PROOF

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ABSTRACT

I will present a proof of Euclid's fifh postulate (I.Post.5) that proves, as an intermediate step, a proposition equivalent to it (I.32); namely, that in any triangle, the sum of the three interior angles of the triangle equals two right angles. The proof that I.32 implies I.Post.5 and vice versa is well-established and will be omitted for the sake of brevity. The proof technique is somewhat unorthodox in that it proves I.33, which states that straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel, before establishing I.32, contrary to the order in which the propositions are demonstrated in Euclid's *Elements.*

Two triangle congruence theorems, namely the side-angle-side (I.4) and side-side-side congruence theorems (I.8) are employed in order to prove I.33 without recourse to I.Post.5 or any of its equivalent formulations. In addition, a parallelogram is constructed by an unorthodox method; namely, by defining the diagonals upon which the parallelogram's sides will be determined prior to the sides themselves. The proof assumes the five common notions stated in Book I of *The Elements* without explicitly making a reference to them when they are used. Furthermore, a figure is presented with color-coded angles and sides, with angles of the same color being equal in measure and sides of ***both the same color and the same number of tick marks*** being equal in length. The sides *GH* and *EJ* enclosed by brackets are indicated to be equal in length, the reason for the different notation being that the tick marks were used in reference to the halves of *GH*, namely *OG* and *OH*. The tick marks then refer to the parts of *GH*, and the bracket refers to the whole of *GH*; the latter is then equated to *EJ* by I.33, which is proven before its use.

PROOF

Let the straight line *EF* falling on the two straight lines *AB* and *CD* make the alternate angles *AEF* and *EFD* equal to one another. Then the straight lines *AB* and *CD* are parallel to one another (I.27). Bisect the finite straight line *EF* at *O* (I.10); we therefore have that *OE* and *OF* are equal to one another. Take the point *G* at random on the straight line *AB* on either side of *E* but not *E* itself. Cut off from *FC* or *FD* the greater, a straight line *FH* equal to *EG* the less such that *FH* and *EG* fall on *EF* from the opposite directions (I.3), and join *OG* and *OH*. Since triangles *GEO* and *HFO* have two sides equal to two sides, namely *EG* equal to *FH* and *OE* equal to *OF*, and have the angles *GEO* and *HFO* contained by the equal sides equal, *OG* and *OH* are equal to one another, and the angles *GOE* and *EGO* are equal to the angles *HOF* and *FHO*, respectively (I.4).

Since *EF* is a straight line, and *HO* is a straight line that falls on *EF* at *O*, the sum of the angles *HOE* and *HOF* is equal to two right angles (I.13). Said in another way, angle *HOE* is equal to the difference between two right angles and angle *HOF*. But angle *HOF* is equal to angle *GOE*, so the sum of the angles *HOE* and *GOE* is equal to two right angles. Therefore, *OG* and *OH* lie in a straight line with one another, namely *GH*, since they are two straight lines not lying on the same side of the straight line *EF* at the point *O* on it that make the sum of the adjacent angles *HOE* and *GOE* equal to two right angles (I.14).

Since *EF* and *GH* are two straight lines that cut one another at *O*, the vertical angles *FOG* and *EOH* are equal to one another. Because *AB* and *CD* are parallel to one another, *EG* and *FH* are also parallel to one another. They are also equal. Join the ends of *GE* and *FH* in the same directions, namely *G* with *F* and *E* with *H*. Since triangles *EOH* and *FOG* have two sides equal to two sides, namely *OH* equal to *OG* and *OE* equal to *OF*, and have the angles *EOH* and *FOG* contained by the equal sides equal, *EH* and *FG* are equal to one another, and the angles *OEH* and *OHE* are equal to the angles *OFG* and *OGF*, respectively (I.4).

But sides *EH* and *FG* are the straight lines joining *GE* and *FH* in the same directions, and *EH* and *FG* are equal to one another. Also, since *GH* is a straight line falling on two straight lines *EH* and *FG* that makes the alternate angles *HGF* and *GHE* equal to one another (*HGF* and *GHE* are equal to each other since they coincide with *OGF* and *OHE,* respectively, with *OGF* and *OHE* themselves being equal to each other), the straight lines *EH* and *FG* are parallel to one another. Therefore, straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Since angles *OFG* and *OFH* are equal to angles *OEH* and *OEG*, respectively, the sum of the angles *OFG* and *OFH* is equal to the sum of the angles *OEH* and *OEG*. But the first sum is equal to the angle *GFH* and the second sum is equal to the angle *HEG*. Therefore, angles *GFH* and *HEG* are equal to one another. We have also previously shown that angles *EGO* and *FHO* are equal to one another, and have also shown that angles *OGF* and *OHE* are equal to one another. Since *EGO* coincides with *EGH*, *FHO* coincides with *FHG*, *OGF* coincides with *HGF*, and *OHE* coincides with *GHE*, angles *EGH* and *FHG* are equal to one another, and angles *HGF* and *GHE* are equal to one another.

Cut off from straight line *HD* the greater a straight line *HJ* equal to *GE* the less (I.3). Since the straight lines *GH* and *EJ* are joined from the ends of equal and parallel lines *HJ* and *GE* (*HJ* and *GE* are parallel since they are each a part of parallel lines *CD* and *AB*, respectively) in the same directions, *GH* and *EJ* are themselves equal and parallel. Since triangles *HEG* and *EHJ* have two sides equal to two sides respectively, namely sides *HJ* and *GE* equal to one another and sides *GH* and *EJ* equal to one another, and also have the common side *EH* equal to *HE*, then they also have all corresponding angles equal to one another; namely angles *GHE* and *JEH* are equal to one another, angles *EGH* and *HJE* are equal to one another, and angles *HEG* and *EHJ* are equal to one another (I.8).

Since we have that both angles *EHJ* and *GFH* are equal to the same angle *HEG*, we have that the angles *EHJ* and *GFH* are equal to one another. We have also previously shown that angles *GHE* and *HGF* are equal to one another. Therefore, the sum of angles *EHJ* and *GHE* is equal to the sum of angles *GFH* and *HGF*. Adding the angle *FHG* to both sums shows that the sum of the angles *EHJ, GHE,* and *FHG* is equal to the sum of the angles *GFH*, *HGF*, and *FHG*. But the first of the two sums is equal to two right angles since *EH* and *GH* are straight lines that stand on the straight line *CD* at the point *H* on it (I.13), and the second of the two sums is the sum of the interior angles of the triangle *HFG*.

Since the sum of the interior angles of the triangle *HFG* and an angle measure of two right angles are both equal to the same quantity, the sum of the interior angles of the triangle *HFG* and two right angles are equal. Therefore, there exists a triangle, namely triangle *HFG*, whose interior angle sum equals two right angles. But the existence of one triangle whose interior angle sum equals two right angles implies that the interior angle sum of all triangles is equal to two right angles (I.32), which in turn implies I.Post.5, by well-established proofs.

FIGURE