**78.75**
68	71	**79.875**
69	71	**81**
70	71	**82.125**
71	71	**83.25**
72	73	**84.375**
73	73	**85.5**
74	79	**86.625**
75	79	**87.75**
76	79	**88.875**
77	79	**90**
78	79	**91.125**
79	79	**92.25**
80	83	**93.375**
81	83	**94.5**
82	83	**95.625**
83	83	**96.75**
84	89	**97.875**
85	89	**99**
86	89	**100.125**
87	89	**101.25**
88	89	**102.375**
89	89	**103.5**
90	97	**104.625**
91	97	**105.75**
92	97	**106.875**
93	97	**108**
94	97	**109.125**
95	97	**110.25**
96	97	**111.375**
97	97	**112.5**
98	101	**113.625**
99	101	**114.75**
100	101	**115.875**
101	101	**117**
102	103	**118.125**
103	103	**119.25**

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104	107	**120.375**
105	107	**121.5**
106	107	**122.625**
107	107	**123.75**
108	109	**124.875**
109	109	**126**
110	113	**127.125**
111	113	**128.25**
112	113	**129.375**
113	113	**130.5**
114	127	**131.625**
115	127	**132.75**
116	127	**133.875**
117	127	**135**
118	127	**136.125**
119	127	**137.25**
120	127	**138.375**
121	127	**139.5**
122	127	**140.625**
123	127	**141.75**
124	127	**142.875**
125	127	**144**
126	127	**145.125**
127	127	**146.25**
128	131	**147.375**
129	131	**148.5**
130	131	**149.625**
131	131	**150.75**
132	137	**151.875**
133	137	**153**
134	137	**154.125**
135	137	**155.25**
136	137	**156.375**
137	137	**157.5**
138	139	**158.625**
139	139	**159.75**
140	149	**160.875**
141	149	**162**
142	149	**163.125**
143	149	**164.25**
144	149	**165.375**
145	149	**166.5**

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146	149	**167.625**
147	149	**168.75**
148	149	**169.875**
149	149	**171**
150	151	**172.125**
151	151	**173.25**
152	157	**174.375**
153	157	**175.5**
154	157	**176.625**
155	157	**177.75**
156	157	**178.875**
157	157	**180**
158	163	**181.125**
159	163	**182.25**
160	163	**183.375**
161	163	**184.5**
162	163	**185.625**
163	163	**186.75**
164	167	**187.875**
165	167	**189**
166	167	**190.125**
167	167	**191.25**
168	173	**192.375**
169	173	**193.5**
170	173	**194.625**
171	173	**195.75**
172	173	**196.875**
173	173	**198**
174	179	**199.125**
175	179	**200.25**
176	179	**201.375**
177	179	**202.5**
178	179	**203.625**
179	179	**204.75**
180	181	**205.875**
181	181	**207**
182	191	**208.125**
183	191	**209.25**
184	191	**210.375**
185	191	**211.5**
186	191	**212.625**
187	191	**213.75**

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188	191	**214.875**
189	191	**216**
190	191	**217.125**
191	191	**218.25**
192	193	**219.375**
193	193	**220.5**
194	197	**221.625**
195	197	**222.75**
196	197	**223.875**
197	197	**225**
198	199	**226.125**
199	199	**227.25**
200	211	**228.375**
201	211	**229.5**
202	211	**230.625**
203	211	**231.75**
204	211	**232.875**
205	211	**234**
206	211	**235.125**
207	211	**236.25**
208	211	**237.375**
209	211	**238.5**
210	211	**239.625**
211	211	**240.75**
212	223	**241.875**
213	223	**243**
214	223	**244.125**
215	223	**245.25**
216	223	**246.375**
217	223	**247.5**
218	223	**248.625**
219	223	**249.75**
220	223	**250.875**
221	223	**252**
222	223	**253.125**
223	223	**254.25**
224	227	**255.375**
225	227	**256.5**
226	227	**257.625**
227	227	**258.75**
228	229	**259.875**
229	229	**261**

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230	233	**262.125**
231	233	**263.25**
232	233	**264.375**
233	233	**265.5**
234	239	**266.625**
235	239	**267.75**
236	239	**268.875**
237	239	**270**
238	239	**271.125**
239	239	**272.25**
240	241	**273.375**
241	241	**274.5**
242	251	**275.625**
243	251	**276.75**
244	251	**277.875**
245	251	**279**
246	251	**280.125**
247	251	**281.25**
248	251	**282.375**
249	251	**283.5**
250	251	**284.625**
251	251	**285.75**
252	257	**286.875**
253	257	**288**
254	257	**289.125**
255	257	**290.25**
256	257	**291.375**
257	257	**292.5**
258	263	**293.625**
259	263	**294.75**
260	263	**295.875**
261	263	**297**
262	263	**298.125**
263	263	**299.25**
264	269	**300.375**
265	269	**301.5**
266	269	**302.625**
267	269	**303.75**
268	269	**304.875**
269	269	**306**
270	271	**307.125**
271	271	**308.25**

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272	277	**309.375**
273	277	**310.5**
274	277	**311.625**
275	277	**312.75**
276	277	**313.875**
277	277	**315**
278	281	**316.125**
279	281	**317.25**
280	281	**318.375**
281	281	**319.5**
282	283	**320.625**
283	283	**321.75**
284	293	**322.875**
285	293	**324**
286	293	**325.125**
287	293	**326.25**
288	293	**327.375**
289	293	**328.5**
290	293	**329.625**
291	293	**330.75**
292	293	**331.875**
293	293	**333**
294	307	**334.125**
295	307	**335.25**
296	307	**336.375**
297	307	**337.5**
298	307	**338.625**
299	307	**339.75**
300	307	**340.875**
301	307	**342**
302	307	**343.125**
303	307	**344.25**
304	307	**345.375**
305	307	**346.5**
306	307	**347.625**
307	307	**348.75**
308	311	**349.875**
309	311	**351**
310	311	**352.125**
311	311	**353.25**
312	313	**354.375**
313	313	**355.5**

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314	317	**356.625**
315	317	**357.75**
316	317	**358.875**
317	317	**360**
318	331	**361.125**
319	331	**362.25**
320	331	**363.375**
321	331	**364.5**
322	331	**365.625**
323	331	**366.75**
324	331	**367.875**
325	331	**369**
326	331	**370.125**
327	331	**371.25**
328	331	**372.375**
329	331	**373.5**
330	331	**374.625**
331	331	**375.75**
332	337	**376.875**
333	337	**378**
334	337	**379.125**
335	337	**380.25**
336	337	**381.375**
337	337	**382.5**
338	347	**383.625**
339	347	**384.75**
340	347	**385.875**
341	347	**387**
342	347	**388.125**
343	347	**389.25**
344	347	**390.375**
345	347	**391.5**
346	347	**392.625**
347	347	**393.75**
348	349	**394.875**
349	349	**396**
350	353	**397.125**
351	353	**398.25**
352	353	**399.375**
353	353	**400.5**
354	359	**401.625**
355	359	**402.75**

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356	359	**403.875**
357	359	**405**
358	359	**406.125**
359	359	**407.25**
360	367	**408.375**
361	367	**409.5**
362	367	**410.625**
363	367	**411.75**
364	367	**412.875**
365	367	**414**
366	367	**415.125**
367	367	**416.25**
368	373	**417.375**
369	373	**418.5**
370	373	**419.625**
371	373	**420.75**
372	373	**421.875**
373	373	**423**
374	379	**424.125**
375	379	**425.25**
376	379	**426.375**
377	379	**427.5**
378	379	**428.625**
379	379	**429.75**
380	383	**430.875**
381	383	**432**
382	383	**433.125**
383	383	**434.25**
384	389	**435.375**
385	389	**436.5**
386	389	**437.625**
387	389	**438.75**
388	389	**439.875**
389	389	**441**
390	397	**442.125**
391	397	**443.25**
392	397	**444.375**
393	397	**445.5**
394	397	**446.625**
395	397	**447.75**
396	397	**448.875**
397	397	**450**

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398	401	**451.125**
399	401	**452.25**
400	401	**453.375**
401	401	**454.5**
402	409	**455.625**
403	409	**456.75**
404	409	**457.875**
405	409	**459**
406	409	**460.125**
407	409	**461.25**
408	409	**462.375**
409	409	**463.5**
410	419	**464.625**
411	419	**465.75**
412	419	**466.875**
413	419	**468**
414	419	**469.125**
415	419	**470.25**
416	419	**471.375**
417	419	**472.5**
418	419	**473.625**
419	419	**474.75**
420	421	**475.875**
421	421	**477**
422	431	**478.125**
423	431	**479.25**
424	431	**480.375**
425	431	**481.5**
426	431	**482.625**
427	431	**483.75**
428	431	**484.875**
429	431	**486**
430	431	**487.125**
431	431	**488.25**
432	433	**489.375**
433	433	**490.5**
434	439	**491.625**
435	439	**492.75**
436	439	**493.875**
437	439	**495**
438	439	**496.125**
439	439	**497.25**

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440	443	**498.375**
441	443	**499.5**
442	443	**500.625**
443	443	**501.75**
444	449	**502.875**
445	449	**504**
446	449	**505.125**
447	449	**506.25**
448	449	**507.375**
449	449	**508.5**
450	457	**509.625**
451	457	**510.75**
452	457	**511.875**
453	457	**513**
454	457	**514.125**
455	457	**515.25**
456	457	**516.375**
457	457	**517.5**
458	461	**518.625**
459	461	**519.75**
460	461	**520.875**
461	461	**522**
462	463	**523.125**
463	463	**524.25**
464	467	**525.375**
465	467	**526.5**
466	467	**527.625**
467	467	**528.75**
468	479	**529.875**
469	479	**531**
470	479	**532.125**
471	479	**533.25**
472	479	**534.375**
473	479	**535.5**
474	479	**536.625**
475	479	**537.75**
476	479	**538.875**
477	479	**540**
478	479	**541.125**
479	479	**542.25**
480	487	**543.375**
481	487	**544.5**

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482	487	**545.625**
483	487	**546.75**
484	487	**547.875**
485	487	**549**
486	487	**550.125**
487	487	**551.25**
488	491	**552.375**
489	491	**553.5**
490	491	**554.625**
491	491	**555.75**
492	499	**556.875**
493	499	**558**
494	499	**559.125**
495	499	**560.25**
496	499	**561.375**
497	499	**562.5**
498	499	**563.625**
499	499	**564.75**
500	503	**565.875**
501	503	**567**
502	503	**568.125**
503	503	**569.25**
504	509	**570.375**
505	509	**571.5**
506	509	**572.625**
507	509	**573.75**
508	509	**574.875**
509	509	**576**
510	521	**577.125**
511	521	**578.25**
512	521	**579.375**
513	521	**580.5**
514	521	**581.625**
515	521	**582.75**
516	521	**583.875**
517	521	**585**
518	521	**586.125**
519	521	**587.25**
520	521	**588.375**
521	521	**589.5**
522	523	**590.625**
523	523	**591.75**

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524	541	**592.875**
525	541	**594**
526	541	**595.125**
527	541	**596.25**
528	541	**597.375**
529	541	**598.5**
530	541	**599.625**
531	541	**600.75**
532	541	**601.875**
533	541	**603**
534	541	**604.125**
535	541	**605.25**
536	541	**606.375**
537	541	**607.5**
538	541	**608.625**
539	541	**609.75**
540	541	**610.875**
541	541	**612**
542	547	**613.125**
543	547	**614.25**
544	547	**615.375**
545	547	**616.5**
546	547	**617.625**
547	547	**618.75**
548	557	**619.875**
549	557	**621**
550	557	**622.125**
551	557	**623.25**
552	557	**624.375**
553	557	**625.5**
554	557	**626.625**
555	557	**627.75**
556	557	**628.875**
557	557	**630**
558	563	**631.125**
559	563	**632.25**
560	563	**633.375**
561	563	**634.5**
562	563	**635.625**
563	563	**636.75**
564	569	**637.875**
565	569	**639**

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566	569	**640.125**
567	569	**641.25**
568	569	**642.375**
569	569	**643.5**
570	571	**644.625**
571	571	**645.75**
572	577	**646.875**
573	577	**648**
574	577	**649.125**
575	577	**650.25**
576	577	**651.375**
577	577	**652.5**
578	587	**653.625**
579	587	**654.75**
580	587	**655.875**
581	587	**657**
582	587	**658.125**
583	587	**659.25**
584	587	**660.375**
585	587	**661.5**
586	587	**662.625**
587	587	**663.75**
588	593	**664.875**
589	593	**666**
590	593	**667.125**
591	593	**668.25**
592	593	**669.375**
593	593	**670.5**
594	599	**671.625**
595	599	**672.75**
596	599	**673.875**
597	599	**675**
598	599	**676.125**
599	599	**677.25**
600	601	**678.375**
601	601	**679.5**
602	607	**680.625**
603	607	**681.75**
604	607	**682.875**
605	607	**684**
606	607	**685.125**
607	607	**686.25**

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608	613	**687.375**
609	613	**688.5**
610	613	**689.625**
611	613	**690.75**
612	613	**691.875**
613	613	**693**
614	617	**694.125**
615	617	**695.25**
616	617	**696.375**
617	617	**697.5**
618	619	**698.625**
619	619	**699.75**
620	631	**700.875**
621	631	**702**
622	631	**703.125**
623	631	**704.25**
624	631	**705.375**
625	631	**706.5**
626	631	**707.625**
627	631	**708.75**
628	631	**709.875**
629	631	**711**
630	631	**712.125**
631	631	**713.25**
632	641	**714.375**
633	641	**715.5**
634	641	**716.625**
635	641	**717.75**
636	641	**718.875**
637	641	**720**
638	641	**721.125**
639	641	**722.25**
640	641	**723.375**
641	641	**724.5**
642	643	**725.625**
643	643	**726.75**
644	647	**727.875**
645	647	**729**
646	647	**730.125**
647	647	**731.25**

6. Breusch Interval Conclusion

It has been proved that *for every positive integer n there is at least one prime number in the Breusch Interval* $\left[n, \frac{9(n+3)}{8}\right]$.

$$\boxed{\forall n \in \mathbb{Z}^{+}, \quad \pi\left[n, \frac{9\,(n+3)}{8}\right] \geq 1}$$

7. Theorems 2, 3 and 4

We just proved that for every positive integer n there is always a prime number in the Breusch Interval $\left[n, \frac{9(n+3)}{8}\right]$.

Remark 2. In this document, whenever we say that a number b is **between** a number a and a number c, it will mean that $a < b < c$, which means that b will never be equal to a or c (the same rule will be applied to intervals). Moreover, the number n that we will use in this document will always be a positive **integer**.

Let us suppose that a is a positive integer. We need to know what value n needs to have so that there exist a prime number p and a prime number q such that $\frac{n}{a} < p < \frac{3n}{2a} < q < \frac{2n}{a}$:

$$\text{if} \qquad \frac{n}{a} < p < \frac{3n}{2a} < q < \frac{2n}{a},$$

$$\text{then} \quad n < ap < \frac{3n}{2} < aq < 2n$$

In order to achieve the goal, we need to know what value n needs to have so that there exist at least one Breusch Interval (and therefore a prime number) between $\frac{n}{a}$ and $\frac{3n}{2a}$ and at least one Breusch Interval between $\frac{3n}{2a}$ and $\frac{2n}{a}$.

1) The integer immediately following the number $\frac{n}{a}$ will be called $\frac{n}{a} + d$. In other words, $\frac{n}{a} + d$ will be the smallest integer that is greater than $\frac{n}{a}$:

- If $\frac{n}{a}$ is an integer, then $\frac{n}{a} + d = \frac{n}{a} + 1$, because in this case we have $d = 1$.
- If $\frac{n}{a}$ is not an integer, then in the expression $\frac{n}{a} + d$ we have $0 < d < 1$, but we do not have any way of knowing the exact value of d if we do not know the value of $\frac{n}{a}$ first.

The reason why we work in this way is because the number n that we will use in the Breusch Interval will always be a positive **integer**.

Now, let us make the calculation:

$$\frac{9\left(\frac{n}{a}+d+3\right)}{8} < \frac{3n}{2a}$$
$$9\left(\frac{n}{a}+d+3\right) < 8 \times \frac{3n}{2a}$$
$$9\left(\frac{n}{a}+d+3\right) < \frac{24n}{2a}$$
$$9\left(\frac{n}{a}+d+3\right) < \frac{12n}{a}$$
$$\frac{9n}{a}+9d+27 < \frac{12n}{a}$$
$$9d+27 < \frac{12n}{a}-\frac{9n}{a}$$
$$9d+27 < \frac{3n}{a}$$

We need to take the largest possible value of d, which is $d = 1$ (if $9 \times 1 + 27 < \frac{3n}{a}$, then $9d + 27 < \frac{3n}{a}$ for all d such that $0 < d \leq 1$):

$$9 \times 1+27 < \frac{3n}{a}$$
$$9+27 < \frac{3n}{a}$$
$$36 < \frac{3n}{a}$$
$$\frac{3n}{a} > 36$$
$$\frac{n}{a} > \frac{36}{3}$$
$$\frac{n}{a} > 12$$

Result 1: $n > 12a$

So, if $n > 12a$ then there is at least one prime number p such that $\frac{n}{a} < p < \frac{3n}{2a}$. This means that if $n > 12a$, then there exists a prime number p such that

$$n < ap < \frac{3n}{2} \quad \text{(we multiply the previous numbers by } a\text{)}$$

2) Now, we need to calculate what value n needs to have so that there is at least one Breusch Interval (and therefore a prime number) between $\frac{3n}{2a}$ and $\frac{2n}{a}$. The integer immediately following the number $\frac{3n}{2a}$ will be called $\frac{3n}{2a} + d$. In other words, $\frac{3n}{2a} + d$ will be the smallest integer that is greater than $\frac{3n}{2a}$.

Let us take into account that $d = 1$ or $0 < d < 1$ depending on whether $\frac{3n}{2a}$ is an integer or not, respectively.

$$\begin{aligned}
\frac{9\left(\frac{3n}{2a}+d+3\right)}{8} &< \frac{2n}{a} \\
9\left(\frac{3n}{2a}+d+3\right) &< 8\times\frac{2n}{a} \\
9\left(\frac{3n}{2a}+d+3\right) &< \frac{16n}{a} \\
\frac{27n}{2a}+9d+27 &< \frac{16n}{a} \\
9d+27 &< \frac{16n}{a}-\frac{27n}{2a} \\
9d+27 &< \frac{2\times 16n-27n}{2a} \\
9d+27 &< \frac{32n-27n}{2a} \\
9d+27 &< \frac{5n}{2a}
\end{aligned}$$

Now, we need to take the largest possible value of d, which is $d = 1$ (if $9\times 1+27 < \frac{5n}{2a}$, then $9d+27 < \frac{5n}{2a}$ for all d such that $0 < d \leq 1$):

$$\begin{aligned}
9\times 1+27 &< \frac{5n}{2a} \\
9+27 &< \frac{5n}{2a} \\
36 &< \frac{5n}{2a} \\
\frac{5n}{2a} &> 36 \\
5n &> 36\times 2a \\
5n &> 72a \\
n &> \frac{72a}{5}
\end{aligned}$$

Result 2: $n > 14.4a$

So, if $n > 14.4a$ then there is at least one prime number q such that $\frac{3n}{2a} < q < \frac{2n}{a}$. This means that if $n > 14.4a$, then there exists a prime number q such that

$$\frac{3n}{2} < aq < 2n \quad \text{(we multiply the previous numbers by } a\text{)}$$

So far we have these two results:

Result 1: $n > 12a$ **Result 2:** $n > 14.4a$

If $n > 14.4a$, then $n > 12a$. To sum up, we have a general result:

General Result: $n > 14.4a$

Theorem 2. *Consequently, if a is a positive integer, then for every positive integer n such that $n > 14.4a$ there is always a pair of prime numbers p and q such that $n < ap < \frac{3n}{2} < aq < 2n$.*

Theorem 3. *According to Theorem 2, we can say that if $n > 14.4$, then there always exist prime numbers r and s such that $n < r < \frac{3n}{2} < s < 2n$.*

If $n > 14.4a$, then there is always a pair of prime numbers p and q such that

$$n < ap < \frac{3n}{2} < aq < 2n, \text{ according to } \textbf{\textit{Theorem 2}}$$

If $n > 14.4$, then there is always a pair of prime numbers r and s such that

$$n < r < \frac{3n}{2} < s < 2n, \text{ according to } \textbf{\textit{Theorem 3}}$$

If $n > 14.4a$ and $a > 1$, then $n > 14.4$. As a consequence, we will state the following theorem.

Theorem 4. *If a is any positive integer greater than 1, then for every positive integer $n > 14.4a$ there are always at least four prime numbers p, q, r and s such that $n < ap < \frac{3n}{2} < aq < 2n$* ***and simultaneously*** *$n < r < \frac{3n}{2} < s < 2n$.*

8. Infinitely Many Primes of the Forms $ap + b$ and $ap - b$

Dirichlet's Theorem states that if a and b are two positive integers which are coprime, then the arithmetic progression $an+b$ contains infinitely many prime numbers. In the expression $an + b$ the number n is considered as a positive integer by some authors ($n \geq 1$), and it is considered as a non-negative integer by others ($n \geq 0$). In this document, n is considered as a positive integer.

In the expression $an+b$, the numbers a and b must be coprime, but these coprime numbers do not need to have different parity (they can have the same or different parity), because the number n can be even as well as odd.

In order for $ap+b$ (where p is a prime number) to be able to generate infinitely many prime numbers, the numbers a and b must be coprime and they must also have different parity. This is because even if a and b are coprime, if these numbers are both odd then $ap+b$ can generate at most one prime number (only when $p=2$).

When we say that the numbers a and b must have different *parity*, we mean that:

- If a is even, then b must be odd.
- If a is odd, then b must be even.

Remark 3. The numbers a and b must be coprime, because if they are not, then neither $ap+b$ nor $ap-b$ can generate infinitely many prime numbers, since only composite numbers will be generated or at most a finite number of primes will be generated. The numbers a and b must also have different parity, otherwise only composite numbers will be generated or at most only one prime number will be generated, as explained before.

In the article ***PRIMOS GEMELOS, DEMOSTRACIÓN KMELLIZA*** [2], **Carlos Giraldo Ospina** proves that if k is any positive even integer, then there are infinitely many prime numbers p such that $p+k$ is also a prime number. **In particular, this author proves that there are infinitely many twin prime numbers, solving one of the unsolved problems in Mathematics.**

We are going to prove that if a and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap+b$ (where p is a prime number) and infinitely many prime numbers of the form $ap-b$.

In order to achieve our goal, we are going to use the same method that C. G. Ospina used to prove Polignac's Conjecture.

Proof.

According to ***Theorem 4***, if $a>1$ is an integer, then for every positive integer $n>14.4a$ there exist prime numbers p and s, such that

$$n < ap < \frac{3n}{2} < s < 2n$$

Definition 4. The numbers ap and s form what we will call *'pair (ap, prime) of order b'*. This is because:

- We have a pair of numbers: a number ap and a prime number s.
- The number ap is followed by the prime number s. In other words, $ap < s$.
- We say that $ap + b = s$. In other words, we say the pair (ap, prime) is 'of order b' because the difference between the numbers forming this pair is b.

Definition 5. The set made up of all positive integers z such that $z > 14.4a$ will be called **Set S(a)**.

Obviously, Set S(a) changes as the value of a changes, that is to say, Set S(a) depends on the value of a.

Examples:

- Set S(2) is the set of all positive integers z such that $z > 14.4 \times 2$. In other words, Set S(2) is made up of all positive integers z such that $z > 28.8$.
- Set S(3) is the set of all positive integers z such that $z > 14.4 \times 3$. In other words, Set S(3) is made up of all positive integers $z > 43.2$.

1. Let us choose any positive integer $a > 1$ and let us suppose that in Set S(a) there are no pairs (ap, prime) of order $b < u$ starting from $n = u$.
   The numbers n and u are positive integers which belong to Set S(a), and the number b is a positive integer such that a and b are coprime and have different parity.
2. Between u and $2u$, $n = u$ (and therefore in the interval $[u, 2u]$) there is at least one pair (ap, prime), according to ***Theorem 4***.
3. The difference between a number ap and a prime number which are located between u and $2u$ is $b < u$.
4. Between u and $2u$, $n = u$ (and therefore in the interval $[u, 2u]$) there is at least one pair (ap, prime) of order $b < u$, according to statements 2. and 3.
5. In Set S(a), starting from $n = u$ there is at least one pair (ap, prime) of order $b < u$, according to statement 4.
6. Statement 5. contradicts statement 1.
7. Therefore, no kind of pair (ap, prime) of any order b can be finite, according to statement 6. □

This means that if $a > 1$ is an integer, then for every positive integer b such that a and b are coprime and have different parity there are infinitely many prime numbers of the form $ap + b$, where p is a prime number. As a consequence, we can state the following theorem.

Theorem 5. *If $a > 1$ and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap + b$, where p is a prime number.*

Theorem 5 proves that there are infinitely many prime numbers of the form $2p+1$, where p is a prime number. This is because the numbers 2 and 1 are coprime and they also have different parity.

This proves that there are infinitely many prime numbers p such that $2p+1$ is also a prime number, which proves there are infinitely many Sophie Germain Prime Numbers. Consequently, one of the unsolved problems in Mathematics is now solved.

Next, we are going to prove another very important theorem.

According to ***Theorem 4***, if $a > 1$ is an integer, then for every positive integer $n > 14.4a$ there exist prime numbers r and q such that

$$n < r < \frac{3n}{2} < aq < 2n$$

Definition 6. The numbers r and aq form what we will call *'pair (prime, aq) of order b'*, because

- We have a pair of numbers: a prime number r and a number aq.
- The prime number r is followed by the number aq. In other words, $r < aq$.
- We say that $r + b = aq$. In other words, we say the pair (prime, aq) is 'of order b' because the difference between the numbers forming this pair is b.

Again, the set made up of all positive integers z such that $z > 14.4a$ will be called **Set S(a)** (see Definition 5.). As we said before, Set S(a) changes as the value of a changes.

1. Let us choose any positive integer $a > 1$ and let us suppose that in Set S(a) there are no pairs (prime, aq) of order $b < u$ starting from $n = u$.
   The numbers n and u are positive integers which belong to Set S(a), and the number b is a positive integer such that a and b are coprime and have different parity.
2. Between u and $2u$, $n = u$ (and therefore in the interval $[u, 2u]$) there is at least one pair (prime, aq), according to ***Theorem 4***.
3. The difference between a prime number and a number aq which are located between u and $2u$ is $b < u$.
4. Between u and $2u$, $n = u$ (and therefore in the interval $[u, 2u]$) there is at least one pair (prime, aq) of order $b < u$, according to statements 2. and 3.

5. In Set S(a), starting from $n = u$ there is at least one pair (prime, aq) of order $b < u$, according to statement 4.
6. Statement 5. contradicts statement 1.
7. Therefore, no kind of pair (prime, aq) of any order b can be finite, according to statement 6.

This means that if $a > 1$ is an integer, then for every positive integer b such that a and b are coprime and have different parity there are infinitely many prime numbers r such that $r + b = aq$. This means that there are infinitely many prime numbers r which are prime numbers of the form $aq - b$, where q is, as we said before, a prime number.

In the expression $aq - b$, we will change letter q for letter p and we will state Theorem 6.

Theorem 6. *If $a > 1$ and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap - b$, where p is a prime number.*

If we consider the case where $a = 1$ and we follow the procedures described in this section (Section 8.), we can easily prove that for every positive even integer k there are infinitely many prime numbers p such that $p + k$ is also a prime number. This is basically how C. G. Ospina already proved Polignac's Conjecture in his paper.

In the expression $p + k$, the number 1 and the positive even integer k are coprime and have different parity ($p + k = 1p + k$). For this reason, **Theorems 5 and 6 can also be applied to the case where $a = 1$**.

9. Landau's Problems

Landau's Problems are four problems in Number Theory concerning prime numbers:

- **Goldbach's Conjecture:** This conjecture states that every positive even integer greater than 2 can be expressed as the sum of two (not necessarily different) prime numbers.
- **Twin Prime Conjecture:** Are there infinitely many prime numbers p such that $p+2$ is also a prime number? This problem was solved by **Carlos Giraldo Ospina (Lic. Matemáticas, USC, Cali, Colombia)**, who proved that if k is any positive even integer, then there are infinitely many prime numbers p such that $p + k$ is also a prime number.
- **Legendre's Conjecture:** Is there always at least one prime number between n^2 and $(n + 1)^2$ for every positive integer n?
- **Primes of the form $n^2 + 1$:** Are there infinitely many prime numbers of the form $n^2 + 1$ (where n is a positive integer)?

Please see the article **Primos Gemelos, Demostración Kmelliza** [2], where C. G. Ospina shows the method he used to solve the Twin Prime Conjecture.

10. Primes of the form $n^2 + 1$

In this document we are going to prove that there are infinitely many prime numbers of the form $n^2 + 1$. In order to achieve our goal, we are going to use the same method that C. G. Ospina used to prove Polignac's Conjecture.

11. Theorems 7, 8 and 9

Let us suppose that n is a positive integer. We need to know what value n needs to have so that there is always a perfect square a^2 such that $n < a^2 < \frac{3n}{2}$.

Remark 4. We say a number is a *perfect square* if it is the square of an integer. In other words, a number x is a perfect square if $\sqrt{x}$ is an integer. Perfect squares are also called *square numbers*.

In general, if m is any positive integer, we need to know what value n needs to have so that there is always a positive integer a such that $n < a^m < \frac{3n}{2}$.

We have $n < a^m < \frac{3n}{2}$. This means that

$$n < a^m \text{ and } a^m < \frac{3n}{2}$$

$$\sqrt[m]{n} < a \text{ and } a < \sqrt[m]{\frac{3n}{2}}$$

To sum up,

$$\sqrt[m]{n} < a < \sqrt[m]{\frac{3n}{2}}$$

As we said before, the number a is a positive integer. Now, the integer immediately following the number $\sqrt[m]{n}$ will be called $\sqrt[m]{n} + d$. In other words, $\sqrt[m]{n} + d$ is the smallest integer that is greater than $\sqrt[m]{n}$:

- If $\sqrt[m]{n}$ is an integer, then $\sqrt[m]{n} + d = \sqrt[m]{n} + 1$, because in this case we have $d = 1$.
- If $\sqrt[m]{n}$ is not an integer, then in the expression $\sqrt[m]{n} + d$ we have $0 < d < 1$, but we do not have any way of knowing the exact value of d if we do not know the value of $\sqrt[m]{n}$ first.

Now, let us make the calculation:

$$\sqrt[m]{n} + d \quad < \quad \sqrt[m]{\frac{3n}{2}}$$

We need to take the largest possible value of d, which is $d = 1$ (if $\sqrt[m]{n} + 1 < \sqrt[m]{\frac{3n}{2}}$, then $\sqrt[m]{n} + d < \sqrt[m]{\frac{3n}{2}}$ for all d such that $0 < d \leq 1$):

$$\begin{aligned}
\sqrt[m]{n} + 1 &< \sqrt[m]{\frac{3n}{2}} \\
1 &< \sqrt[m]{\frac{3n}{2}} - \sqrt[m]{n} \\
1 &< \sqrt[m]{\frac{3}{2}n} - \sqrt[m]{n} \\
1 &< \sqrt[m]{1.5n} - \sqrt[m]{n} \\
1 &< \sqrt[m]{1.5}\,\sqrt[m]{n} - \sqrt[m]{n} \\
1 &< \sqrt[m]{n}\left(\sqrt[m]{1.5} - 1\right) \\
\frac{1}{\sqrt[m]{1.5} - 1} &< \sqrt[m]{n} \\
\left(\frac{1}{\sqrt[m]{1.5} - 1}\right)^m &< n \\
n &> \left(\frac{1}{\sqrt[m]{1.5} - 1}\right)^m \\
n &> \frac{1^m}{\left(\sqrt[m]{1.5} - 1\right)^m} \\
n &> \frac{1}{\left(\sqrt[m]{1.5} - 1\right)^m}
\end{aligned}$$

This means that if m is a positive integer, then for every positive integer $n > \frac{1}{\left(\sqrt[m]{1.5}-1\right)^m}$ there is at least one positive integer a such that $n < a^m < \frac{3n}{2}$. Now, if $n > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m}$ then $n > \frac{1}{\left(\sqrt[m]{1.5}-1\right)^m}$.

Theorem 7. *Consequently, if m is a positive integer, then for every positive integer $n > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m}$ there is at least one positive integer a such that $n < a^m < \frac{3n}{2}$.*

Now we are going to prove that if m is any positive integer, then $\frac{1}{\left(\sqrt[m]{1.5}-1\right)^m} > 1$.

Proof.

$$\begin{aligned}
\frac{1}{\left(\sqrt[m]{1.5}-1\right)^m} &> 1\\
1 &> 1\left(\sqrt[m]{1.5}-1\right)^m\\
1 &> \left(\sqrt[m]{1.5}-1\right)^m\\
\sqrt[m]{1} &> \sqrt[m]{1.5}-1\\
1 &> \sqrt[m]{1.5}-1\\
1+1 &> \sqrt[m]{1.5}\\
2 &> \sqrt[m]{1.5}\\
2^m &> 1.5
\end{aligned}$$

It is very easy to verify that $2^m > 1.5$ for every positive integer m. Consequently, if m is any positive integer, then $\frac{1}{\left(\sqrt[m]{1.5}-1\right)^m} > 1$. □

If $\frac{1}{\left(\sqrt[m]{1.5}-1\right)^m} > 1$, then $\frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m} > 14.4$. This means that $\frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m} > 14.4$ for every positive integer m.

Theorem 8. *As a consequence, if m is any positive integer and $n > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m}$, then $n > 14.4$.*

In Section 7. it was proved that for every positive integer $n > 14.4$ there exist prime numbers r and s such that $n < r < \frac{3n}{2} < s < 2n$. This true statement is called ***Theorem 3***.

Theorem 9. *According to Theorems 3, 7 and 8, if m is a positive integer, then for every positive integer $n > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m}$ there exist a positive integer a and a prime number s such that $n < a^m < \frac{3n}{2} < s < 2n$.*

12. Theorems 10, 11 and 12

Now, if m is a positive integer, let us calculate what value n needs to have so that there is always a positive integer a such that $\frac{3n}{2} < a^m < 2n$:

We have

$$\frac{3n}{2} < a^m < 2n$$

This means that

$$\frac{3n}{2} < a^m \text{ and } a^m < 2n$$
$$\sqrt[m]{\frac{3n}{2}} < a \text{ and } a < \sqrt[m]{2n}$$

To sum up,

$$\sqrt[m]{\frac{3n}{2}} < a < \sqrt[m]{2n}$$

The number a is a positive integer. Now, the integer immediately following the number $\sqrt[m]{\frac{3n}{2}}$ will be called $\sqrt[m]{\frac{3n}{2}}+d$. In other words, $\sqrt[m]{\frac{3n}{2}}+d$ is the smallest integer that is greater than $\sqrt[m]{\frac{3n}{2}}$:

- If $\sqrt[m]{\frac{3n}{2}}$ is an integer, then $\sqrt[m]{\frac{3n}{2}} + d = \sqrt[m]{\frac{3n}{2}} + 1$, because in this case we have $d = 1$.
- If $\sqrt[m]{\frac{3n}{2}}$ is not an integer, then in the expression $\sqrt[m]{\frac{3n}{2}} + d$ we have $0 < d < 1$, but we do not have any way of knowing the exact value of d if we do not know the value of $\sqrt[m]{\frac{3n}{2}}$ first.

Let us make the calculation:

$$\sqrt[m]{\frac{3n}{2}} + d \quad < \quad \sqrt[m]{2n}$$

We need to take the largest possible value of d, which is $d = 1$ (if $\sqrt[m]{\frac{3n}{2}} + 1 < \sqrt[m]{2n}$, then $\sqrt[m]{\frac{3n}{2}} + d < \sqrt[m]{2n}$ for all d such that $0 < d \leq 1$):

$$\begin{aligned}
\sqrt[m]{\frac{3n}{2}} + 1 \quad &< \quad \sqrt[m]{2n} \\
1 \quad &< \quad \sqrt[m]{2n} - \sqrt[m]{\frac{3n}{2}} \\
1 \quad &< \quad \sqrt[m]{2n} - \sqrt[m]{\frac{3}{2}n} \\
1 \quad &< \quad \sqrt[m]{2n} - \sqrt[m]{1.5n} \\
1 \quad &< \quad \sqrt[m]{2}\,\sqrt[m]{n} - \sqrt[m]{1.5}\,\sqrt[m]{n} \\
1 \quad &< \quad \sqrt[m]{n}\left(\sqrt[m]{2} - \sqrt[m]{1.5}\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{\sqrt[m]{2}-\sqrt[m]{1.5}} &< \sqrt[m]{n} \\
\left(\frac{1}{\sqrt[m]{2}-\sqrt[m]{1.5}}\right)^m &< n \\
\frac{1^m}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} &< n \\
\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} &< n \\
n &> \frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}
\end{aligned}
$$

This means that if m is a positive integer, then for every positive integer $n > \frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ there is at least one positive integer a such that $\frac{3n}{2} < a^m < 2n$. Now, if $n > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ then $n > \frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$.

Theorem 10. *Consequently, if m is a positive integer, then for every positive integer $n > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ there is at least one positive integer a such that $\frac{3n}{2} < a^m < 2n$.*

Now we are going to prove that if m is any positive integer, then $\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 1$.

Proof.

$$
\begin{aligned}
\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} &> 1 \\
1 &> 1\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m \\
1 &> \left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m \\
\sqrt[m]{1} &> \sqrt[m]{2}-\sqrt[m]{1.5} \\
1 &> \sqrt[m]{2}-\sqrt[m]{1.5} \\
1+\sqrt[m]{1.5} &> \sqrt[m]{2}
\end{aligned}
$$

$\sqrt[m]{1.5} > 1$ because $1.5 > 1^m$, that is to say, $1.5 > 1$.

$\sqrt[m]{2} \leq 2$ because $2 \leq 2^m$.

This means that

$$1+\sqrt[m]{1.5} > 2 \geq \sqrt[m]{2}$$

which proves that

$$1+\sqrt[m]{1.5} \quad > \quad \sqrt[m]{2}$$

Therefore, if m is any positive integer, then $\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 1$. □

If $\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 1$, then $\frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 14.4$. This means that $\frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 14.4$ for every positive integer m.

Theorem 11. *As a consequence, if m is any positive integer and $n > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$, then $n > 14.4$.*

Theorem 12. *According to Theorems 3, 10 and 11, if m is a positive integer, then for every positive integer $n > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ there exist a prime number r and a positive integer a such that $n < r < \frac{3n}{2} < a^m < 2n$.*

13. Infinitely many prime numbers of the form n^2+1

According to ***Theorem 9***, for every positive integer $n > \frac{14.4}{\left(\sqrt{1.5}-1\right)^2}$ there exist a positive integer a and a prime number s such that $n < a^2 < \frac{3n}{2} < s < 2n$.

Definition 7. The numbers a^2 and s form what we will call *'pair (perfect square, prime) of order k'*. This is because:

- We have a pair of numbers: a perfect square a^2 and a prime number s.
- The perfect square a^2 is followed by the prime number s. In other words, $a^2 < s$.
- We say that $a^2 + k = s$. In other words, we say the pair (perfect square, prime) is 'of order k' because the difference between the numbers forming this pair is k.

Now we need to define some other concepts:

Definition 8. The set made up of all positive integers z such that $z > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m}$ will be called **Set A(m)**. Examples:

- Set A(2) is the set of all positive integers z such that $z > \frac{14.4}{\left(\sqrt{1.5}-1\right)^2}$. In other words, Set A(2) is made up of all positive integers z such that $z > 285.09...$

- Set A(3) is the set of all positive integers z such that $z > \frac{14.4}{\left(\sqrt[3]{1.5}-1\right)^3}$.
  In other words, Set A(3) is made up of all positive integers $z > 4751.47...$

Definition 9. The set made up of all positive integers z such that $z > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ will be called **Set B(m)**.

Let us prove that there are infinitely many prime numbers of the form $n^2 + 1$. In order to achieve the goal, we are going to use the same method that C. G. Ospina used in his article ***Primos Gemelos, Demostración Kmelliza*** [2] to prove Polignac's Conjecture.

1. Let us suppose that in Set A(2) there are no pairs (perfect square, prime) of order $k < u$ starting from $n = u$.
   The numbers n and u are positive integers which belong to Set A(2).
2. Between u and $2u$, $n = u$, there is at least one pair (perfect square, prime), according to ***Theorem 9***.
3. The difference between two integers located between u and $2u$ is $k < u$.
4. Between u and $2u$, $n = u$, there is at least one pair (perfect square, prime) of order $k < u$, according to statements 2. and 3.
5. In Set A(2), starting from $n = u$ there is at least one pair (perfect square, prime) of order $k < u$, according to statement 4.
6. Statement 5. contradicts statement 1.
7. Therefore, no kind of pair (perfect square, prime) of any order k can be finite, according to statement 6.

Remark 5. It is already known that for every positive integer k the polynomial $n^2 + k$ is irreducible over $\mathbb{R}$ and thus irreducible over $\mathbb{Z}$, since every second-degree polynomial whose discriminant is a negative number is irreducible over $\mathbb{R}$.

According to statement 7., for every positive integer k there are infinitely many pairs (perfect square, prime) of order k. This means that pairs (perfect square, prime) of order 1 can not be finite. In other words, prime numbers of the form $n^2 + 1$ can not be finite.

Theorem 13. *All this proves that for every positive integer k there are infinitely many prime numbers of the form $n^2 + k$.*

In general, if we use **Set A(m)**, **Set B(m)** and Theorems 9 and 12 and we use the same method, we could prove that there are infinitely many prime numbers of the form $n^m + k$ and infinitely many prime numbers of the form

$n^m - k$ for certain values of m and k (we only have to take into account the cases where the polynomials $n^m + k$ and $n^m - k$ are irreducible over $\mathbb{Z}$).

14. Final Conclusion

We will restate the three most important theorems that were proved in this document (see Sections 8. and 13.):

Theorem 5: If a and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap + b$, where p is a prime number.

Theorem 6: If a and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap - b$, where p is a prime number.

Theorem 13: For every positive integer k there are infinitely many prime numbers of the form $n^2 + k$, which means there are infinitely many primes of the form $n^2 + 1$.

Appendix A. New conjecture

Conjecture 1. *If n is any positive integer and we take n consecutive integers located between n^2 and $(n+1)^2$, then among those n integers there is at least one prime number. In other words, if $a_1, a_2, a_3, a_4, \ldots, a_n$ are n consecutive integers such that $n^2 < a_1 < a_2 < a_3 < a_4 < \ldots < a_n < (n+1)^2$, then at least one of those n integers is a prime number. This conjecture will be called **Conjecture C**.*

- **Legendre's Conjecture**

It is very easy to verify that the amount of integers located between n^2 and $(n+1)^2$ is equal to $2n$.

Proof.

$$\begin{aligned}(n+1)^2 - n^2 &= 2n+1\\ n^2+2n+1-n^2 &= 2n+1\\ 2n+1 &= 2n+1\end{aligned}$$

We need to exclude the number $(n+1)^2$ because we are taking into consideration the integers that are greater than n^2 and smaller than $(n+1)^2$:

$$2n+1-1 = 2n \qquad \square$$

According to this, between n^2 and $(n+1)^2$ there are two groups of n consecutive integers each that do not have any integer in common. Example for $n = 3$:

$$(3)^2 \quad \underbrace{\underbrace{10 \quad 11 \quad 12}_{\substack{\text{Group A}\\ (n\text{ consecutive integers})}} \quad \underbrace{13 \quad 14 \quad 15}_{\substack{\text{Group B}\\ (n\text{ consecutive integers})}}}_{2n\text{ consecutive integers}} \quad (3+1)^2$$

Group A and **Group B** do not have any integer in common. According to **Conjecture C**, Group A contains at least one prime number and Group B also contains at least one prime number, which means that between 3^2 and $(3+1)^2$ there are at least **two** prime numbers. This is true because the numbers 11 and 13 are both prime numbers.

All this means that if Conjecture C is true, then there are at least **two** prime numbers between n^2 and $(n+1)^2$ for every positive integer n. **As a result, if Conjecture C is true, then Legendre's Conjecture is also true.**

- **Brocard's Conjecture**

This conjecture states that if p_n and p_{n+1} are consecutive prime numbers greater than 2, then between $(p_n)^2$ and $(p_{n+1})^2$ there are at least four prime numbers.

Since $2 < p_n < p_{n+1}$, we have $p_{n+1} - p_n \geq 2$. This means that there is at least one positive integer a such that $p_n < a < p_{n+1}$. As a result, there is at least one positive integer a such that $(p_n)^2 < a^2 < (p_{n+1})^2$.

Conjecture C states that between $(p_n)^2$ and a^2 there are at least two prime numbers and that between a^2 and $(p_{n+1})^2$ there are also at least two prime numbers. In other words, if Conjecture C is true then there are at least four prime numbers between $(p_n)^2$ and $(p_{n+1})^2$. **As a consequence, if Conjecture C is true then Brocard's Conjecture is also true.**

- **Andrica's Conjecture**

This conjecture states that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ for every pair of consecutive prime numbers p_n and p_{n+1} (of course $p_n < p_{n+1}$).

Obviously, every prime number is located between two consecutive perfect squares. If we take any prime number p_n, which is obviously located between n^2 and $(n+1)^2$, two things may happen:

Case 1. The number p_n is located among the first n consecutive integers that are located between n^2 and $(n+1)^2$. These n integers form what we call *Group A*, and the following n integers form what we call *Group B*, as shown in the following graphic:

$$n^2 \quad < \quad \underbrace{\underbrace{\bullet \quad \bullet \quad \ldots \quad \bullet \quad \bullet}_{\substack{\text{Group A} \\ (n \text{ consecutive integers})}} \quad \underbrace{\bullet \quad \bullet \quad \ldots \quad \bullet \quad \bullet}_{\substack{\text{Group B} \\ (n \text{ consecutive integers})}}}_{2n \text{ consecutive integers}} \quad < \quad (n+1)^2$$

If p_n is located in Group A and Conjecture C is true, then p_{n+1} is either located in Group A or in Group B. In both cases we have $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$, because $\sqrt{(n+1)^2} - \sqrt{n^2} = 1$ and the numbers $\sqrt{p_{n+1}}$ and $\sqrt{p_n}$ are closer to each other than $\sqrt{(n+1)^2}$ in relation to $\sqrt{n^2}$.

Case 2. The prime number p_n is located in Group B.

If p_n is located in Group B and Conjecture C is true, it may happen that p_{n+1} is also located in Group B. In this case it is very easy to verify that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$, as explained before.

Otherwise, if p_{n+1} is not located in Group B, then p_{n+1} is located in Group C. In this case the largest value p_{n+1} can have is $p_{n+1} = (n+1)^2 + n + 1 = n^2 + 2n + 1 + n + 1 = n^2 + 3n + 2$ and the smallest value p_n can have is $p_n = n^2 + n + 1$ (in order to make the process easier, we are not taking into account that in this case the numbers p_n and p_{n+1} have different parity, so they can not be both prime at the same time).

This means that the largest possible difference between $\sqrt{p_{n+1}}$ and $\sqrt{p_n}$ is $\sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{n^2+3n+2} - \sqrt{n^2+n+1}$. Let us look at the following graphic:

$$n^2 < \quad \ldots \quad \overbrace{\underbrace{\triangle \quad \bullet \quad \ldots \quad \bullet \quad \bullet}_{\substack{\text{Group B}\\ (n \text{ consecutive integers})}} < (n+1)^2 < \underbrace{\bullet \quad \bullet \quad \ldots \quad \bullet \quad \bullet \quad \square}_{\substack{\text{Group C}\\ (n+1 \text{ consecutive integers})}}}^{\text{maximum distance between } p_n \text{ and } p_{n+1}}$$

$$\begin{aligned} \triangle &= n^2+n+1 = p_n \\ \square &= n^2+3n+2 = p_{n+1} \end{aligned}$$

It is easy to prove that $\sqrt{n^2+3n+2} - \sqrt{n^2+n+1} < 1$.

Proof.

$$\begin{aligned}
\sqrt{n^2+3n+2} - \sqrt{n^2+n+1} &< 1 \\
\sqrt{n^2+3n+2} &< 1 + \sqrt{n^2+n+1} \\
n^2+3n+2 &< \left(1 + \sqrt{n^2+n+1}\right)^2 \\
n^2+3n+2 &< 1 + 2\sqrt{n^2+n+1} + n^2 + n + 1 \\
n^2+3n+2-n^2-n-1 &< 1 + 2\sqrt{n^2+n+1} \\
2n+1 &< 1 + 2\sqrt{n^2+n+1} \\
2n &< 2\sqrt{n^2+n+1} \\
n &< \frac{2\sqrt{n^2+n+1}}{2} \\
n &< \sqrt{n^2+n+1} \\
n^2 &< n^2+n+1
\end{aligned}$$

which is true for every positive integer n. □

We can see that even when the difference between p_{n+1} and p_n is the largest possible difference, we have $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$. If the difference between p_{n+1} and p_n were smaller, then of course it would also happen that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$.

According to Cases 1. and 2., if Conjecture C is true then Andrica's Conjecture is also true.

To conclude, if Conjecture C is true, then Legendre's Conjecture, Brocard's Conjecture and Andrica's Conjecture are all true.

- **Possible new interval**

It is easy to verify that if Conjecture C is true, then in the interval $\left[n^2+n+1,\;\; n^2+3n+2\right]$ (see the graphics shown before) there are at least two prime numbers for every positive integer n.

The number n^2+n+1 will always be an odd integer.

Proof.

• If n is even, then n^2 is also even. Then we have

$$(\textit{even integer} + \textit{even integer})\ +\ 1\ =\ \textit{even integer} + \textit{odd integer}\ =$$
$$=\ \textit{odd integer}$$

• If n is odd, then n^2 is also odd. Then we have

$$(\textit{odd integer}\ +\ \textit{odd integer})\ +\ 1\ =\ \textit{even integer}\ +\ \textit{odd integer}\ =$$
$$=\ \textit{odd integer}$$ □

Since the number n^2+n+1 will always be an odd integer, then it may be prime or not.

Now, the number n^2+3n+2 can never be prime because this number will always be an even integer (and it will be greater than 2).

Proof.

• If $n=1$ (smallest value n can have), then $n^2+3n+2=1+3+2=6$.

• If n is even, then n^2 and $3n$ are both even integers. The number 2 is also an even integer, and we know that

$$\textit{even integer}\ +\ \textit{even integer}\ +\ \textit{even integer}\ =\ \textit{even integer}$$

• If n is odd, then n^2 and $3n$ are both odd integers, and we know that

$$(odd\ integer\ +\ odd\ integer)\ +\ even\ integer\ =$$
$$=\ even\ integer\ +\ even\ integer\ =\ even\ integer \qquad \square$$

From all this we deduce that if Conjecture C is true, then the maximum distance between two consecutive prime numbers is the one from the number n^2+n+1 to the number $n^2+3n+2-1 = n^2+3n+1$, which means that in the interval $[n^2+n+1,\ \ n^2+3n+1]$ there are at least two prime numbers. In other words, in the interval $[n^2+n+1,\ \ n^2+3n]$ there is at least one prime number.

The difference between the numbers n^2+n+1 and n^2+3n is $n^2+3n-(n^2+n+1) = n^2+3n-n^2-n-1 = 2n-1$. In addition to this, $\left\lfloor\sqrt{n^2+n+1}\right\rfloor = n$. This means that in the interval $\left[n^2+n+1,\ \ n^2+n+1+2\left\lfloor\sqrt{n^2+n+1}\right\rfloor-1\right]$ there is at least one prime number. In other words, if $a = n^2+n+1$ then the interval $[a,\ \ a+2\lfloor\sqrt{a}\rfloor-1]$ contains at least one prime number.

The symbol $\lfloor\rfloor$ represents the *floor function.* The floor function of a given number is the largest integer that is not greater than that number. For example, $\lfloor x\rfloor$ is the largest integer that is not greater than x.

Now, if Conjecture C is true, then the following statements are all true:

Statement 1. *If a is a perfect square, then in the interval $[a,\ \ a+\lfloor\sqrt{a}\rfloor]$ there is at least one prime number.*

Statement 2. *If a is an integer such that $n^2 < a \le n^2+n+1 < (n+1)^2$, then in the interval $[a,\ \ a+\lfloor\sqrt{a}\rfloor-1]$ there is at least one prime number.*

Statement 3. *If a is an integer such that $n^2 < n^2+n+2 \le a < (n+1)^2$, then in the interval $[a,\ \ a+2\lfloor\sqrt{a}\rfloor-1]$ there is at least one prime number.*

We know that $a+2\lfloor\sqrt{a}\rfloor-1 \ge a+\lfloor\sqrt{a}\rfloor$.

Proof.

$$a+2\left\lfloor\sqrt{a}\right\rfloor-1 \ge a+\left\lfloor\sqrt{a}\right\rfloor \quad\Leftrightarrow\quad 2\left\lfloor\sqrt{a}\right\rfloor-1 \ge \left\lfloor\sqrt{a}\right\rfloor \quad\Leftrightarrow$$

$$\Leftrightarrow\quad 2\left\lfloor\sqrt{a}\right\rfloor \ge \left\lfloor\sqrt{a}\right\rfloor+1 \quad\Leftrightarrow$$

$$\Leftrightarrow\quad \left\lfloor\sqrt{a}\right\rfloor+\left\lfloor\sqrt{a}\right\rfloor \ge \left\lfloor\sqrt{a}\right\rfloor+1 \quad\Leftrightarrow\quad \left\lfloor\sqrt{a}\right\rfloor \ge 1,$$

which is true for every positive integer a. $\square$

And we also know that $a + 2\left\lfloor \sqrt{a} \right\rfloor - 1 > a + \left\lfloor \sqrt{a} \right\rfloor - 1$.

Proof.

$$a + 2\left\lfloor \sqrt{a} \right\rfloor - 1 > a + \left\lfloor \sqrt{a} \right\rfloor - 1 \quad \Leftrightarrow \quad 2\left\lfloor \sqrt{a} \right\rfloor > \left\lfloor \sqrt{a} \right\rfloor,$$

which is obviously true for every positive integer a. □

All this means that the interval $[a,\ \ a + 2\left\lfloor \sqrt{a} \right\rfloor - 1]$ can be applied to the number a from Statement 1., to the number a from Statement 2. and to the number a from Statement 3.

Therefore, if n is any positive integer and Conjecture C is true, then in the interval $[n,\ \ n + 2\left\lfloor \sqrt{n} \right\rfloor - 1]$ there is at least one prime number (we change letter a for letter n). According to this, we can also say that if Conjecture C is true then in the interval $[n,\ \ n + 2\sqrt{n} - 1]$ there is always a prime number for every positive integer n.

Now... how can we prove Conjecture C?

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References

[1] Iwaniec, H. and Pintz, J. *Primes in Short Intervals*. Monatsh. f. Math. 98, 115-143, 1984.

[2] Carlos Giraldo Ospina. *Primos Gemelos, Demostración Kmelliza* (in Spanish). `http://matematicainsolita2.8m.com/ar90.htm`

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