Features of the Hilbert Book Model.

The HBM is a simple Higgsless model of fundamental physics that is strictly based on the axioms of traditional quantum logic. It uses a sequence of instances of an extension of a quaternionic separable Hilbert space that each represents a static status quo of the whole universe.

# Features of the Hilbert Book Model

Hans van Leunen

# The Hilbert Book Model

# FEATURES OF THE HILBERT BOOK MODEL

Colophon

Written by Ir J.A.J. van Leunen
The subject of this book is a new model of physics

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# FEATURES OF THE HILBERT BOOK MODEL

## *ACKNOWLEDGEMENTS*

I thank my wife Albertine, who tolerated me to work days and nights on a subject that can only be fully comprehended by experts in this field. For several years she had to share me with my text processor. She stimulated me to bring this project to a feasible temporary end, because this project is in fact a never ending story.

I also have to thank my friends and discussion partners that listened to my lengthy deliberations on this non society chitchat suitable subject and patiently tolerated that my insights changed regularly.

## *DETAILS*

The Hilbert Book Model is the result of a still ongoing research project.

That project started in 2009.

This version added sections on the progression factor, progression interval, the Higgs phenomenon and the covariant derivative.

The continuing status of the project can be followed at
http://www.crypts-of-physics.eu

The author's e-print site is:
http://vixra.org/author/j_a_j_van_leunen.
This book is accompanied by a slide show at
http://vixra.org/abs/1202.0033

*The nice thing about laws of physics is that they repeat themselves. Otherwise they would not be noticed. The task of physicists is to notice the repetition.*

# Contents

**Preface**

I started the Hilbert Book Model during my studies in physics in the sixties on the Technical University of Eindhoven (TUE).

In the first two years the lectures concerned only classical physics. In the third year quantum physics was introduced. I had great difficulty in understanding why the methodology of doing physics changed drastically. So I went to the teacher, which was an old nearly retired professor and asked him:

"Why is quantum mechanics done so differently from classical mechanics?".

His answer was short. He stated":

"The reason is that quantum mechanics is based on the superposition principle".

I quickly realized that this was part of the methodology and could not be the reason of the difference in methodology. So I went back and told him my concern. He told me that he could not give me a better answer and if I wanted a useful answer I should research that myself. So, I first went to the library, but the university was quite new and its library only contained rather old second hand books, which they got from other institutions. Next I went to the city's book shops. I finally found a booklet from P. Mittelstaedt: (Philosophische Probleme der modernen Physik, BI Hochschultaschenbücher, Band 50, 1963) that contained a chapter on quantum logic. Small particles obey a kind of logic that differs from classical logic. As a result their dynamic behavior differs from the behavior of larger objects. I concluded that this produced the answer that I was looking for.

I searched further and encountered papers from Garret Birkhoff and John von Neumann that explained the correspondence between quantum logic and separable Hilbert spaces. That produced a more conclusive answer to my question.

The lectures also told me that observables were related to eigenvalues of Hermitian operators. These eigenvalues are real numbers. However, it was clearly visible that nature has a 3+1D structure. So I tried to solve that discrepancy as well. After a few days of puzzling I discovered a number system that had this 3+1D structure and I called them compound numbers. I went back to my professor and asked him why such compound numbers were not used in physics. Again he could not give a reasonable answer.

When I asked the same question to a much younger assistant professor he told me that these numbers were discovered more than a century earlier by William Rowan Hamilton when he was walking with his wife over a bridge in Dublin. He was so glad about his discovery that he carved the formula that treats the multiplication of these numbers into the sidewall of the bridge. The inscription has faded away, but it is now molded in bronze and fixed to the same wall. The numbers are known as quaternions. So, I went to the library and searched for papers on quaternions.

In those years C. Piron wrote his papers on quaternionic Hilbert spaces. That information completed my insight in this subject. I finalized my physics study with an internal paper on quaternionic Hilbert spaces.

The university was specialized in applied physics and not in theoretical physics. This did not stimulate me to proceed with the subject. Next, I went into a career in industry where I used my knowledge of physics in helping to analyze intensified imaging and in assisting with the design of night vision equipment and X-ray image intensifiers. That put me with my nose on the notion of quanta.

The output window of image intensifiers did not show radiation. Instead they showed clouds of impinging quanta. In those times I had not much opportunity to deliberate on that fact. However, after my retirement I started to rethink the matter. That was the instant that the Hilbert Book Model continued further.

Thus, in a few words: The Hilbert Book Model tries to explain the existence of quanta. It does that by starting from traditional quantum logic.

The Hilbert Book Model is a Higgsless model of physics. It explains how elementary particles get their mass by using an approach that differs from the Higgs mechanism..

You will find the model to be in many aspects controversial and non-conventional. That is why the author took great efforts in order to keep the model self-consistent.

Its main purpose is to get insight into the possibilities of the physical toolkit.

*If a mathematical theory is self-consistent, then there is a realistic chance that nature somewhere somehow uses it.*

*If that theory is compatible with traditional quantum logic, then there is a* large *chance that nature will use it.*

This drives my intuition.

HvL

# PART I

# The fundaments

**Abstract**

The fundaments of quantum physics are still not well established. This book tries to find the cracks in these fundaments and explores options that were left open. This leads to unconventional solutions and a new model of physics.

In order to optimize self-consistency, the model is strictly based on the axioms of traditional quantum logic. Traditional quantum logic is lattice isomorphic to the set of closed subspaces of an infinite dimensional separable Hilbert space. It means that the separable Hilbert space can be used as the realm in which quantum physics will be modeled. However, this would result in a rather

primitive model. It can easily be shown that this model cannot implement dynamics and does not provide fields.

First, the model is extended such that it can represent fields. This results in a model that can represent a static status quo of the whole universe. The most revolutionary introduction in the Hilbert Book Model is the representation of dynamics by a sequence of such static models in the form of a sequence of extended separable Hilbert spaces.

At this point the Hilbert Book Model already differs significantly from conventional physics. Conventional quantum physics does not strictly hold to the axioms of traditional quantum logic, handles fields in a different way and implements dynamics differently.

Conventional quantum physic stays with complex state functions[1]. In contrast the HBM also explores the full potential of quaternionic state functions. As a consequence the HBM offers two different views on the undercrofts of quantum physics. The complex state function offers a wave dynamics view. The quaternionic state function opens a fluid dynamics view. The fluid dynamics view is unprecedented.

Since the switch from complex to quaternionic quantum state function does not affect physical reality, the two views will both be correct

[1] The HBM uses the name state function for the quantum state of an object or system rather than the usual term wave function because the state function may characterize flow behavior as well as wave behavior.

The quaternionic state functions enable the exploration of the geometry of elementary particles in which quaternionic sign flavors play an important role.

In the HBM elementary particles and physical fields are generated via the coupling of two sign flavors of the same quaternionic probability amplitude distribution (QPAD).

The quantum fluid dynamic view opens insight in the effect of the state functions on space curvature.

# 1 FUNDAMENTS

The most basic fundaments consist of quantum logic, its lattice isomorphic companion, the separable Hilbert space and the extensions of these basic models such that they incorporate fields.

## 1.1 Logic model

In order to safeguard a high degree of consistency, the author has decided to base the Hilbert Book Model on a consistent set of axioms. It is often disputed whether a model of physics can be strictly based on a set of axioms. Still, what can be smarter than founding a model of physics on the axioms of classical logic?

Since in 1936 John von Neumann[2] wrote his introductory paper on quantum logic the scientific community knows that nature cheats with classical logic. Garret Birkhoff and John von Neumann showed that nature is not complying with the laws of classical logic. Instead nature uses a logic in which exactly one of the laws is weakened when it is compared to classical logic. As in all situations where rules are weakened, this leads to a kind of anarchy. In those areas where the behavior of nature differs from classical logic, its composition is a lot more complicated. That area is the site of the very small items. Actually, that area is in its principles a lot more fascinating than the cosmos. The cosmos conforms, as far as we know, nicely to classical logic. In scientific circles the weakened logic that is discussed here is named traditional quantum logic.

As a consequence the Hilbert Book Model will be strictly based on the axioms of traditional quantum logic. However, this choice

[2] http://en.wikipedia.org/wiki/John_von_Neumann#Quantum_logics

immediately reveals its constraints. Traditional quantum logic is not a nice playground for the mathematics that characterizes the formulation of most physical laws. Lucky enough, von Neumann encountered the same problem and together with Garret Birkhoff[3] he detected that the set of propositions of quantum logic is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space. The realm of a Hilbert space is far more suitable for performing the mathematics of quantum physics than the domain of traditional quantum logic. Some decades later Constantin Piron[4] proved that the inner product of the Hilbert space must by defined by numbers that are taken from a division ring. Suitable division rings are the real numbers, the complex numbers and the quaternions[5]. The Hilbert Book Model also considers the choice with the widest possibilities. It uses both complex and quaternionic Hilbert spaces. However, quaternions play a decisive role in the design of the Hilbert Book Model. Higher dimension hypercomplex numbers may suit as eigenvalues of operators or as values of physical fields, but for the moment the HBM can do without these numbers. Instead, at the utmost quaternions will be used for those purposes.

## 1.2 State functions

Now we have a double model that connects logic with a flexible mathematical toolkit. But, this solution does not solve all requirements. Neither quantum logic nor the separable Hilbert space can handle physical fields and they also cannot handle dynamics. In or-

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[3] http://en.wikipedia.org/wiki/John_von_Neumann#Lattice_theory

[4] C. Piron 1964; _Axiomatique quantique_

[5] http://en.wikipedia.org/wiki/Quaternion

der to enable the implementation of fields we introduce quantum state functions[6]. The HBM does not fit quantum state functions inside a separable Hilbert space, but instead it attaches these functions to selected Hilbert space vectors[7].

It is a mathematical fact that both the real numbers and the rational numbers contain an infinite amount of elements. It is possible to devise a procedure that assigns a label containing a different natural number to every rational number. This is not possible for the real numbers. Technically this means that the set of real numbers has a higher cardinality than the set of rational numbers. In simple words it means that there are far more real numbers than there are rational numbers. Still both sets can densely cover a selected continuum, such as a line. However, the rational numbers leave open places, because infinite many real numbers fit between each pair of rational numbers.

Complex numbers and quaternions have the same cardinality as the real numbers. They all form a continuum. Rational numbers have the same cardinality as the integers and the natural numbers. They all form (infinite) countable sets.

Now take the fact that the set of observations covers a continuum and presume that the observed objects form a countable set. This poses a problem when the proper observation must be attached to a selected observed object. The problem is usually over-

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[6] The HBM uses quantum state function rather than wave function. In this document "quantum state function" is often simplified to "state function", which must not be exchanged with the classical notion of state function.

[7] It is well known that modulus squared integrable functions form a separable Hilbert space $L^2$. These measurable functions act as vectors, rather than as fields.

determined and in general it is inconsistent. The problem can only be solved when a little inaccuracy is allowed in the value of the observations[8].

In quantum physics this inaccuracy is represented by the (quantum) state function, which is a probability amplitude distribution. It renders the inaccuracy stochastic.

Thus state functions solve the fact that separable Hilbert spaces are countable and as a consequence can only deliver a countable number of eigenvectors for the particle location operator, while the observation of a location is taken from a continuum.

Usually state functions are taken to be complex probability amplitude distributions (CPAD's), but it is equally well possible to use quaternionic probability amplitude distributions. These include the full functionality of CPAD's. With the selection of QPAD's automatically a set of fields is attached to the Hilbert space. The real part becomes a scalar field that can be interpreted as a charge density distribution. The imaginary part becomes a vector field that can be interpreted as a current density distribution.

Each state function links the eigenvector of the particle location operator to a continuum. That continuum is the parameter space of the state function. Next we introduce ***Palestra*** as the parameter space that is shared by all state functions. It is possible to attach the Palestra as eigenspace to a location operator that resides in the Gelfand triple of the separable Hilbert space.

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[8] The situation is comparable to the case where a set of linear equations must be solved, while is known that the set of possible solutions is smaller.

For some objects (the wave-like objects) not the quantum state function and the Palestra are used but instead the canonical conjugate equivalents are used. The corresponding operators are the canonical conjugates of the location operators.

State functions extend over Palestra. Further, they superpose. They all consist of the same stuff. In a superposition it cannot be determined what part of the superposed value belongs to what state function. It is possible to define local QPAD's that are defined as the superposition of tails of a category of state functions. Such constructed QPAD's will be called background QPAD's.

It must be noticed that state functions neither belong to the separable Hilbert space nor belong to the Gelfand triple. They just form links between these objects.

The attachment of state functions extends the separable Hilbert space and connects it in a special way to its Gelfand triple. Due to the isomorphism of the lattice structures, the quantum logic is extended in a similar way. This leads to a reformulation of quantum logic propositions that makes them incorporate stochastically inaccurate observations instead of precise observations. The logic that is extended in this way will be called extended quantum logic. The separable Hilbert space that is extended in this way will be called extended separable Hilbert space.

## 1.3 The sandwich

The introduction of state functions concludes the modeling of the extended Hilbert space. We now have constructed a ***sandwich*** consisting of a separable Hilbert space, a set of state functions and the Gelfand triple that belongs to the Hilbert space.

The target for the sandwich is that it represents everything that is present or will become present in universe. It must contain all ingredients from which everything in universe can be generated.

In view of the existence of dark matter and dark energy, this is a very strong requirement. It will be difficult to prove. Instead we put it in the form of a postulate:

*"The sandwich contains all ingredients from which everything in universe can be generated."*

It will be shown that elementary particles can be generated as a result of couplings of state function QPAD's and background QPAD's where the QPAD's are sign flavors of the same base QPAD. The coupling is characterized by properties. These properties form the sources of corresponding physical fields.

With other words the presumption that all ingredients for generating particles and physical fields are present in the sandwich is probably fulfilled. As a consequence each HBM page represents a static status quo of the universe.

### 1.3.1 Alternative approaches

The model could have introduced QPAD's by collecting them in a special $L^2$ space of measurable quaternionic distributions and embed that space in a wider separable Hilbert space.

QED and QCD still take another approach and treat fields as operators rather than as Hilbert vectors.

The HBM takes another approach. Because they are so special, I have taken the fields apart. The HBM defines them as links between Hilbert vectors of a separable Hilbert space and continuum eigenspaces of operators in the Gelfand triple of that Hilbert space. For

the HBM that model works fine and offers easy and helpful insight. It does not raise conflicts.

The most important effect is that the link with quantum logic is held upright and the extension can be mimicked in that logic. Another advantage is that the HBM approach produces a clear difference between the primary fields that are directly related to quantum state functions and the secondary fields that represent the influences of the properties of the couplings of primary fields. The secondary fields are physical fields that play a fundamentally different role than the primary fields.

## 1.4 Model dynamics

The fact that the separable Hilbert space is not capable of implementing progression is exposed by the fact that the Schrödinger picture and the Heisenberg picture are both valid views of quantum physical systems, despite the fact that these views attribute the time parameter to different actors[9]. This can only be comprehended when the progression parameter is a characteristic of the whole representation[10].

If everything that is present or will be present in universe can be derived from ingredients that are available in the sandwich, then the sandwich represents a status quo of the full universe. If this assumption is correct, then implementing dynamics is simple.

A model that implements dynamics will then consist of a sequence of the described sandwiches. The model will resemble a

[9] See: History

[10] It turns out that the progression interval is a physical invariant

book. One sandwich represents one HBM page. The page counter is the progression parameter.

The whole Hilbert Book Model consists of an ordered sequence of sandwiches that each includes the Gelfand triple including its Hilbert space and the attached QPAD's. The progression parameter acts as page number of the book.

In the resulting Hilbert Book Model the progression is made in universe wide steps.

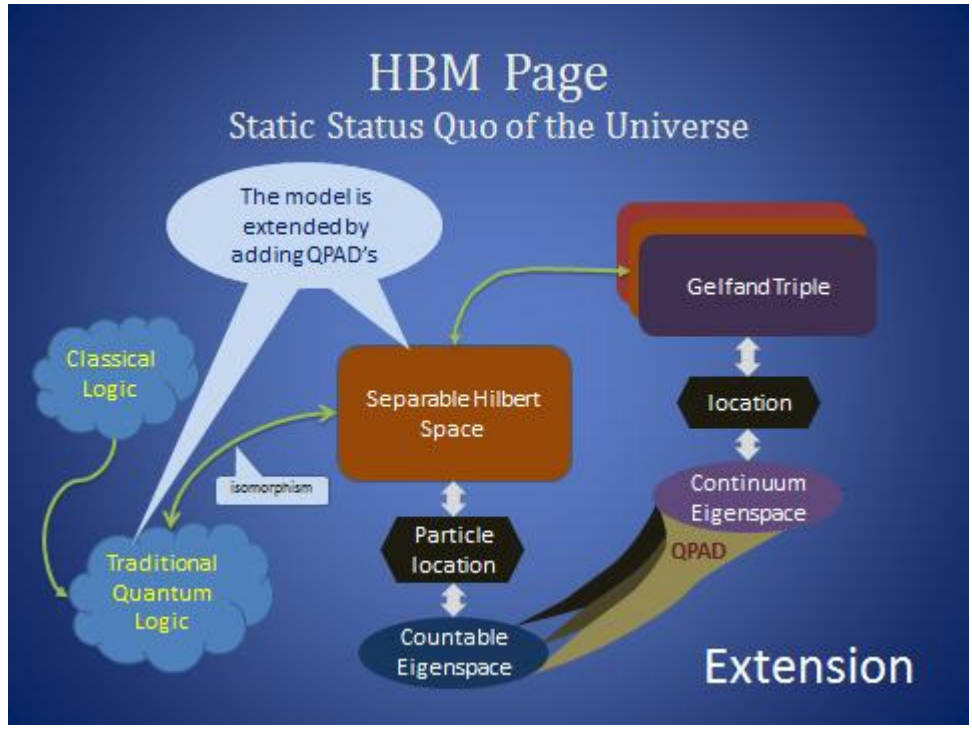


Figure 1: Structure of the Hilbert Book Model

## 1.5 Progression

Time exists in two different forms. The first form is proper time and its clock is ticking at the location of an observed item. The second form is coordinate time and its clock is ticking at the location of the observer. Together with space these two notions of time form

the notion of spacetime. Both clock ticks indicate progression. This progression is independent of the observer and it is independent of a selected observed item. Like space, progression is a global characteristic of the whole universe. Thus space and progression can be combined into a 1+3D space. This space does not have a Minkowski signature like spacetime has. It has Euclidean signature. The corresponding 1+3D location fits in a quaternion.

Apart from a constant factor and a shift, progression and proper time have the same value. When the interval is time-like an infinitesimal spacetime interval is equal to c times the corresponding infinitesimal proper time interval. In that case, the infinitesimal spacetime interval is apart from a constant factor equal to an infinitesimal progression interval. The infinitesimal spacetime interval is a physical invariant. Thus also the infinitesimal progression interval is a physical invariant.

Dynamics occurs in steps that cover all space in universe. In a selected HBM page the progression parameter has a fixed value. However, taken over a sequence of pages the quaternionic progression-space location varies in its progressive real part as well as in its spatial imaginary part.

In the HBM, special features of spacetime only come into play when an observer investigates another item. Its features become apparent when the reference frame of the observer moves with respect to the reference frame of the observed object. The features that appear in that case are: Length contraction, time dilation and change of mass.

The fact that progression occurs in universe wide steps includes that something exists that assures continuity between the HBM

pages. That something is formed by the quantum state functions. In their quaternionic format they contain currents. Thus, even a static QPAD contains moving charges. That movement constructs the link between subsequent pages.

Without the quantum state functions the dynamics of the HBM would be chaotic. These functions glue the Hilbert spaces together. This requires the quantum state functions to be quaternionic. Only the imaginary parts of these functions represent the currents that store the preconditions for the next progression step.

## 1.6 Progression factor

The Hamiltonian has real and non-negative eigenvalues. In conventional complex number based quantum physics the Hamiltonian can be considered as the generator of a unitary operator that appears as a factor in the (complex) quantum state function. Its canonical conjugate is the progression parameter[11]. However, with quaternionic quantum state functions the choice of the unitary operator poses problems, because the generator gets a spatial direction. In fact in the HBM only the modulus of the generator is relevant. In the HBM, the progression factor plays no role.

However, if one wants to hold onto the Hamiltonian as a generator, than even in quaternionic quantum physics, it becomes valid to interpret the Hamiltonian $H$ as

$$H = i\hbar \frac{\partial}{\partial t} \tag{1}$$

[11] Dirac; "The principles of quantum mechanics", Fourth edition, section 31.

where $i$ is the complex imaginary base number and $t$ is the progression parameter. On the other hand this gives rise to conflicts and mix-ups between the complex and the quaternionic imaginaries.

## 1.7 Quaternionic versus complex Hilbert space

The link between the eigenvector of the particle location operator and the continuum eigenspace of the location operator in the Gelfand triple must be a probability amplitude distribution. It can be implemented by a quaternionic probability amplitude distribution (QPAD), as well by a complex probability amplitude distribution (CPAD). A complex state function will connect the particle location operators in complex Hilbert space to the location operator in the corresponding Gelfand triple. QPAD's will be used in combination with a quaternionic Hilbert space.

| **Complex quantum physics** | **Quaternionic quantum physics** |
|---|---|
| Complex Hilbert space | Quaternionic Hilbert space |
| Complex inner product | Quaternionic inner product |
| Complex coefficients | Quaternionic coefficients |
| Hermitian location operator | Normal location operator |
| Real eigenvalues | Quaternionic eigenvalues |
| Complex state function | Quaternionic state function |
| CPAD | QPAD |
| Palestra | Palestra |
| Gelfand triple | Gelfand triple |
| Quantum wave dynamics | Quantum fluid dynamics |

The selection for a complex number based approach has some consequences for features that are related to quaternions and that

complex numbers do not provide. These features must be simulated by using spinors and matrices that mimic quaternion features via complex state functions and Hermitian operators. If these simulations are implemented properly, then the choice for QPAD versus CPAD is a purely mathematical decision. In any case the choice does not influence physical reality.

One important fact is that the complex number based quantum state functions leave the freedom of an arbitrarily selectable phase factor. The quaternionic quantum state functions do not allow such factor. It would immediately affect the incorporated vector field. As a consequence gauge transformation pose problems with quaternionic quantum state functions. On the other hand, complex quantum state functions do not incorporate fields. In quaternionic quantum physics gauge transforms must be restricted to cases with one dimensional spatial freedom.

The selection between CPAD's and QPAD's changes the view that human's may have on physics. It appears that the choice for QPAD's instead of CPAD's turns the theory of fundamental physics from the familiar complex number based wave dynamics theory into a quaternion based fluid dynamics theory.

Within the quaternion oriented theory the usefulness of the toolkit of conventional complex number oriented physics will reduce to special one-dimensional cases. This includes one-parametric oscillations and rotations.

Inside Palestra occur features such as charge density distributions, current density distributions, sources, drains, compression re-

gions and decompression regions. These features[12] can be described by QPAD's, but cannot be described by CPAD's. For that reason the QPAD approach is far richer than the CPAD approach.

The fact that such a small change in strategy has such great and unexpected impact signals that the fundamentals of physics are still not well understood.

The HBM approach is also richer than the GRT approach. With respect to GRT, the HBM offers a much more detailed analysis of what happens in the undercrofts of physics.

The implementation of physical fields via the attachment of QPAD's to eigenvectors in the separable Hilbert space is a crucial departure from conventional physical methodology. Conventional quantum physics uses complex probability amplitude distributions (CPAD's), rather than QPAD's[13]. Quantum Field Theory[14], in the form of QED[15] or QCD[16], implements physical fields in a quite different manner.

## 1.8 Spacetime versus quaternions

Apart from a constant factor and a shift, progression and proper time have the same value. In principle these values can be synchronized.

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[12] The HBM currently neglects vortexes.

[13] http://en.wikipedia.org/wiki/Probability_amplitude

[14] http://en.wikipedia.org/wiki/Quantum_field_theory

[15] http://en.wikipedia.org/wiki/Quantum_electrodynamics

[16] http://en.wikipedia.org/wiki/Quantum_chromodynamics

Spacetime features, like proper time, coordinate time, the Minkowski space, time dilatation and length compression only play a role when the reference frame of the observer moves with respect to the observed item.

In this case the dynamics at short range can be considered as a one dimensional displacement. In these cases the complex number based approach of quantum physics may work better (read easier) than the quaternionic approach. However, due to their multidimensional nature, fields are nearly always easier treated in a quaternionic approach. Only beams of radiation might be treated easier with a complex approach. The configuration of elementary particles can be better analyzed by using quaternionic quantum physics, because it uses quantum fluid dynamics.

Due to the fact that spacetime as a 3+1D space has a Minkowski signature, it is useless to try to implement spacetime with quaternions. However it fits well in a complex number based construct. On the other hand, if the interval is time-like, the spacetime interval $d\varsigma$ is directly related to the proper time interval $d\tau$ and to the progression interval $d\tau$ [17].

$$d\varsigma = c\, d\tau$$

[17] We use the same symbol τ for progression and for proper time.

| Quantity | Spacetime | quaternion |
|---|---|---|
| signature | Minkowski | Euclidean |
| Nabla | $\nabla = \left\{\frac{i}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right\}$ | $\nabla = \{\nabla_0, \boldsymbol{\nabla}\}$ |
| Field | $\psi = \{ic\psi_0, \boldsymbol{\psi}\}$ | $\psi = \{\psi_0, \boldsymbol{\psi}\}$ |
| Equation | $\varphi = \nabla\psi$ | $\varphi = \nabla\psi$ |

Arbab I. Arbab uses the complex spacetime approach in his papers about the quaternionic continuity equation[18].

## 1.9 Advantages of QPAD's

The choice for QPAD's appears very advantageous. The real part of the QPAD can be interpreted as scalar field in the form of a "charge" density distribution. Similarly the imaginary part of the QPAD can be interpreted as a vector field in the form of a "current" density distribution. The squared modulus of the value of the QPAD can be interpreted as the probability of the presence of the carrier of the "charge". The "charge" can be any property of the carrier or it represents a collection of the properties of the carrier. In this way, when the state function is represented by a QPAD the equation of motion becomes a continuity equation[19].

Since the state function QPAD's attach Hilbert eigenvectors to a value in a continuum the carriers can be interpreted as tiny patches of that continuum. The transport of these patches can be responsible for the local compression or decompression of the continuum space. With other words, via this mechanism QPAD's may influence the local curvature in that continuum.

[18] http://arxiv.org/abs/1003.0071

[19] Also called balance equation.

Interpreting the carriers as tiny patches of the parameter space is a crucial step. It transfers the dynamics into quantum fluid dynamics. Only via this step it becomes possible that QPAD's influence local curvature.

All state function QPAD's share their parameter space. The shared affine parameter space is called Palestra. This space can be curved. It can be represented by a quaternionic distribution. This quaternionic distribution has a flat parameter space. We need the affine version of the parameter spaces because the center locations of the QPAD's will differ.

## 1.10 Sign flavors

In most cases where quaternionic distributions are used, the fact that quaternions possess two independent types of sign selections is ignored. The first sign selection type, the conjugation, inverts the sign of all three imaginary base vectors. It is a combination of the sign switch of the whole number and the sign switch of the real part. The second sign selection type, the reflection, inverts the sign of a single imaginary base vector. The reflection can be taken in three independent directions. When relevant these directions are color coded. The sign selections in a quaternionic distribution are all similar. Individually, the conjugation and the reflection switch the handedness of the external vector product in the product of two quaternions that are taken from the same quaternionic distribution. The sign selection of the parameter space is usually taken as the reference for the sign selections of the quaternionic distributions. When a quaternionic distribution has the same sign selection as its parameter space has, then it will be called a base quaternionic

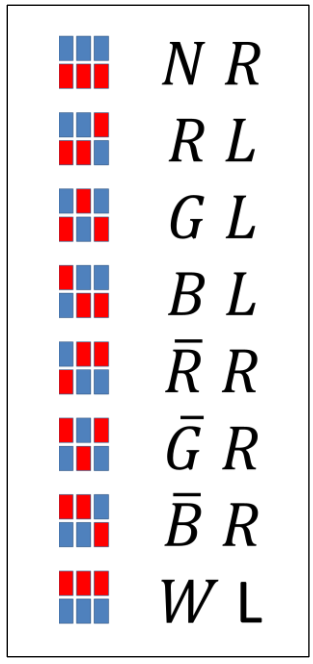


Figure 2: Sign selections

distribution. The sign selection of the parameter space is supposed to be isotropic. The two isotropic sign selections of the distribution are candidate to be the base sign selection.

For each QPAD, the mixture of conjugation and colored reflections produces eight different sign flavors[20]. This adds a significant amount of functionality to quaternionic distributions. In quantum physics the sign flavors play a crucial role. In conventional physics this role is hidden in complex probability amplitude distributions (CPAD's), alpha, beta and gamma matrices and in spinors.

## 1.11 Virtual carriers and interactions

Primary QPAD's are quaternionic distributions of the probability of presence of ***virtual*** "charge" carriers. They have no other geometrical significance than that they are tiny patches of the parameter space. The "charge" may stand for an ensemble of properties. Both state function QPAD's as well as constructed background QPAD's are primary QPAD's

The couplings of primary QPAD's result in elementary particles[21]. The properties that characterize this coupling form the sources of second category physical fields. Second category physical fields have ***actual*** charges as their sources and particles as their charge carriers. Second category physical fields concern a single property of the carrier. They are known as the physical field that relates to that property. Their presence can be observed (detected).

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[20] R, G, B are colors. N is normal. W is white.
L is left handed. R is right handed.

[21] See: Particle physics

*The Hilbert Book Model does not use the notion of a virtual particle. Instead the role of carriers in primary QPAD's is used for this purpose.*

Static primary QPAD's cannot be observed directly. Their existence can only be derived from the existence of second category physical fields.

In the Hilbert Book Model the "implementation" of forces via the exchange of virtual particles is replaced by the mutual influencing of the corresponding QPAD's. This influence is instantiated via the fact that the concerned primary QPAD's superpose and that their currents feed/supply other QPAD's. Further, elementary particles can interact with free QPAD's that have the same sign flavor as their quantum state function.

## 1.12 QPAD-sphere

QPAD's are quaternionic amplitude distributions and can be interpreted as a combination of a scalar "charge" density distribution and a vectorial "current" density distribution. The currents in a static QPAD consist of uniformly moving charge carriers. The separable Hilbert space is a static environment. The currents can only change between HBM pages. When the state function of a particle is represented by a primary QPAD, then this gives a special interpretation of that state function. A very special kind of primary QPAD is a local background QPAD[22]. It represents the local superposition of the tails of the state functions of a category of distant elementary particles.

[22] See: Special QPAD's

The QPAD's that act as state functions may be imagined in a region that glues the eigenspace of the location operator that resides in the Gelfand triple to the Hilbert eigenvectors of the particle location operator that resides in the separable Hilbert space. Constructed QPAD's such as the background QPAD's also reside in this region. They may be coupled to state function QPAD's. In that case the coupling generates a particle. This region is called the QPAD-sphere and contains potential "streams" of space patches that are superfluous in the eigenspace in the Gelfand triple and that fail in the corresponding eigenspace in the separable Hilbert space. Both eigenspaces are considered to be affine spaces. The actual streaming takes place in Palestra. Coupled QPAD's act as pumps that circulate space patches in the QPAD-sphere[23].

[23] See: figure 3, the QPAD-sphere

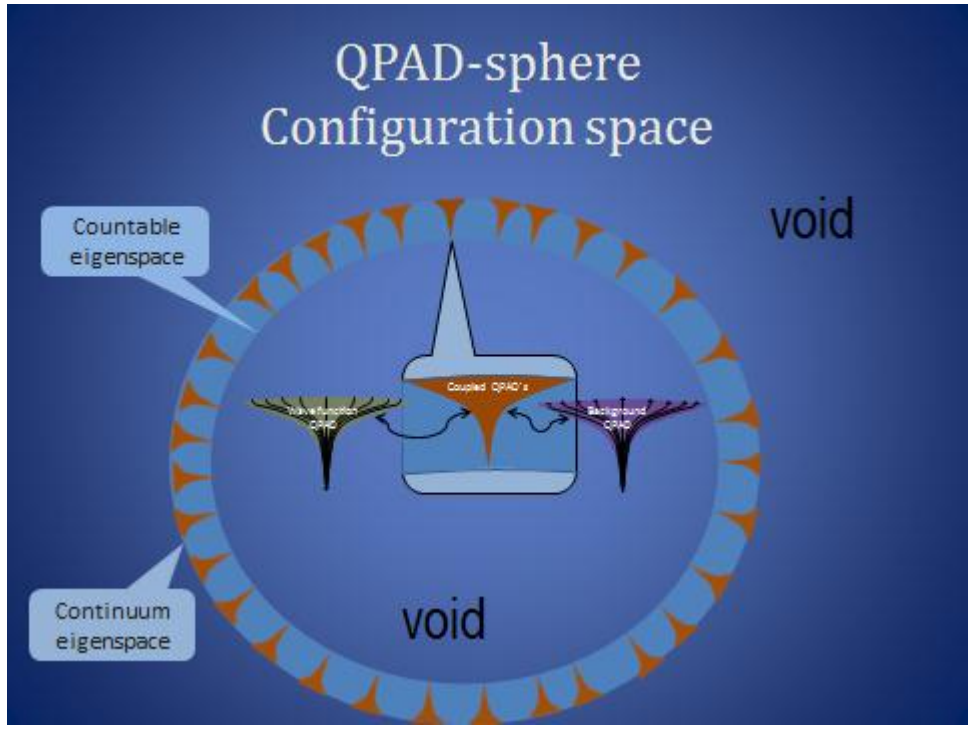


Figure 3: The QPAD-sphere

Now let this situation be managed by a mathemagician that must re-compute the situation at regular instances. He gets his location information from the continuum eigenspace in the Gelfand triple and must find the proper eigenvector in the Hilbert space for each particle. He has a countable number of potential target locations but a far larger number of potential input locations. He solves this by

allowing a stochastic inaccuracy between the continuum based input location and his final result location. Thus each particle possesses a normal distribution of potential locations. The mathemagician implements his solution by stealing potential positions that belong to distant particles and adding them close to the center of the normal distribution of the potential positions of the local particle that he currently investigates. This process pumps potential positions around. Since every action of the mathematician is independent from previous or later actions, the process behaves as a Poisson process.

The next step is best visualized using affine eigenspaces. In this case a **sphere** results with a thin atmosphere in which thermal streams of eigenspace patches circulate such that at the location of particles the atmosphere is compressed. Like the air molecules in the earth's atmosphere the eigenspace patches are circulating. Each particle has its own state function QPAD, which is denser on its center than on its tail. The circulation takes place due to the fact that eigenspace patches are taken from the tails of the state functions of distant particles and added to the QPAD of the local particle.

A coupled local state function pumps space patches taken from the tails of other state functions to the drain at its center and supplies them to a background QPAD, which spreads them over its surround. In this way it reclaims space patches that were supplied by distant sources and supplies them in the form of local space patches. Thus the local background QPAD acts as a source where the local coupled state function acts as a drain of space patches. The process that does this can be characterized as a Poisson process. The result of the stream of space patches is a local **space curvature**.

## 1.13 The HBM Palestra

We introduce a special kind of space. This space can curve. Its curvature can change. So, the space can move. It is space that moves with respect to a flat coordinate system. If you attach a separate coordinate system to the moving space, then you can describe what is happening. For example it can be described by a quaternionic distribution. Use only the imaginary part of the values and the imaginary part of the parameter. Now you have a distribution that describes your special space. You can use this space as the parameter space of still another function. For example you can use it as parameter space of a CPAD or a QPAD or any other field.

In the Hilbert Book Model all state function QPAD's and the fields that are derived from them or from their couplings, share the same parameter space. For that reason this common parameter space will be given a special name; ***the HBM Palestra***[24]. This shared parameter space spreads universe wide. It is the place where universe is located.

The HBM Palestra does not correspond to the historic notions of ether[25] that were used in physics. Instead it is defined by the descriptions that are given in this section.

In general QPAD's are no more than special kinds of quaternionic distributions. In the HBM some of the primary QPAD's have a special interpretation as quantum state functions of elementary particles. In the HBM CPAD's will also use the HBM Palestra as their

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[24] The name Palestra is suggested by Henning Dekant's wive Sarah. It is a name from Greek antiquity. It is a public place for training or exercise in wrestling or athletics

[25] http://en.wikipedia.org/wiki/Aether_theories

parameter space. If CPAD's are used as quantum state functions, then the Hilbert space will use complex numbers for its inner product. CPAD's are especially suited to handle one dimensional or one parametric problems.

The parameter space of a QPAD can be interpreted as a quaternionic distribution. It has itself a parameter space, which is formed by a flat 3D continuum. The Palestra is taken from the eigenspace of a location operator that resides in the Gelfand triple of the separable Hilbert space. Only the imaginary part of the quaternionic distribution is used. It can be considered as a 3D Riemannian manifold. The local metric defines the local curvature. What occurs in this manifold is described by the QPAD's. It is also possible to use the real part of the Palestra. However, in that case the value represents the progression parameter and is a constant throughout the imaginary part of the Palestra.

The Palestra is the playground of all what happens in fundamental physics. It is governed by a special kind of fluid dynamics (QFD). Things like charge density distributions, current density distributions, sources, drains, compressed regions and decompressed regions occur in this space. The QPAD's are not the transporters. They only describe the transport process. The action takes place in their shared parameter space, which is the Palestra. That's how these QPADS's can influence each other.

### 1.13.1 Metric

The Palestra is a continuous and compact space. It is characterized by a continuous quaternionic distribution $\wp$ that has a single sign flavor. This means that in the Palestra no reflections occur. $\wp$ defines local distances. Every location of the Palestra is characterized by a metric tensor, which defines at every location for every direction an infinitesimal distance. The infinitesimal progression interval plays an important role in this local distance and is a invari-

ant with respect to transformations that represent continuous symmetries. These transformations form a Lie group. The infinitesimal distance in the curved Palestra is defined as[26]

$$ds(x) = ds^{\nu}(x)e_{\nu} = \sum_{\mu=0\ldots3} \frac{\partial \wp}{\partial x_{\mu}} dx_{\mu} = q^{\mu}(x)\, dx_{\mu} \tag{1}$$

The quaternionic distribution $\wp(x)$ specifies the metric of the Palestra. $x_{\mu}$ are coordinates in the **flat** parameter space of $\wp(x)$.

The infinitesimal distance $ds$ is quaternion valued. The $dx_{\mu}$ are real valued. The infinitesimal distance specifies the step along the four quaternionic base axes when in the parameter space of $\wp(x)$ infinitial steps $dx_{\mu}$ are taken along the $x_{\mu}$ axes.

$$ds_{\nu} = q_{\mu\nu}(x)dx_{\mu};\ x = x_0 + x^{\nu}\boldsymbol{e}_{\nu};\nu = 1,2,3 \tag{2}$$

We define the factors $q^{\mu}(x)$ as derivatives of the quaternionic distribution $\wp(x)$

$$q^{\mu}(x) = q_{\mu 0}(x) + q^{\mu\nu}(x)\boldsymbol{e}_{\nu} = \frac{\partial \wp}{\partial x_{\mu}} \tag{3}$$

They belong to the flat parameter space of the quaternionic distribution $\wp(x)$. The quaternion values are all expressed in the same coordinate system, which has base numbers $\boldsymbol{e}_{\nu}$ : $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The flat parameter space of $\wp(x)$ also uses these coordinates. The intervals $dx_{\mu}$ can also be considered to be taken in that flat parameter space.

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[26] This approach differs from the approach of Mendel Sachs.

$ds_\nu$ is an interval taken in the Palestra that is measured in the coordinates of the flat parameter space. Without curvature it equals $e^\mu\ dx_\mu$. This corresponds to special relativity.

$dx_0$ is the infinitesimal spacetime interval $d\text{ş}$. (In this discussion we suppose it to be time-like). Apart from a constant factor it equals the progression interval $d\tau$. It is directly related to the proper time interval $d\tau$.

$$dx_0 = c\ d\tau \tag{4}$$

$\tau$ is the proper time. It is measured at the location of the observed item. Thus it is independent of who is observing.

$ds$ appears to correspond to the coordinate time interval $c\ dt$.

$$d\text{ş}^2 = dx_0^2 = c^2 d\tau^2 = c^2\ dt^2 - q_\mu q_\nu^* dx^\mu dx^\nu; \quad \mu, \nu = 1, 2, 3 \tag{5}$$

The coordinate time interval $dt$ follows from

$$ds\ ds^* = c^2 dt^2 = c^2\ d\tau^2 + q_\mu q_\nu^* dx^\mu dx^\nu;\ \mu, \nu = 1, 2, 3 \tag{6}$$

$$= q_\mu q_\nu^* dx^\mu dx^\nu;\ \mu, \nu = 0, 1, 2, 3$$

$t$ is the coordinate time. It is measured at the location of the observer. It includes the time that information needs to arrive from the observed item to the location of the observer.

The infinitesimal progression interval $d\tau$ is a model invariant of the HBM. The infinitesimal spacetime interval $d\text{ş}$ is a physical invariant. For that reason the infinitesimal spacetime interval $d\text{ş}$ is used for the definition of the local spacetime metric tensor.

By using the spacetime interval for defining the metric, the Palestra becomes a pseudo-Riemannian manifold with a Minkowski signature.

When instead the coordinate time interval is used as the controlling interval for the metric, the Palestra is a Riemannian manifold with an Euclidean signature. The coordinate time interval is not a physical invariant.

### 1.13.2 Lie groups

When the spacetime interval is used as a physical invariant, then the corresponding Lie group is the Einstein group[27]. The Einstein group relates to a 1+3D pseudo Riemannian space with Minkowski signature.

### 1.13.3 Path of a QPAD

Even when a QPAD travels a geodesic, the curvature and the torque of the path go together with the existence of extra fields[28].

## 1.14 Dynamics control

What happens in Palestra can be distinguished in two types of dynamics;

- the dynamics of particles and
- the dynamics of fields.

The dynamics of particles is controlled by quantum mechanics.
The dynamics of fields is controlled by quantum fluid dynamics.

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[27] http://en.wikipedia.org/wiki/Einstein_group#The_Einstein_group
See: Symmetry in electrodynamics; M. Sachs

[28] Appendix; Path of the quantum state function

### 1.14.1 Quantum mechanics

Quantum mechanics is well treated by conventional quantum physics. It concerns movements of particles and oscillations of particles or oscillations inside conglomerates. It is usually treated in the realm of a complex Hilbert space. The typical governing equation is the Schrödinger equation.

### 1.14.2 Quantum fluid dynamics

Quantum fluid dynamics relies on the use of QPAD's as state functions. It concerns movements of fields. It is treated in the realm of a quaternionic Hilbert space. The typical governing equation is an equivalent of the quaternionic format of the Dirac equation. The generic equation is the elementary coupling equation and it is best interpreted as balance equation.

What happens in the HBM Palestra is controlled by Quantum Fluid Dynamics (QFD). QFD differs from conventional fluid dynamics in that in QFD the charge density distributions and current density distributions describe probable locations and paths in their own parameter space, while in conventional fluid dynamics these distributions describe actual locations and paths that occur in an considered medium such as a gas or liquid. That is why in QFD the charge density distributions and current density distributions are combined in quaternionic probability amplitude distributions, while in conventional fluid dynamics they are located in scalar and vector fields.

### 1.14.3 Elementary coupling

Elementary coupling forms the base of particle generation. The elementary coupling equation couples two QPAD's $\psi$ and $\varphi$.

$$\nabla\psi = m\,\varphi \tag{1}$$

Here $\nabla$ is the quaternionic nabla and $m$ is a coupling factor.

This equation immediately delivers a formula for the coupling factor $m$.

$$\int_{Palestra} \varphi^{*}\nabla\psi = m \int_{Palestra} \varphi^{*}\varphi = m \tag{2}$$

When compared to the Dirac equation and the Majorana equation the two QPAD's are sign flavors of the same basic QPAD. With eight sign flavors for each QPAD, this results in 64 different elementary coupling equations[29]. However, when $\psi = \varphi$ it can be shown that $m$ must be zero. In that case the QPAD must either be zero or it must oscillate.

Thus the restricted elementary coupling equation results in 56 particles and 8 waves. The particles are attached to an eigenvector of the (particle) location operator and the waves are attached to an eigenvector of the canonical conjugate of the (wave) location operator.

In many cases the generated particles can be attributed to members of the standard model. However, the standard model shows generations that do not yet show in this simple model. It means that there are three versions of m or with other words there are three generations of the pair $\{\psi, \varphi\}$. It might be that the restriction that $\psi$ and $\varphi$ are sign flavors of the same base QPAD is too strong.

The elementary coupling provides the generated particle with four properties:

- The coupling factor m

[29] See table of elementary particles.

- An electric charge
- A spin
- A color charge

In higher order couplings the first three properties are preserved. In fact these properties are sources of new fields. All color charges appear to combine to white. Higher order couplings couple elementary particles with each other and with fields.

## 1.15 Fields

In the HBM fields consist of

1. Oscillating QPAD's (waves)
2. Fields coupled in elementary particles
    a. State function QPAD's
    b. Constructed background QPAD's
3. Coupling property fields

This leads to two categories of physical fields

- Oscillating QPAD's (waves)
- Coupling property fields

## 1.16 Quaternionic versus complex quantum physics

Quaternionic quantum physics provides a different view on the configuration and behavior of fields and tiny particles than complex quantum physics provides. The choice does not affect physical reality. Nature stays the same.

Quaternionic quantum physics treats its subject by applying quantum fluid dynamics. It is well suited for the analysis of quasi-static multidimensional problems such as the geometry of fields and elementary particles.

Complex quantum physics is better suited for the analysis of situations where only a single spatial dimension is involved, such as oscillations and movements along a path.

# 2 History

In its first years, the development of quantum physics occurred violently. Little attention was paid to a solid and consistent foundation. The development could be characterized as delving in unknown grounds. Obtaining results that would support applications was preferred above a deep understanding of the fundamentals.

The first successful results were found by Schrödinger and Heisenberg. They both used a quantization procedure that converted a common classical equation of motion into a quantum mechanical equation of motion. Schrödinger used a state function that varied as a function of its time parameter, while operators do not depend on time. Heisenberg represented the operators by matrices and made them time dependent, while their target vectors were considered to be independent of time. This led to the distinction between the Schrödinger picture and the Heisenberg picture.

Somewhat later John von Neumann and others integrated both views in one model that was based on Hilbert spaces. Von Neumann also laid the connection of the model with quantum logic. However, that connection was ignored in many of the later developments. Due to the restrictions that are posed by separable Hilbert spaces, the development of quantum physics moved to other types of function spaces. The Hilbert Book Model chooses a different way. It keeps its reliance on quantum logic, but attaches fields, including state functions, in the region between the Hilbert space and its Gelfand triple.

Due to the integration of both pictures in a single Hilbert space, it becomes clear that the Schrödinger picture and the Heisenberg picture represent two different views of the same situation. It ap-

pears to be unimportant were time is put as a parameter. The important thing is that the time parameter acts as a progression indicator. This observation indicates that the validity of the progression parameter covers the whole Hilbert space. With other words, the Hilbert space itself represents a static status quo. Conventional quantum physics simply ignores this fact.

In those days quaternions played no role. The vector spaces and functions that were used all applied complex numbers and observables were represented with self-adjoint operators. These operators are restricted to real eigenvalues.

Quaternions were discovered by the Irish mathematician Sir William Rowan Hamilton[30] in 1843. They were very popular during no more than two decades and after that they got forgotten. Only in the sixties of the twentieth century, supported by the discovery of Constantin Piron that a separable Hilbert space ultimately may use quaternions for its inner product, a short upswing of quaternions occurred. But quickly thereafter they fell into oblivion again. Currently most scientists never encountered quaternions. The functionality of quaternions is taken over by complex numbers and a combination of scalars and vectors and by a combination of Clifford algebras, Grassmann algebras, Jordan algebras, alpha-, beta- and gamma-matrices and by spinors. The probability amplitude functions were taken to be complex rather than quaternionic. Except for the quaternion functionality that is hidden in the α, β, γ matrices, hardly any attention was given to the possible sign selections of quaternion imaginary base vectors and as a consequence the sign flavors of quaternionic distributions stay undetected. So, much of the typical functionality of quaternions still stays obscured.

[30] http://en.wikipedia.org/wiki/William_Rowan_Hamilton

The approach taken by quantum field theory departed significantly from the earlier generated foundation of quantum physics that relied on its isomorphism with quantum logic. Both QED and QCD put the quantum scene in non-separable function spaces. The state function is only seen as a complex probability amplitude distribution. Spinors and gamma matrices are used to simulate quaternion behavior. Physical fields are commonly seen as something quite different from state functions. However they are treated in a similar way.

The influence of Lorentz transformations[31] gives scientists the impression that space and time do not fit in a quaternion but instead in a spacetime quantity that features a Minkowski signature. Length contraction, time dilation and space curvature have made it improbable that progression would be seen as a universe wide parameter[32].

These developments cause a significant deviation between the approach that is taken in contemporary physics and the line according which the Hilbert Book Model is developed.

## 2.1 Criticism

Due to its unorthodox approach and controversial methods the Hilbert Book Model has drawn some criticism

### 2.1.1 Model

*Question:*

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Appendix; Lorentz transformation

[32] However progression and proper time appear to be the same kind of parameter and are physical invariants.

The separable Hilbert space has clearly some nasty restrictions. Why can quantum physics not be completely done in the realm of a rigged Hilbert space?

*Answer:*

In that case there is no fundamental reason for a separate introduction of fields in QP. It will also not be possible to base QP on traditional quantum logic (TQL), because the isomorphism that exists between TQL and separable Hilbert spaces (SHS's) does not exist between TQL and a rigged Hilbert space (RHS). See figure 1.

In the HBM the quaternionic probability amplitude distributions (QPAD's) on which fields are based, link the SHS with its Gelfand triple {Φ;SHS;Φ'}, which is a RHS. However, the QPAD's are not part of the SHS and are not part of the RHS. The HBM can be pictured as:

$$TQL \Leftrightarrow SHS \Rightarrow \{QPAD's\} \Rightarrow RHS \equiv \{\Phi;SHS;\Phi'\} \tag{1}$$

The isomorphism $\Leftrightarrow$ is replaced by incongruence $\Leftarrow\neq\Rightarrow$ in

$$TQL \Leftarrow\neq\Rightarrow RHS \tag{2}$$

### 2.1.2 Quaternions

*Remark1:*

A tensor product between to quaternionic Hilbert spaces cannot be constructed. So it is better to stay with complex Hilbert spaces.

*Remark2:*

The notion of covariant derivative, which is an important concept on quantum field theory, offers problems with quaternionic distributions, so it is better to stay with a complex representation.

This is due to the fact that for quaternionic distributions in general[33]:

$$\nabla(f\ g) \neq\ f\ \nabla\ g\ +\ (\nabla f)\ g \tag{1}$$

*Response:*

The HBM proves that solutions exist that do not apply tensor products or covariant derivatives.

In fact the subject can be reversed:

*If a methodology is in conflict with a quaternionic approach, then it must not be applied as a general methodology in quantum physics. A complex number based method can best be applied in special, one dimensional or one parametric cases.*

### 2.1.3 Quaternionic versus complex probability amplitude distributions

*Remark 1*

Conventional physics solves everything by using complex probability amplitude distributions (CPAD's).

*Remark 2*

It is sufficient to stay with that habitude. Quaternionic probability amplitude distributions (QPAD's) might be unphysical.

*Response:*

QPAD's extend the functionality of CPAD's and make it possible to interpret equations of motion as balance equations. They can be considered as a combination of a scalar potential and a vector

[33] The quaternionic nabla can be split in its parts see below: Differentiation

potential or as the combination of a charge density distribution and a current density distribution.

Their sign flavors enable the interpretation of spinors as a set of sign flavors that belong to the same base QPAD. This throws new light on the Dirac and Majorana equations.

Coupling of QPAD's can be related to local curvature. This interpretation is impossible with CPAD's.

## 2.2 Consequence

The application of the HBM requests from physicists that they give up some of the conventional methodology and learn new tricks.

The HBM allows using complex number based quantum physics aside quaternion based quantum physics. It introduces fields as objects that are separated from the separable Hilbert space as well as from the corresponding Gelfand triple.

# PART II

## The Hilbert Book Model

# 3 INGREDIENTS

The most intriguing ingredients of the model are quaternions, quaternionic distributions and quaternionic probability amplitude distributions (QPAD's), both equipped with their sign flavors.

## 3.1 Role of the particle locator operator

The particle locator operator $\mathfrak{S}$ is one of the operators for which the eigenvectors are coupled to a continuum called Palestra that is related to the eigenspace of a corresponding location operator that resides in the Gelfand triple.

Palestra may be curved. It means that this continuum is a quaternionic function of the eigenvalues of another location operator. The eigenspace of that operator is flat. It is covered by the number space of the quaternions. Palestra has the same sign flavor as this parameter space. Without curvature its parameters and the corresponding values are equal.

For each eigenvector of the particle locator operator Palestra acts as parameter space for the QPAD that connects this eigenvector with the eigenspace of the corresponding location operator that resides in the Gelfand triple. It means that Palestra is identical with that eigenspace. The QPAD can be visualized as a fuzzy funnel that drops stochastically inaccurate observation values onto the particle.

The particle location operator has a canonical conjugate, which is the corresponding particle momentum operator. This corresponds to a different particle eigenvector, a different QPAD, a different corresponding eigenspace in the Gelfand triple and a different corresponding operator. For example the parameter space of the new QPAD is the canonical conjugate of Palestra and the corresponding momentum operator in the Gelfand triple is the canonical conjugate

of the discussed location operator. Without any curvature the old and the new QPAD would be each other's Fourier transform.

## 3.2 QPAD's

All elementary particles correspond to different eigenvectors of the particle locator operator $\mathfrak{S}$.

Each elementary particle has its own QPAD that acts as its state function.

Taken over a set of subsequent page numbers two pictures are possible. In both pictures existing particles are characterized by their own state function QPAD. Thus for particles their state function QPAD is a stable factor.

According to the Heisenberg picture the QPAD is static between subsequent HBM pages, but links to different eigenvectors of the particle location operator.

According to the Schrödinger picture the QPAD varies between subsequent HBM pages, but links to the same eigenvector of the particle location operator.

In both pictures a location observation must deliver the same result.

If we want to categorize particles, then we must categorize their state function QPAD's. But, we also must take the background QPAD in account to which the state function QPAD is coupled.

Some QPAD's stay uncoupled. These QPAD's must oscillate[34]. They form elementary waves. All elementary waves correspond

[34] See Free QPAD's

to different eigenvectors of the wave locator operator $\widetilde{\mathfrak{S}}$, which is the canonical conjugate of $\mathfrak{S}$.

If a QPAD has a well-defined location in configuration space, then it does not have a well-defined location in the canonical conjugate space. So for this investigation we better consider both locations together.

A possible strategy is to use the superposition of the QPAD and its Fourier transform. This solution distinguishes QPAD's that, apart from a scalar, are invariant under Fourier transformation[35].

An important category of invariants is formed by QPAD's that have the shape $f(r) = C/r$, where $C$ is a constant and $r$ is the distance from the central location[36][37].

Further it is sensible to introduce for each category the notion of an average QPAD. Determining the average QPAD either involves integration over full Palestra or it involves integration over the canonical conjugate of Palestra.

Another procedure constructs the superposition of all tails of a category of QPAD's at a certain location. This produces a background QPAD. For each location such a background QPAD exists, but it may differ per location. However, when a large category is involved, it can only differ very marginally for neighboring locations.

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[35] See Appendix; Functions invariant under Fourier transform

[36] See http://en.wikipedia.org/wiki/Hankel_transform#Some_Hankel_transform_pairs

[37] Also see http://en.wikipedia.org/wiki/Bertrand's_theorem

## 3.3 Uncertainty

The fact that QPAD's are probability amplitude distributions makes them the main source of the indeterminism in the HBM. However, the fact that the particle location operator uses a QPAD and the momentum operator uses the Fourier transform introduces another source of indeterminism which is similar to the Heisenberg's uncertainty. It is due to the properties of the Fourier transform. As a consequence the location operator and the momentum operator do not commute. This is equivalent to the Heisenberg uncertainty principle.

## 3.4 Helmholtz decomposition

A static QPAD consists of a charge density distribution and an independent current density distribution. The charge density distribution corresponds to a curl free vector field and the current density distribution corresponds to a divergence free vector field. Any change in the currents goes together with extra field components that do not correspond to the charge density distribution or the current density distribution.

Inside a single HBM page the QPAD's are static. Only taken over a range of HBM pages the Helmholtz decomposition is violated.

## 3.5 QPAD vizualization

If I was a good artist and I was asked to give an artist impression of the magic wand of a magician, then I would not paint a dull rod or a stick with a star at the end. Instead I would draw a very thin glass rod that has a sparkling fuzzy ball at its tip. A static view of that ball would be like:

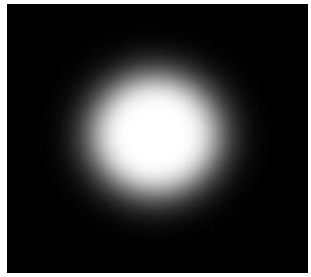

Figure 4: Typical isotropic QPAD

A dynamic view would show how the sparkles move inside out (source) or outside in (drain). This fuzzy ball is how a simple QPAD may look like when we could see it. In a static QPAD, uniformly moving virtual charge carriers replace the sparkles. The movement need neither be purely parallel nor purely radial, but in a static QPAD it must be uniform. The carriers may be interpreted as tiny patches that are taken from the continuum background that forms the parameter space of the QPAD.

Even when they are attenuated or spatially or temporally spread by a binomial process, Poisson processes create a result that has a Poisson distribution. The output of an efficient Poisson process has a density distribution that comes close to a Gaussian distribution. A typical example QPAD might show such a Gaussian density distribution. When actual electrical charges would be distributed this way, then this distribution creates a potential that has the shape of an error function. Already at a short distance from its center this function decreases very close to a 1/r dependence. At that distance the local potential would be the same as when a single large charge was put at the center. Instead our example has virtual charges. So it represents a single virtual charge at the center (or an ensemble of

properties of the carrier of that single charge). The raised potential is also virtual. Still it describes our example QPAD.

The source that creates the above described fuzzy ball may be characterized as a Poisson process.[38]

## 3.6 QPAD categories

In the HBM QPAD's exist in several categories:

- A primary QPAD is linked to an eigenvector of an operator in separable Hilbert space.
    - Coupled state function QPAD's are linked to the particle location operator
    - Free QPAD's are linked to the canonical conjugate of the particle location operator, they must oscillate and form waves.
- A composed QPAD is constructed from a superposition of primary QPAD's

## 3.7 Special composed QPAD's

We will consider a special ensemble of primary QPAD's $\{\psi_i(r, q_i)\}$.

- The $\psi_i(r, 0)$ are normalized: $\int_V |\psi_i(r, 0)|^2 \, dV = 1$.
- The $\psi_i(r, 0)$ must be spherically symmetric.
- From a given minimal distance their modulus must decrease with radius $r$ as $1/r$.

The special composed QPAD's are Fourier transform invariant and conform to Bertrand's theorem[39].

[38] See: What image intensifiers reveal

### 3.7.1 The average QPAD

The ensemble $\{\psi_i(r,0)\}$ of the special composed QPAD's has an average $\Psi(r,0)$

## 3.8 The background QPAD

Background QPAD's are composed QPAD's. A given background QPAD depends on the selection criterion that is used in the composition. Every sign flavor might have its own background QPAD.

The ensemble $\{\psi_i(q_i)\}$ is distributed randomly over the center points $\{q_i\}$ in an affine parameter space. At a given point P in this space the superposition of all tails of the members of the ensemble $\{\psi_i(q_i)\}$ will be constructed.

This superposition will be renormalized and then indicated by $\Phi(r,P)$.

Thus,

$$\int_V |\Phi(r,P)|^2 \, dV = 1 \tag{1}$$

In this superposition the largest contribution comes from the $\psi_i(q_i)$ for which the $q_i$ is farthest from P. Further the directions of the imaginary part are reversed with respect to the directions in the $\psi_i(q_i)$.

Especially at long distances, all differences are smoothed away via an averaging process.

The result is that for a background QPAD that consist of the superposition of the tails of all QPAD's in universe:

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[39] http://en.wikipedia.org/wiki/Bertrand%27s_theorem

$$\Phi(r, P) = \Psi^*(r, P) \tag{2}$$

We will interpret $\Phi(r, 0)$ as the background QPAD.

The approach taken here, shows similarity with the approach of Dennis Sciama in his paper: "On the origin of inertia"[40].

#### 3.8.1.1 Uniform movement

Due to its construction the location as well as the shape of the background QPAD is very stable. In the HBM the background QPAD is reconstructed at every subsequent page. As long as the whole QPAD remains static, it can be reconstructed at a displaced location. Thus, as long as this movement is contained in its current density distribution, the background QPAD can move freely in a uniform way.

In principle every location has its own background QPAD. However, it makes only sense to couple the locations of particles with a local background QPAD. If this coupling is strong then it acts as a sticky resistance against acceleration of the coupled particle.

### 3.8.2 Isotropy

Source QPAD's are isotropic. Drain QPAD's are also isotropic and are the conjugate of a corresponding source QPAD.

Anisotropic QPAD's are hybrids. In one or two dimensions they correspond to source QPAD's. In the other dimensions they correspond to a drain QPAD.

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[40] See: http://www.adsabs.harvard.edu/abs/1953MNRAS.113...34S

Oscillating QPAD's oscillate between source modes and drain modes. They usually can be described by spherical or linear harmonic (quantum) oscillators. A special case of the oscillating QPAD is a plain wave.

## 3.9 Physical fields

Physical fields appear in two categories[41]

- The first category physical field concerns free primary QPAD's.
- The second category physical field relate to properties of coupled QPAD's

Each of the members of the second category of physical fields represents the influences of a single property of a coupled pair of QPAD's.

The two categories are fundamentally different. They behave differently[42].

## 3.10 Inertia

Inertia[43] is based on the fact that all particles in universe influence a selected local particle. Since this influence is isotropic it usually does not disturb the particle. This condition holds as long as the particle is located stationary or moves uniformly. In that case the connected QPAD is static. However, when the particle accelerates, according to field theory, this goes together with the existence

[41] For more details see chapter 4.

[42] This becomes apparent at the event horizon of black holes.

[43] Appendix; The universe of items

of an extra field that becomes part of the particle's QPAD and that counteracts the acceleration.

The background QPAD that is coupled to the state function QPAD at the right side of the elementary coupling equation[44] represents the influence of the universe on the local particle. It represents the superposition of all tails of a category of state functions of particles that exist in universe. For that reason it is the source of inertia. This is shown in the elementary coupling equation.

$$\nabla\psi = m\,\varphi \tag{1}$$

For the restricted elementary coupling equation the QPAD's $\psi$ and $\varphi$ are sign flavors of the same base QPAD $\phi$, which on its turn has the same sign flavor as Palestra has.

For elementary fermions the coupled field $\varphi$ equals the isotropic background field $\Phi$.

For electrons the state function $\psi$ equals $\Psi = \Phi^*$

The formula (1) holds for all massive elementary particle types. It holds for elementary fermions as well as for the elementary bosons that couple to other sign flavors[45] than $\Phi$.

If the restricted elementary coupling equation holds then, for every QPAD sign flavor $\psi^x$ that acts as a state function of an elementary particle, exists a background QPAD $\psi^y$ that has the same base QPAD $\psi$. However $\psi^*$ will also suit as base QPAD. The corresponding particle type is fully characterized by the pair $\{\psi^x, \psi^y\}$

These facts are in detail treated in the section on particle physics.

---

[44] See Elementary particles

[45] See Quaternions; Sign selections

According to equation (1) the state function QPAD is rather strongly coupled to the local background QPAD. The equation describes the situation of an independently moving particle.

In atoms the electrons oscillate around the nucleus. There these particles move more freely. They are still coupled to the centrally located background QPAD, but the coupling is rather loose. Equation (1) does not describe that situation. Instead of the Dirac equation, the Klein-Gordon equation fits more appropriately.

The background QPAD's play the role that is thought for the Higgs field[46].

## 3.11 Coupling and curvature

When a local state function QPAD is coupled to a local background QPAD, then the background QPAD can be considered as a source that supplies space patches to the local state function QPAD. On its turn the differential of the local state function QPAD acts as a drain. Thus, the coupled system pumps space taken from the rest of the universe to the locations where the local state function QPAD varies. These flowing space patches form part of the space that the eigenspace $E_G$ of the location operator that resides in the Gelfand triple has extra with respect to the eigenspace $E_H$ of the particle location operator that resides in the separable Hilbert space. Otherwise said: the eigenspace $E_G$ compresses via the state function QPAD to the eigenspace $E_H$ and expands back via the coupled background QPAD to eigenspace $E_G$. The observer that uses $E_H$ as his observation space, experiences a local compression of his observation space at the location of the observed QPAD. This can be expressed by a local curvature of the observation space.

[46] Sign flavors may complicate this picture.

As extra detail can be said that the source and the drain act as Poisson processes that cause a Gaussian distribution of the patches of space that flow in/out the drain/source location.

The moving space patches may be interpreted as virtual carriers of the properties that characterize the coupling event.

The coupling properties themselves act as sources of second category physical fields. These are known as second category physical fields. In the GRT based Kerr-Newman metric equation[47] similar properties act as sources of curvature.

[47] The Kerr-Newman equation is NOT part of the HBM. It is only used for comparison.

## 3.12 Hyper-complex numbers

Hyper-complex numbers form categories that are ordered with respect to their dimension. The dimension $D$ takes the form $D = 2^n$, where $n$ is a non-negative integer. A hyper-complex number of dimension $D$ can be obtained from a pair of hyper-complex numbers of dimension $D - 1$ via a construction algorithm. Several construction algorithms exist. The most popular is the Cayley-Dickson construction[48]. A less known construction algorithm is the $2^n$-on construction of Warren Smith[49]. This construction delivers numbers that in the higher dimensions retain better arithmetic capabilities. Up and including the octonions the two construction algorithms deliver the same numbers. The sedions differ from the $2^4$-ons.

In their lower m dimensions the $2^n$-ons behave similarly to the $2^m$-ons.

The $2^n$-ons have n independent imaginary base vectors. As a consequence the $2^n$-ons feature n independent sign selections or when independent directions are considered it are $2^n$ independent sign selections.

Both construction methods ignore these sign selections. Sign selections play a crucial role in this paper.

## 3.13 Quaternions

A quaternion is a 1+3 dimensional hyper-complex number. It has a one dimensional real part and a three dimensional imaginary part. As a result, it can be seen as the combination of a real scalar and a three dimensional vector.

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[48] http://en.wikipedia.org/wiki/Cayley%E2%80%93Dickson_construction

[49] Appendix; $2^n$-ons, See scorevoting.net/WarrenSmithPages/homepage/nce2.ps

$$q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \tag{1}$$

The quaternions form a division ring[50]. According to the Frobenius theorem[51], the only finite-dimensional division algebras over the reals are the reals themselves, the complex numbers, and the quaternions.

The coefficients $\{q_m\}$ are real numbers. Bi-quaternions exist that have complex coefficients, but these do not form a division ring.

### 3.13.1 Sign selections

The quaternions possess two independent sign selection types. When directions are reckoned they constitute four independent sign selections. The conjugation $q \Leftrightarrow q^*$ inverts the sign of all imaginary base vectors. It acts isotropic.

$$q^* = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3 \tag{1}$$

The reflection $q \Leftrightarrow q^{\scriptscriptstyle 1}$ inverts a single imaginary base vector and for
that reason it acts anisotropic.

$$q^{\scriptscriptstyle 1} = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 - \boldsymbol{k}q_3 \tag{2}$$

Here, the base vector $\boldsymbol{k}$ is selected arbitrarily.

---

[50] http://en.wikipedia.org/wiki/Division_ring

[51] http://en.wikipedia.org/wiki/Frobenius_theorem_(real_division_algebras)

The four sign selections can be mixed. They generate eight sign states.

Thus, if the three independent directions in which reflections can occur are also taken into account, then eight different sign selections are possible.

These sign selections are color coded as is shown in figure 2.

Individually the conjugation and the reflection both flip the handedness of the external vector product of the imaginary part when both factors use the same sign selections.

### 3.13.2 Habits

The addition works as in all division rings, however the product of two quaternions does not commute.

#### 3.13.2.1 Product rule

The product rule is best expressed in the form of real scalars and 3D vectors:

$$ab = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{1}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{2}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{i}(a_2 b_3 - a_3 b_2) + \boldsymbol{j}(a_3 b_1 - a_1 b_3) + \boldsymbol{k}(a_1 b_2 - a_2 b_1) \tag{3}$$

$$\boldsymbol{i}\,\boldsymbol{j} = \pm \boldsymbol{k} \tag{4}$$

#### 3.13.2.2 Norm

The norm or modulus is defined by:

$$|a| = \sqrt{a_0 a_0 + \langle \boldsymbol{a}, \boldsymbol{a} \rangle} \tag{5}$$

## 3.14 Quaternionic distributions

Several forms of quaternionic distributions exist. Two forms are relevant for the HBM.

A curved coordinate system can be related to a flat coordinate system via a quaternionic distribution. The flat coordinate system

plays the role of parameter space. On its turn the curved coordinate system can also play the role of a parameter space. It does that for quaternionic probability amplitude distributions (QPAD's). In the HBM, Palestra is the parameter space that is shared by all QPAD's , all CPAD's and all derived fields.

A quaternionic probability amplitude distribution [52] is a quaternionic distribution. Its value can be split in a real part that can be interpreted as a charge density distribution and an imaginary part that can be interpreted as a current density distribution. The squared modulus of the value can be interpreted as the probability density of the presence of the carrier of the charge. The charge can be any property of the carrier or it stands for the ensemble of the properties of the carrier.

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[52] http://en.wikipedia.org/wiki/Probability_amplitude treats complex probability amplitude distributions.

### 3.14.1 Sign flavors

The quaternions that form the values of a quaternionic distribution must all feature the same set of sign selections. This fact attaches a sign flavor to each quaternionic distribution. Quaternionic distributions come in eight sign flavors[53]. We indicate color by an extra index: $i = R, G, B$. $\bar{R}, \bar{G}, \bar{B}$ are anti-colors. See figure. The right column shows the handedness of the distribution. We will use the symbol $\psi$ or $\psi^{⓪}$ for the sign flavor of the quaternionic distribution that has the same sign flavor as its parameter space. However, selecting $\psi^{⑦}$ as base instead of $\psi^{⓪}$ may work as well. Since $\psi^{⑦}$ acts as background QPAD for fermions, it might be a better choice.

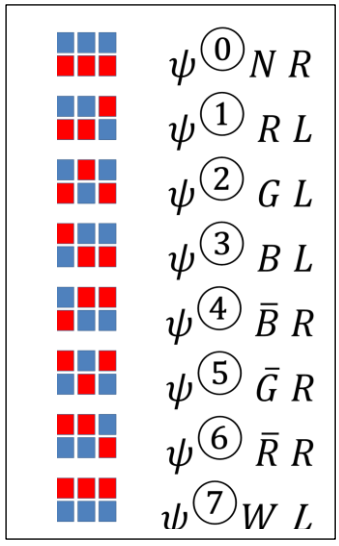


Figure 5: Sign flavors.

We will use

$$\psi^{⑦} = \psi^*;\ \psi^{⓪} = \psi \tag{1}$$

[53] The notion of "sign flavor" is used because for elementary particles "flavor" already has a different meaning.

Often the symbols $\psi$ and $\psi^*$ will be used instead of the symbols $\psi^{⓪}$ and $\psi^{⑦}$.

### 3.14.2 QD multiplication

What happens when quaternions from different sign flavors will be multiplied?

1. First a reference sign flavor is selected.
2. This sign flavor is taken to be the sign flavor of the distribution that will receive the result.
3. The factors are first brought to this reference sign selection.
4. In this process nothing changes in the values of the quaternions.
5. After that the multiplication takes place.
6. The result is delivered in the reference sign flavor.

With other words the multiplication takes place with the handedness that is defined in the target distribution.

### 3.14.3 Differentiation and Fourier transform

A quaternionic distribution f(q) can be differentiated[54].

$$g(q) = \nabla f(q) = \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left( \pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \quad (1)$$

[54] For more details, see Appendix; Quaternionic distributions,

The colored $\mp$ and $\pm$ signs refer to the influence of conjugation of $f(q)$ on quaternionic multiplication. The $\pm$sign refers to the influence of reflection of $f(q)$.

In the Palestra $\nabla_0$ represents the derivative with respect to progression. It is not the derivative with respect to coordinate time.

In this section, the parameter q is supposed to be taken from a non-curved parameter space. With that precondition, in Fourier space differentiation becomes multiplication with the canonical conjugate coordinate $k$ and therefore the equivalent equation becomes:

$$\begin{aligned} \tilde{\mathrm{g}}(k) = \mathrm{k}\tilde{f}(k) & \\ = \mathrm{k}_0\tilde{f}_0(k) \mp \langle \mathbf{k}, \tilde{\boldsymbol{f}}(k) \rangle \pm \mathrm{k}_0\tilde{\boldsymbol{f}}(k) & \\ + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k)\right) & \end{aligned} \tag{2}$$

For the imaginary parts holds:

$$\mathbf{g}(q) = \pm\nabla_0 \boldsymbol{f}(q) + \mathbf{\nabla} f_0(q) \pm \left(\pm \mathbf{\nabla} \times \boldsymbol{f}(q)\right) \tag{3}$$

$$\tilde{\mathbf{g}}(k) = \pm\mathrm{k}_0\tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k)\right) \tag{4}$$

### 3.14.4 Extra freedom

The solution $f(q)$ of

$$g(q) = \nabla f(q) \tag{1}$$

Is determined apart from a gauge term $h(q)$

$$g(q) = \nabla(f(q) + h(q)) \tag{2}$$

Where

$$\nabla h(q) = 0 \tag{3}$$

This leads to three equations:

$$\nabla_0 h_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{h}(q) \rangle = \mathbf{0} \tag{4}$$

$$\pm \nabla_0 \boldsymbol{h}(q) + \boldsymbol{\nabla} h_0(q) = 0 \tag{5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{h}(q) = 0 \tag{6}$$

This leads to the existence of $\Lambda_0$

$$\boldsymbol{h}(q) = \boldsymbol{\nabla} \Lambda_0(q) \tag{7}$$

### 3.14.5 Spinors and matrices

In contemporary physics complex probability amplitude distributions (CPAD's) are used rather than QPAD's. Spinors and matrices are used to simulate QPAD behavior for CPAD's.

A spinor $[\psi]$ is a 1×4 matrix consisting of CPAD's that represent the sign flavors of a QPAD. Sometimes the spinor is represented as a 1×2 matrix.

The $\boldsymbol{\alpha}$ and $\beta$ matrices influence the elements of spinor $[\psi]$.

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{3}$$

(4)

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{5}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

$\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ represent imaginary base vectors of the simulated quaternion. $\beta$ represents the conjugation action for the spinor.

A relation exist between $\alpha_1, \alpha_2, \alpha_3$ and the Pauli[55] matrices $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7}$$

$$1 \mapsto I, \qquad \boldsymbol{i} \mapsto \sigma_1, \qquad \boldsymbol{j} \mapsto \sigma_2, \qquad \boldsymbol{k} \mapsto \sigma_3 \tag{8}$$

This combination is usually represented in the form of gamma matrices.

In Dirac representation, the four contravariant gamma matrices are

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

[55] http://en.wikipedia.org/wiki/Pauli_matrices

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \qquad \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It is useful to define the product of the four gamma matrices as follows:

$$\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{10}$$

The gamma matrices as specified here are appropriate for acting on Dirac spinors written in the Dirac basis; in fact, the Dirac basis is defined by these matrices. In the Dirac basis[56]:

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{11}$$

This corresponds with $\alpha_k = \gamma^k, \beta = \gamma^5$.
Apart from the Dirac basis, a Weyl basis exists

$$\gamma^0 = \gamma^\beta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \tag{12}$$

The Weyl basis has the advantage that its chiral projections[57] take a simple form:

[56] http://en.wikipedia.org/wiki/Gamma_matrices#Dirac_basis

$$\psi_L = ½\,(1 - \gamma^5)[\psi] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}[\psi] \tag{13}$$

$$\psi_R = ½\,(1 + \gamma^5)[\psi] = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}[\psi] \tag{14}$$

$$[\psi^*] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}[\psi] \tag{15}$$

### 3.14.6 Continuity equation

When applied to a quaternionic probability amplitude distribution (QPAD), the equation for the differentiation leads to a continuity equation[58].

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of the corresponding charge[59] is given by the continuity equation:

Total change within $V$ = flow into $V$ + production inside $V$ (1)

$$\frac{d}{dt}\int_V \rho_0\, dV = \oint_S \hat{\boldsymbol{n}}\rho_0 \frac{\boldsymbol{v}}{c}\, dS + \int_V s_0\, dV \tag{2}$$

$$\int_V \nabla_0\rho_0\, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho} \rangle\, dV + \int_V s_0\, dV \tag{3}$$

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[57] http://en.wikipedia.org/wiki/Chirality_(physics)

[58] See: http://en.wikipedia.org/wiki/Reynolds_transport_theorem

[59] Also see Noether's laws: http://en.wikipedia.org/wiki/Noether%27s_theorem

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface $S$, $\boldsymbol{v}(t,\boldsymbol{q})$ is the velocity at which the charge density $\rho_0(t,\boldsymbol{q})$ enters volume $V$ and $s_0$ is the source density inside $V$. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{4}$$

It is the flux (flow per unit area and unit time) of $\rho_0$ .

The combination of $\rho_0(t,\boldsymbol{q})$ and $\boldsymbol{\rho}(t,\boldsymbol{q})$ is a quaternionic skew field $\rho(t,\boldsymbol{q})$ and can be seen as a probability amplitude distribution (QPAD).

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{5}$$

$\rho(t,\boldsymbol{q})\rho^*(t,\boldsymbol{q})$ can be seen as an overall probability density distribution of the presence of the carrier of the charge. $\rho_0(t,\boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(t,\boldsymbol{q})$ is the current density distribution.

The conversion from formula (2) to formula (3) uses the Gauss theorem[60]. This results in the law of charge conservation:

$$\begin{aligned} s_0(t,\boldsymbol{q}) &= \nabla_0 \rho_0(t,\boldsymbol{q}) \\ &\quad \mp \langle \boldsymbol{\nabla}, \left(\rho_0(t,\boldsymbol{q})\boldsymbol{v}(t,\boldsymbol{q}) + \boldsymbol{\nabla} \times \boldsymbol{a}(t,\boldsymbol{q})\right)\rangle \\ &= \nabla_0 \rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q}) + \boldsymbol{A}(t,\boldsymbol{q})\rangle \\ &= \nabla_0 \rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}(t,\boldsymbol{q}), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \\ &\quad \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle\, \rho_0(t,\boldsymbol{q}) \end{aligned} \tag{6}$$

[60] http://en.wikipedia.org/wiki/Divergence_theorem

$$\mp\langle \boldsymbol{\nabla}, \boldsymbol{A}(t,\boldsymbol{q})\rangle$$

The blue colored $\pm$ indicates quaternionic sign selection through conjugation of the field $\rho(t,\boldsymbol{q})$. The field $\boldsymbol{a}(t,\boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a}(t,\boldsymbol{q})\rangle = 0 \tag{7}$$

$\boldsymbol{A}(t,\boldsymbol{q}) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \boldsymbol{a}(t,\boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(t,\boldsymbol{q})$.

Equation (6) represents a balance equation for charge density. What this charge actually is, will be left in the middle. It can be one of the properties of the carrier or it can represent the full ensemble of the properties of the carrier.

This only treats the real part of the full equation. The full continuity equation runs:

$$\begin{aligned} s(t,\boldsymbol{q}) = \nabla\rho(t,\boldsymbol{q}) &= s_0(t,\boldsymbol{q}) + \boldsymbol{s}(t,\boldsymbol{q}) \\ &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q})\rangle \pm \nabla_0\boldsymbol{\rho}(t,\boldsymbol{q}) \\ &\qquad + \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \\ &\qquad \pm \left(\pm\boldsymbol{\nabla} \times \boldsymbol{\rho}(t,\boldsymbol{q})\right) \\ &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}(t,\boldsymbol{q}), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \\ &\qquad \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle\, \rho_0(t,\boldsymbol{q}) \\ &\qquad \pm\nabla_0\boldsymbol{v}(t,\boldsymbol{q}) + \nabla_0\rho_0(t,\boldsymbol{q}) \\ &\qquad\qquad + \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \end{aligned} \tag{8}$$

$$\pm\big(\pm(\rho_0(t,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(t,\boldsymbol{q})-\boldsymbol{v}(t,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\big)$$

$$s_0(t,\boldsymbol{q}) = 2\nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{v}(q),\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \mp \langle\boldsymbol{\nabla},\boldsymbol{v}(t,\boldsymbol{q})\rangle\,\rho_0(t,\boldsymbol{q}) \tag{9}$$

$$\boldsymbol{s}(t,\boldsymbol{q}) = \pm\nabla_0\boldsymbol{v}(t,\boldsymbol{q}) \pm \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \tag{10}$$

$$\pm\Big(\pm\big(\rho_0(t,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(t,\boldsymbol{q})-\boldsymbol{v}(t,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\big)\Big)$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. Conjugation also causes a switch of handedness. It changes the sign of all three imaginary base vectors.

In its simplest form the full continuity equation runs:

$$\phi = \nabla\psi$$

Thus the full continuity equation specifies a quaternionic distribution $\phi$ as a differential $\nabla\psi$. We can transform this equation in a coupling equation[61] by introducing a coupling factor $m$:

$$\nabla\psi = m\,\varphi$$

With other words, the coupling equation is a full continuity equation, where $m\,\varphi$ is the source term and $\nabla\psi$ is the correspond-

[61] See Elementary coupling equation.

ing drain. The qualification “full” means that both the real and the imaginary parts of the sources and drains are reckoned.

## 3.15 Hilbert space

The unit sphere of the separable Hilbert space $\mathbf{H}$ is an affine space. All unit size eigenvectors end in this sphere.

A normal operator in the separable Hilbert space has set of unit size eigenvectors that span the complete Hilbert space as an orthogonal base. For normal operators each eigenvector corresponds to an eigenvalue.

This does not hold for the particle location operator and it also does not hold for the wave location operator. The wave location operator is the canonical conjugate of the wave position operator. For these operators the eigenvectors do not correspond directly with a fixed eigenvalue. In the quaternionic Hilbert space this link is a QPAD. In the complex Hilbert space this link is a CPAD. Further the eigenvectors do not span the whole Hilbert space.

| Particle location | Wave location |
|---|---|
| The particle location operator is a position operator. | The wave location operator is a momentum operator. |
| The particle location operator links a particle to its position | The wave location operator links a wave to its momentum. |
| For the particle location operator the eigenvectors are linked to the eigenvalue via a quantum state function | For the wave location operator the eigenvectors are linked to the eigenvalue via a quantum wave function. |
| Quantum state functions link eigenvectors of the particle location operator to a position eigenvalue. | Quantum wave functions link eigenvectors of the wave location operator to a momentum eigenvalue. |
| The parameter space of the quantum state function is the Palestra | The parameter space of the quantum wave function is the canonical conjugate of Palestra |

Taken over a set of subsequent page numbers two pictures are possible. In both pictures existing particles are characterized by their own quantum state function. Each elementary wave is characterized with its own quantum wave function, which is the canonical conjugate of the quantum position function.

According to the Heisenberg picture the quantum state function is static between subsequent HBM pages, but links to different eigenvectors of the particle location operator.

According to the Schrödinger picture the quantum state function varies between subsequent HBM pages, but links to the same eigenvector of the particle location operator.

In both pictures a location observation must deliver the same result.

According to this view the state functions and wave functions carry the post conditions for the previous HBM page and the preconditions for the next HBM page. They control temporal continuity. This is best expressed in the Schrödinger picture.

The eigenvectors of the locator operators are exceptional. They do not have a fixed eigenvalue. Instead these eigenvectors attach to a linking CPAD or QPAD, which on its turn is attached to the considered object. The eigenvalues of the location operator are not directly coupled to an eigenvector of that operator. Instead the linking probability amplitude distribution (PAD) intermediates and delivers the eigenvalue. The eigenvalue is taken from the HBM Palestra. The HBM Palestra is the parameter space that is shared by all state function QPAD's. The Palestra corresponds to the eigenspace of an operator that houses in the Gelfand triple $\mathbf{\Phi}$ of the Hilbert space $\mathbf{H}$. A quaternionic distribution maps the Palestra to a coordinate system that overlays the number system of the quaternions. That flat

space is the eigenspace of still another operator that resides in the Gelfand triple.

The Pauli principle introduces extra restrictions. Fermions that possess the same properties cannot obtain the same location value. In the Hilbert space the eigenvectors of the particle location operator that are attached to fermions are mutually orthogonal, but do not form a base of the full Hilbert space.

When stepping from one HBM page to the next page, the state function QPAD's *install the tendency to keep the corresponding eigenvectors together such that the corresponding expected eigenvalues form a coherent and ordered set*.

The coupling between the eigenvectors of the particle locator operator and the Palestra is not precise. It is stochastic and its minimal size is of the order of the Planck-length. This granularity is due to the fact that the Hilbert space is separable and as a consequence the set of eigenvalues is countable. That is why the granules in the surface of a black hole have this size. The surface of a black hole is an image of the unit sphere of a subspace of the Hilbert space where the eigenvectors of the particle location operator form an orthogonal base and all attached QPAD's pack the eigenvalues together in an optimal way.

Via the PAD, the eigenvectors of the particle locator operator all touch such a granule. The relation with quantum logic means that the Hilbert vector stands for a proposition that has a yes/no value. In case of the Hilbert vectors that are attached to the granules the yes/no value represents group membership. Thus each granule represents a bit of information.

For the eigenvectors of the particle locator operator a densest packaging exists. It means that in that condition the QPAD's have shrunk to their smallest possible location difference.

# 4 PARTICLE PHYSICS

This chapter treats the **first level coupling**. The result of that coupling are first level particles. These particles are solely created out of coupled primary QPAD's and annihilate back into primary QPAD's[62]. That is why these particles are called elementary particles. The zero-level of coupling stands for no coupling. It results in elementary waves.

This section will not separate different particle generations.

## 4.1 Elementary coupling equation

First level particles appear to obey a special kind of (full) continuity equation. The generic equation is called the elementary coupling equation. In this continuity equation the source term is represented by the coupled background QPAD $\varphi$. It is coupled to the state function QPAD $\psi$.

$$\nabla\psi = m\,\varphi \tag{1}$$

Here $\nabla$ is the quaternionic nabla and $m$ is a coupling factor.

$\nabla_0$ represents the derivative with respect to progression.

The equation is a full continuity equation. $m\,\varphi$ is the source term and $\nabla\psi$ is the corresponding drain.

This equation immediately delivers a formula for the coupling factor $m$.

$$\int_{Palestra} \varphi^* \nabla\psi = m \int_{Palestra} \varphi^* \varphi = m \tag{2}$$

[62] In this chapter only QPAD's are discussed. CPAD's do not carry sign flavors.

For the antiparticle holds the conjugated equation.

$$(\nabla\psi)^* = m\,\varphi^* \tag{3}$$

Apart from the conjugation, it looks exactly the same equation as (1), but the Palestra does not take part in the conjugation. In comparison with ”normal” elementary particles, the antiparticles act as if progression is reversed.

### 4.1.1 Extra freedom

Each solution $\psi$ of

$$\nabla\psi = m\,\varphi \tag{1}$$

is also solution of

$$\nabla(\psi + \chi) = m\,\varphi \tag{2}$$

where

$$\nabla\chi = 0 \tag{3}$$

This is a gauge transformation of $\psi$.

### 4.1.2 Origin

The elementary coupling equation did not fall from heaven. It is possible to use the following reasoning (which is not historical)

$$\Phi = \nabla\psi$$

defines quaternionic differentiation. $\nabla$ is the quaternionic nabla and $\Phi$ and $\psi$ are quaternionic distributions. The whole equation can be split in a real part and an imaginary part.

$$\nabla\psi = \Phi$$

is the form of the corresponding continuity equation[63]. It is also known as balance equation. Here $\Phi$ represents the source term. It is better known as the real part of the equation.

$$\nabla_0\psi_0 \; - \; \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle \; = \; \Phi_0$$

The term $\langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle$ concerns the inner product of $\nabla$ and $\psi$.

Continuity equations are known in integral format and in differential format.

The next step is adding the coupling factor $m$. This results in the elementary coupling equation.

$$\nabla\psi = m\,\Phi$$

The real parts of the quaternionic distributions are scalar fields.

The imaginary parts of the quaternionic distributions are vector fields.

Now the final step is the interpretation of the real parts as charge distributions, the imaginary parts as current distribution and the charge carriers as tiny patches of the parameter space of the distributions.

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[63] Appendix; Differentiation; Continuity equation

This interpretation transforms $\Phi$ and $\psi$ into quaternionic probability amplitude distributions (QPAD's) and the equations in equations that describe quantum physical phenomena.

The last equation has a striking similarity with the quaternionic format of the Dirac equation and with the quaternionic format of the Majorana equation. This means that $\psi$ represents a quantum state function of a particle. The equation couples $\psi$ and $\Phi$ into a pair $\{\psi, \Phi\}$, which represents the particle.

## 4.2 Restricted elementary coupling

Restricted elementary coupling means that the coupled QPAD's are sign flavors of the same base QPAD. Both the Dirac equation and the Majorana[64] equation belong to this type of elementary coupling equation.

For all restricted particles hold:

$$\nabla\psi^{x} = m\,\psi^{y} \tag{1}$$

$$\nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi^{x}} \rangle = m\,\psi_0 \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\psi^{x}} + \boldsymbol{\nabla}\psi_0^{x} + \nabla_0\boldsymbol{\psi^{x}} = m\,\boldsymbol{\psi^{y}} \tag{3}$$

For the antiparticle:

$$\nabla^{*}\psi^{x*} = m\,\psi^{y*} \tag{4}$$

The Fourier transform equivalents are

[64] The Majorana particle is supposed to consist of the combination of a neutrino and an anti-neutrino.

$$P_0\tilde{\psi}_0 - \langle \boldsymbol{P}, \tilde{\boldsymbol{\psi}}^{\boldsymbol{x}} \rangle = m\,\tilde{\psi}_0 \tag{5}$$

$$\boldsymbol{P} \times \tilde{\boldsymbol{\psi}}^{\boldsymbol{x}} + \boldsymbol{P}\tilde{\psi}_0 + P_0\tilde{\boldsymbol{\psi}}^{\boldsymbol{x}} = m\,\tilde{\boldsymbol{\psi}}^{\boldsymbol{y}} \tag{6}$$

$P_0$ is generally known as the Hamiltonian[65]. $\boldsymbol{P}$ is the momentum.

$$\int_V \psi^{⓪}\psi^{⑦}\,dV = 1 \tag{7}$$

$$\int_V \psi^{x}\psi^{x*}\,dV = g^{x};\; x = ⓪, ①, ②, ③, ④, ⑤, ⑥, ⑦ \tag{8}$$

The factor $g^x$ is real and non-negative.
Further, the equation for coupling factor $m$ is:

$$\int_V (\psi^{y*}\nabla\psi^{x})\,dV = m\int_V (\psi^{y*}\psi^{y})\,dV = m\int_V |\psi^{y}|^2\,dV \tag{9}$$

An equivalent of the Lagrangian may look like

$$\mathcal{L} \stackrel{?}{=} \psi^{y*}\,\nabla\psi^{x} - m\,\psi^{y*}\psi^{y} \tag{10}$$

### 4.2.1 Dirac equation

The best known equation of motion for elementary fermions is the Dirac equation. It is written using spinors and matrices.

[65] However, $\nabla_0$ is the derivative with respect to progression (not coordinate time)

The Dirac equation for a free moving electron or positron is known as:

$$\nabla_0[\psi] + \langle \boldsymbol{\nabla}, \boldsymbol{\alpha} \rangle [\psi] = m\beta[\psi] \tag{1}$$

The Dirac matrices $\boldsymbol{\alpha}$ and $\beta$ give the spinor $[\psi]$ the function of a pair of QPAD's.

$\nabla_0$ represents derivative with respect to progression.

This spinor equation can be converted into two quaternionic equations that act on the QPAD's $\psi_R$ and $\psi_L$:

$$\nabla_0 \psi_R + \boldsymbol{\nabla} \psi_R = m\psi_L \tag{2}$$

$$\nabla_0 \psi_L - \boldsymbol{\nabla} \psi_L = m\psi_R \tag{3}$$

In the mass term the coupling factor $m$ couples $\psi_L$ and $\psi_R$. When $m = 0$ then $\psi_L$ and $\psi_R$ are not coupled. Further:

$$\psi_R = \psi_L^* = \psi_0 + \boldsymbol{\psi} \tag{4}$$

In the left term of equations (2) and (3), $\psi_R$and $\psi_L$ represent the state function of the particle. In the right side appears the background field. In that sense $\{\psi_R, \psi_L\}$ and $\{\psi_L, \psi_R\}$ represent each other's antiparticle.

Reformulating these equations gives

$$\nabla \psi^{⓪} = m\, \psi^{⑦} \tag{5}$$

$$\nabla_0(\psi_0 + \boldsymbol{\psi}) + \boldsymbol{\nabla}(\psi_0 + \boldsymbol{\psi}) = m(\psi_0 - \boldsymbol{\psi}) \tag{6}$$

For the conjugated equation holds

$$\nabla^{*}\psi^{⑦} = m\,\psi^{⓪} \tag{7}$$

$$\nabla_0(\psi_0 - \boldsymbol{\psi}) - \boldsymbol{\nabla}(\psi_0 - \boldsymbol{\psi}) = m(\psi_0 + \boldsymbol{\psi}) \tag{8}$$

This implements the reverse flip. The corresponding particle is the antiparticle.

$$\{\psi^{⓪}, \psi^{⑦}\} \leftrightarrow \{\psi^{⑦}, \psi^{⓪}\} \tag{9}$$

Both flips switch the handedness.

Equations (5) and (7) are each other's quaternionic conjugate. However, it must be noticed that all terms are conjugated, including the nabla operator, but the parameter space stays untouched. Thus equation (7) differs from equation (5).

Summing the equations gives via

$$\boldsymbol{\nabla\psi} = \boldsymbol{\nabla} \times \boldsymbol{\psi} - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle \tag{10}$$

the result

$$\nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle = m\,\psi_0 \tag{11}$$

The difference gives

$$\nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 + \boldsymbol{\nabla} \times \boldsymbol{\psi} = -m\,\boldsymbol{\psi} \tag{12}$$

Just reversing the sign flavors does not work. For the same QPAD $\psi$, the corresponding equation will contain extra terms:

$$\nabla\psi^{⑦} = \nabla_0(\psi_0 - \boldsymbol{\psi}) + \boldsymbol{\nabla}(\psi_0 - \boldsymbol{\psi}) \qquad (13)$$
$$= \nabla_0\psi_0 - \nabla_0\boldsymbol{\psi} + \boldsymbol{\nabla}\psi_0 - \boldsymbol{\nabla}\boldsymbol{\psi}$$

$$= (m\,\psi_0 + \langle\boldsymbol{\nabla},\boldsymbol{\psi}\rangle) - (-m\,\boldsymbol{\psi} - \boldsymbol{\nabla}\psi_0) + \boldsymbol{\nabla}\psi_0 - (\boldsymbol{\nabla}\times\boldsymbol{\psi} - \langle\boldsymbol{\nabla},\boldsymbol{\psi}\rangle)$$

$$= m\,\psi + 2\langle\boldsymbol{\nabla},\boldsymbol{\psi}\rangle + 2\boldsymbol{\nabla}\psi_0$$

Thus if the reverse equation fits, then it will concern another QPAD configuration $\psi'$ that will not fit the original equation.

The pair $\{\psi'^{⑦}, \psi'^{⓪}\}$ that fits equation:

$$\nabla\psi'^{⑦} = m\,\psi'^{⓪} \qquad (14)$$

represents a different particle than the electron $\{\psi^{⓪}, \psi^{⑦}\}$, which obeys equation (5). It also differs from the positron $\{\psi^{⑦}, \psi^{⓪}\}$, which obeys equation (7).

Where the electron couples to the general background QPAD $\psi^{⑦}$, the new particle couples to the conjugate $\psi^{⓪}$ of the general background QPAD $\psi^{⑦}$. The new particle is called a shadow of the positron.

#### 4.2.1.1 Correspondence principle

The Dirac equation obeys the correspondence principle.

$$\nabla\psi = m\,\psi^* \qquad (1)$$

$$\nabla^*\nabla\psi = m\,\nabla^*\psi^* = m\,(\nabla\psi)^* = m\,(m\,\psi) \qquad (2)$$

$$\nabla^2\psi = m^2\,\psi \tag{3}$$

This corresponds to the Klein Gordon equation.

$\nabla_0$ represents derivative with respect to progression rather than with respect to coordinate time.

#### 4.2.1.2 The coupling factor

Multiplying both sides of the equation of motion for the electron:

$$\nabla\psi^{⓪} = m\,\psi^{⑦} \tag{1}$$

with $\psi^{⓪}$ and then integrate over the full parameter space gives:

$$\int_V \psi^{⓪}\nabla\psi^{⓪}\,dV \tag{2}$$
$$= m\int_V \psi^{⓪}\psi^{⑦}\,dV = m\int_V \left|\psi^{⓪}\right|^2 dV$$
$$= m$$

Thus, the coupling factor $m$ can be computed from the QPAD $\psi^{⓪}$.

### 4.2.2 The Majorana equation

The Majorana equation deviates from the Dirac equation in that is applies another sign flavor of the state function QPAD $\psi$. That other sign flavor is still coupled to the general background QPAD $\psi^{⑦}$.

$$\nabla\psi^{①} = m_n\,\psi^{⑦} \tag{1}$$

The conjugated equation defines the antiparticle.

$$\nabla^{*}\psi^{⑥} = m_n\,\psi^{⓪} \tag{2}$$

The particle is represented by the ordered pair $\{\psi^{①}, \psi^{⑦}\}$. The corresponding flip does not switch the handedness.

The Majorana particle is supposed to consist of the combination of a neutrino, and an anti-neutrino. The HBM takes the assumption that equation (1) holds for a single elementary particle.

The equation offers two interpretations.

A straight forward interpretation would classify the particle that corresponds to equation (1) as an anti-up-quark with charge -⅔ e.

Equation (2) will then correspond to an up-quark.

The second interpretation takes the point of view that only a switch of handedness can provide charge for the particle. This interpretation classifies the particle as an neutrino, which is a neutral-particle. Equation (2) will then hold for the anti-neutrino.

The coupling factor $m_n$ for the particle follows from:

$$\begin{aligned}\int_V \psi^{⓪}\nabla\psi^{①}\,dV &\\ &= m_n \int_V \psi^{⓪}\psi^{⑦}\,dV\\ &= m_n \int_V \left|\psi^{⓪}\right|^2 dV = m_n\end{aligned} \tag{3}$$

The pairs $\{\psi^{①}, \psi^{⑦}\}$, $\{\psi^{②}, \psi^{⑦}\}$ and $\{\psi^{③}, \psi^{⑦}\}$ give similar results. They correspond to different colors (R,G,B) and to corresponding anti-particles $\{\psi^{⑥}, \psi^{⓪}\}$, $\{\psi^{⑤}, \psi^{⓪}\}$ and $\{\psi^{④}, \psi^{⓪}\}$

It might be that neutrinos and up-quarks share the same elementary quantum equation. There must exist a reason why neutrinos are not charged, while up-quarks have charge. That reason is not obvious from the sign flavors of the participating fields.

### 4.2.3 The next particle type

We have exploited:

$$\nabla\psi^{⓪} = m\,\psi^{⑦} \tag{1}$$

and

$$\nabla\psi^{①} = m_n\,\psi^{⑦} \tag{2}$$

With equivalents $\{\psi^{①}, \psi^{⑦}\}$, $\{\psi^{②}, \psi^{⑦}\}$ and $\{\psi^{③}, \psi^{⑦}\}$.
The next possibility would be:

$$\nabla\psi^{④} = m_d\,\psi^{⑦} \tag{3}$$

The conjugated equation is:

$$\nabla^{*}\psi^{③} = m_d\,\psi^{⓪} \tag{4}$$

The particle is represented by the ordered pair $\{\psi^{④}, \psi^{⑦}\}$. The corresponding flip does switch the handedness. Again the state function QPAD is coupled to the general background QPAD $\psi^{⑦}$.

Like the electron, this particle will have charge, but its charge will be three times lower, because only one instead of three imaginary base vectors cause the switch in handedness. Of course, this is an opportunistic interpretation, but it seems to fit when we assume that the particle is a down quark with charge equal to -⅓e.

The formula for the coupling factor $m_d$ is:

$$\int_V \psi^{⓪} \nabla \psi^{④}\, dV = m_d \int_V \psi^{⓪} \psi^{⑦}\, dV = m_d \int_V \left|\psi^{⓪}\right|^2 dV = m_d \tag{5}$$

The coupling that constitutes the down quark is anisotropic. This fact introduces a new kind of charge, which is called color charge. The color is related to the direction of the reflection of the state function QPAD.

The pairs $\{\psi^{④}, \psi^{⑦}\}$, $\{\psi^{⑤}, \psi^{⑦}\}$ and $\{\psi^{⑥}, \psi^{⑦}\}$ give similar results. They represent the three down quark colors. The corresponding antiparticles are $\{\psi^{③}, \psi^{⓪}\}$, $\{\psi^{②}, \psi^{⓪}\}$ and $\{\psi^{①}, \psi^{⓪}\}$.

In summary the down quarks have the following properties:

- Location
  - Position
  - Momentum
- Electric charge

- Spin
- Color charge

For down quarks the color charge is relevant. This is due to the fact that three down quarks can combine in a baryon[66]. Without the existence of color charge the Pauli principle would forbid that.

## 4.3 Massles bosons

This paragraph treats the zero-level of coupling.

### 4.3.1 No coupling

The last possible form in which the state function couples to the background field $\psi^{⑦}$ is:

$$\nabla\psi^{⑦} = m\,\psi^{⑦} \tag{1}$$

The formula for the coupling factor $m$ is:

$$\int_V \psi^{⓪}\nabla\psi^{⑦}\,dV = m\int_V \psi^{⓪}\psi^{⑦}\,dV = m\int_V |\psi^{⓪}|^2\,dV = m \tag{2}$$

$$\int_V \psi^{⓪}\nabla\psi^{⑦}\,dV = ½\int_V \nabla|\psi|^2\,dV = 0 \tag{3}$$

[66] See: Hadrons

Presence does not leak. So,

$$m = 0. \tag{4}$$

With other words a QPAD does not couple to itself. What results is a free QPAD.

$\psi^{⓪}$ and $\psi^{⑦}$ deliver photons. The other sign flavors deliver gluons.

### 4.3.2 The free QPAD

When for sign flavor $\psi^x$ the coupling factor $m$ is zero then:

$$\nabla\psi^x = 0 \tag{1}$$

$$\nabla_0\psi_0^x = \langle \boldsymbol{\nabla}, \boldsymbol{\psi^x} \rangle \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\psi^x} + \boldsymbol{\nabla}\psi_0^x + \nabla_0\boldsymbol{\psi^x} = \mathbf{0} \tag{3}$$

It means that a change $\nabla_0\boldsymbol{\psi^x}$ in the speed of the current goes together with a rotation $\boldsymbol{B}$ of the current

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{\psi^x} \tag{4}$$

and/or a new field $\mathfrak{E}$:

$$\mathfrak{E} = -\boldsymbol{\nabla}\psi_0^x \tag{5}$$

For comparison, in the equations of Maxwell[67] the field $\boldsymbol{E}$ is defined as:

[67] http://en.wikipedia.org/wiki/Maxwell%27s_equations#Potential_formulation

$$\boldsymbol{E} = -\boldsymbol{\nabla}\psi_0^x - \nabla_0\boldsymbol{\psi^x} = \mathfrak{E} - \nabla_0\boldsymbol{\psi^x} \tag{6}$$

In those equations $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field. However here these fields have a more general meaning.

Thus equation (3) means:

$$\boldsymbol{B} = \boldsymbol{E} \tag{7}$$

More interesting is the corollary

$$\nabla_0\boldsymbol{B} = \nabla_0\boldsymbol{E} \tag{8}$$

$$\begin{aligned}\nabla_0\boldsymbol{B} = \boldsymbol{\nabla}\times\nabla_0\boldsymbol{\psi^x} &= \boldsymbol{\nabla}\times\mathfrak{E} - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\psi^x} \\ &= -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\psi^x}\end{aligned} \tag{9}$$

$$= \boldsymbol{\nabla^2\psi^x} - \boldsymbol{\nabla}\langle\boldsymbol{\nabla},\boldsymbol{\psi^x}\rangle = \boldsymbol{\nabla^2\psi^x} - \boldsymbol{\nabla}\nabla_0\psi_0^x$$

$$\nabla_0\boldsymbol{E} = -\boldsymbol{\nabla}\nabla_0\psi_0^x - \nabla_0^2\boldsymbol{\psi^x} \tag{10}$$

Thus

$$\boldsymbol{\nabla^2\psi^x} = -\nabla_0^2\boldsymbol{\psi^x} \tag{11}$$

Or

$$\boldsymbol{\nabla^2\psi^x} = \boldsymbol{0} \tag{12}$$

Further

$$\begin{aligned}\nabla_0^2\psi_0^x = \langle\boldsymbol{\nabla},\nabla_0\boldsymbol{\psi^x}\rangle &= -\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\times\boldsymbol{\psi^x}\rangle - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\psi_0^x\rangle \\ &= -\boldsymbol{\nabla^2}\psi_0^x\end{aligned} \tag{13}$$

Thus:

$$\nabla^2 \psi^x = 0 \tag{14}$$

With other words a free (= not coupled) $\psi^x$ is either harmonic or it is static. The static condition corresponds to the ground state.

This holds for all QPAD's.

#### 4.3.2.1 Spin

Free QPAD's have spin[68].

$$\boldsymbol{\Sigma}_{field} = \int_V \boldsymbol{E}(q) \times \boldsymbol{\psi}^x(q) dV \tag{1}$$

Elementary bosons appear to have the same value. It is not clear why in couplings this spin reduces to half this value for elementary fermions.

### 4.3.3 Extra freedom

Each solution $\psi^x$ of the restricted elementary equation is determined up to a solution $\chi^x$ with $\nabla \chi^x = 0$.

$$\nabla(\psi^x + \chi^x) = m\, \psi^y \tag{1}$$

This is a gauge transformation.

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[68] Bo Thidé (13-09-2011):
http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf ,formula (4.85a)

## 4.4 Consideration

We have now exhausted all possibilities for coupling a QPAD sign flavor to the general background QPAD $\psi^{⑦}$. Above we could link the analyzed particles to electrons, down quarks and either up-quarks or neutrinos. Their antiparticles were treated as well. The investigated particles are elementary fermions. (*We did not state or explain why they are fermions*.)

We also analyzed the situation that a QPAD is coupled to itself. That situation leads to zero coupling factor, which means no coupling. The free QPAD's are bosons and correspond to photons or gluons. (*We did not state or explain why these objects are bosons*.)

The ordered pair $\{\psi^x, \psi^y\}$ represents a category of elementary particle types.

For antiparticles all participating fields and the nabla operator conjugate.

We can now try to establish the apparent rules of the game. The rules are:

1. If the state function QPAD is coupled to the general background QPAD $\psi^{⑦}$, then the particle is a fermion. Otherwise, it is a boson. For antiparticles this rule must be adapted.
2. A handedness switch can have one of two consequences
    - If the coupling takes place between two QPAD sign flavors with the same handedness, then the corresponding particle may be electrically charged.

- If the coupling takes place between two QPAD sign flavors with the same handedness, then the corresponding particle is not electrically charged.

3. The electric charge depends on the number and on the direction of the base vectors that differ.
4. The electric charge count for each difference is ±⅓e.
5. The color charge depends on the direction(s) of anisotropy of the coupling.

The first rule indicates that it has sense to take $\psi^{⑦}$ as the base sign flavor.

From the known elementary particles members exist that are not yet covered. They are $W_+$ and $W_-$ bosons and Z bosons. We like to proceed in a similar way, but the coupling with the isotropic background QPAD $\psi^{⑦}$ is used up. Now let us try other couplings.

We already encountered one, the ordered pair of sign flavors $\{\psi^{⑦}, \psi^{⓪}\}$ that obeys

$$\nabla\psi^{⑦} = m\,\psi^{⓪} \tag{1}$$

The coupling changes the handedness, so the particle is charged. It has much in common with the positron. Still it is not the antiparticle of the electron, because its equation of motion differs. It might exist, but then it probably hides behind the positron. It is the shadow of the positron.

In the following we may mark particle pairs as ‘anti’. In those cases we may also indicate the shadow of the antiparticle.

## 4.5 Anisotropic coupling fields

We have explored all particles that make use of the isotropic background QPAD $\psi^{⑦}$. These particles appear to be fermions. Next we like to explore particles that couple to anisotropic backgrounds. These particles will appear to be bosons. It means that they all have integer valued spin. *It is not explained why these particles are bosons or have full integer spin.* We derive their identity from their electric charge and from the fact that they are bosons.

### 4.5.1 The cross-sign flavor equations

These equations describe the situation that couples $\{\psi^{⑥}, \psi^{①}\}$ , $\{\psi^{⑤}, \psi^{②}\}$ or $\{\psi^{④}, \psi^{③}\}$ and vice versa.

$$\nabla \psi_i^{⑥} = m_{W_+} \psi_i^{①} \tag{1}$$

The conjugated equation is:

$$\nabla^* \psi^{①} = m_{W_-} \psi^{⑥} \tag{2}$$

Another form is

$$\nabla \psi^{①} = m_{W_-} \psi^{⑥} \tag{3}$$

The conjugated equation is:

$$\nabla^* \psi^{⑥} = m_{W_+} \psi^{①} \tag{4}$$

The sign flavor switch affects three imaginary base vectors and flips the handedness. As a consequence the particles have a full electric charge. It concerns two particle types, the $W^-$ and the $W^+$ bosons. These bosons carry electrical charges.

The $W_+$ and $W_-$ bosons are considered to be each other's anti-particle. It is also possible that they hide between each other's anti-particle.

$$\psi_i^{⑥} \nabla\psi_i^{⑥} = m_{W_+}\psi_i^{⑥} \psi_i^{①} \tag{5}$$

$$\int_V \left(\psi_i^{⑥}\nabla\psi_i^{⑥}\right) dV = m_{W_+}\int_V \left(\psi_i^{⑥} \psi_i^{①}\right) dV = m_{W_+}g \tag{6}$$

$$\psi_i^{①} \nabla\psi_i^{①} = m_{W_-}\psi_i^{①} \psi_i^{⑥} \tag{7}$$

$$\int_V \left(\psi_i^{①} \nabla\psi_i^{①}\right) dV = m_{W_-}\int_V \left(\psi_i^{⑥} \psi_i^{①}\right) dV = m_{W_-}g \tag{8}$$

Similar reasoning holds for pairs $\{\psi^{⑤}, \psi^{②}\}$ and $\{\psi^{④}, \psi^{③}\}$.

### 4.5.2 The Z boson

The particle that obeys:

$$\nabla\psi^{⑦} = m_Z \psi^{①} \tag{1}$$

The handedness is kept the same. Two base vectors change direction. It is a either a neutral boson or a boson with charge ⅔ e.

The pairs $\{\psi^{⑦}, \psi^{②}\}$ and $\{\psi^{⑦}, \psi^{③}\}$ behave similar. Corresponding antiparticles are $\{\psi^{⓪}, \psi^{⑥}\}$, $\{\psi^{⓪}, \psi^{⑤}\}$ and $\{\psi^{⓪}, \psi^{④}\}$.

$$\int_V \left(\psi_i^{⑥}\, \nabla\psi^{⑦}\right)\, dV \;= m_Z \int_V \left(\psi_i^{⑥}\, \psi_i^{①}\right)\, dV = m_Z\, g \tag{2}$$

Another possibility is:

$$\nabla\psi^{⓪} \;= m_{Z?}\, \psi^{⑥} \tag{3}$$

The pairs $\{\psi^{⓪}, \psi^{⑤}\}$ and $\{\psi^{⓪}, \psi^{④}\}$ behave similar. Corresponding antiparticles are $\{\psi^{⑦}, \psi^{①}\}$, $\{\psi^{⑦}, \psi^{②}\}$ and $\{\psi^{⑦}, \psi^{③}\}$.

## 4.6 Resulting particles

Under the resulting particles fall 12 multi-color couplings.
$\{\psi^{①}, \psi^{②}\}, \{\psi^{①}, \psi^{③}\}, \{\psi^{②}, \psi^{①}\}, \{\psi^{②}, \psi^{③}\}, \{\psi^{③}, \psi^{①}\}, \{\psi^{③}, \psi^{②}\},$
$\{\psi^{④}, \psi^{⑤}\}, \{\psi^{④}, \psi^{⑥}\}, \{\psi^{⑤}, \psi^{③}\}, \{\psi^{⑤}, \psi^{⑥}\}, \{\psi^{⑥}, \psi^{④}\}$ and $\{\psi^{⑥}, \psi^{⑤}\}$
With these go 12 antiparticles, which form the same set.

The multi-color couplings are anisotropic QPAD's that are coupled to other anisotropic QPAD's that only differ with respect to their color. They show great resemblance with mesons.

Further, we have XX particles via pairs
$\{\psi^{⓪}, \psi^{①}\}, \{\psi^{⓪}, \psi^{②}\}$ and $\{\psi^{⓪}, \psi^{③}\}$.
They are -1/3 e charged bosons and have antiparticles
$\{\psi^{⑦}, \psi^{⑥}\}, \{\psi^{⑦}, \psi^{⑤}\}$ and $\{\psi^{⑦}, \psi^{④}\}$.
We have Y particles via pairs
$\{\psi^{③}, \psi^{⑤}\}$ and $\{\psi^{③}, \psi^{⑥}\}$.
They are -1/3 e charged bosons and have antiparticles
$\{\psi^{④}, \psi^{②}\}$ and $\{\psi^{④}, \psi^{①}\}$.

We have YY particles via pairs
$\{\psi^{⓪}, \psi^{④}\}, \{\psi^{⓪}, \psi^{⑤}\}$ and $\{\psi^{⓪}, \psi^{⑥}\}$.
They are 1/3 e charged bosons and have antiparticles
$\{\psi^{⑦}, \psi^{③}\}, \{\psi^{⑦}, \psi^{②}\}$ and $\{\psi^{⑦}, \psi^{①}\}$.

Finally, we have ZZ particles via pairs
$\{\psi^{②}, \psi^{④}\}$ and $\{\psi^{②}, \psi^{⑥}\}$.
They are -1/3 e charged bosons and have antiparticles
$\{\psi^{⑤}, \psi^{③}\}$ and $\{\psi^{⑤}, \psi^{①}\}$.

## 4.7 Antiparticles

Just like the universe is filled with a huge number of particles, it is also filled with a huge number of antiparticles. This anti-world has its own kind of background QPAD's. It represents the local superposition of the tails of the state functions of a category of antiparticles in universe. Otherwise the antiparticles would not sense the same kind of inertia that particles do.

## 4.8 Shadow particles

Several particle types have properties and behavior that are similar to the properties and behavior of antiparticle types of other particle types.

For example $\{\psi^{⑦}, \psi^{⓪}\}$ is hidden behind the positron which is the antiparticle of the electron $\{\psi^{⓪}, \psi^{⑦}\}$.

The $W_+$ boson $\{\psi^{⑥}, \psi^{①}\}$ hides behind the antiparticle of the $W_-$ boson $\{\psi^{①}, \psi^{⑥}\}$ and vice versa.

## 4.9 Table of elementary particles

The elementary coupling equation couples pairs $\{\psi^x, \psi^y\}$ of quaternionic probability amplitude distributions. 64 of those ordered pairs exist. 56 particles and 8 waves.
If a particle is marked as 'anti', it concerns the shadow of the anti-particle.
At many points a striking agreement with the standard model exists. However, also some striking disagreements exist. For example if neutrinos and Z-bosons are supported then up-quarks are not supported as elementary particles.

| nr | pair | type | +/- | R?L? | e-charge | color | description |
|---|---|---|---|---|---|---|---|
| 1 | $\{\psi^{⓪}, \psi^{⓪}\}$ | boson | 0 | RR | 0 | none | photon |
| 2 | $\{\psi^{⓪}, \psi^{①}\}$ | boson | -1 | RL | -1/3 e | $R$ | XX |
| 3 | $\{\psi^{⓪}, \psi^{②}\}$ | boson | -1 | RL | -1/3 e | $G$ | XX |
| 4 | $\{\psi^{⓪}, \psi^{③}\}$ | boson | -1 | RL | -1/3 e | $B$ | XX |
| 5 | $\{\psi^{⓪}, \psi^{④}\}$ | boson | -2 | RR | 0 | $\bar{B}$ | Anti Z |
| 6 | $\{\psi^{⓪}, \psi^{⑤}\}$ | boson | -2 | RR | 0 | $\bar{G}$ | Anti Z |
| 7 | $\{\psi^{⓪}, \psi^{⑥}\}$ | boson | -2 | RR | 0 | $\bar{R}$ | Anti Z |
| 8 | $\{\psi^{⓪}, \psi^{⑦}\}$ | fermion | -3 | RL | -e | none | electron |
| 9 | $\{\psi^{①}, \psi^{⓪}\}$ | fermion | 1 | LR | 1.3 e | $R$ | anti-down quark |
| 10 | $\{\psi^{①}, \psi^{①}\}$ | boson | 0 | LL | 0 | $RR$ | gluon |
| 11 | $\{\psi^{①}, \psi^{②}\}$ | boson | 0 | LL | 0 | $RG$ | multicolor |
| 12 | $\{\psi^{①}, \psi^{③}\}$ | boson | 0 | LL | 0 | $RB$ | multicolor |
| 13 | $\{\psi^{①}, \psi^{④}\}$ | boson | 1 | LR | 1/3 e | $R\bar{B}$ | YY |
| 14 | $\{\psi^{①}, \psi^{⑤}\}$ | boson | 1 | LR | 1/3 e | $R\bar{G}$ | YY |
| 15 | $\{\psi^{①}, \psi^{⑥}\}$ | boson | -3 | LR | -e | $R\bar{R}$ | $W_-$ |
| 16 | $\{\psi^{①}, \psi^{⑦}\}$ | fermion | -2 | LL | 0 | $R$ | neutrino |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | $\{\psi^{②},\psi^{⓪}\}$ | fermion | 1 | LR | 1.3 e | $G$ | Anti-down quark |
| 18 | $\{\psi^{②},\psi^{①}\}$ | boson | 0 | LL | 0 | $GR$ | multicolor |
| 19 | $\{\psi^{②},\psi^{②}\}$ | boson | 0 | LL | 0 | $GG$ | gluon |
| 20 | $\{\psi^{②},\psi^{③}\}$ | boson | 0 | LL | 0 | $GB$ | multicolor |
| 21 | $\{\psi^{②},\psi^{④}\}$ | boson | -1 | LR | -1/3 e | $G\bar{B}$ | ZZ |
| 22 | $\{\psi^{②},\psi^{⑤}\}$ | boson | -3 | LR | -e | $G\bar{G}$ | $W_{-}$ |
| 23 | $\{\psi^{②},\psi^{⑥}\}$ | boson | -1 | LR | -1/3 e | $G\bar{R}$ | ZZ |
| 24 | $\{\psi^{②},\psi^{⑦}\}$ | fermion | -2 | LL | 0 | $G$ | neutrino |
| 25 | $\{\psi^{③},\psi^{⓪}\}$ | fermion | 1 | LR | 1.3 e | $B$ | Anti-down quark |
| 26 | $\{\psi^{③},\psi^{①}\}$ | boson | 0 | LL | 0 | $BR$ | multicolor |
| 27 | $\{\psi^{③},\psi^{②}\}$ | boson | 0 | LL | 0 | $BG$ | multicolor |
| 28 | $\{\psi^{③},\psi^{③}\}$ | boson | 0 | LL | 0 | $BB$ | gluon |
| 29 | $\{\psi^{③},\psi^{④}\}$ | boson | -3 | LR | -e | $B\bar{B}$ | $W_{-}$ |
| 30 | $\{\psi^{③},\psi^{⑤}\}$ | boson | -1 | LR | -1/3 e | $B\bar{G}$ | Y |
| 31 | $\{\psi^{③},\psi^{⑥}\}$ | boson | -1 | LR | -1/3 e | $B\bar{R}$ | Y |
| 32 | $\{\psi^{③},\psi^{⑦}\}$ | fermion | -2 | LL | 0 | $B$ | neutrino |
| 33 | $\{\psi^{④},\psi^{⓪}\}$ | fermion | 2 | RR | 0 | $\bar{B}$ | Anti-neutrino |
| 34 | $\{\psi^{④},\psi^{①}\}$ | boson | 1 | RL | 1/3 e | $\bar{B}R$ | Anti-Y |
| 35 | $\{\psi^{④},\psi^{②}\}$ | boson | 1 | RL | 1/3 e | $\bar{B}G$ | Anti-Y |
| 36 | $\{\psi^{④},\psi^{③}\}$ | boson | 3 | RL | e | $\bar{B}$ | $W_{+}$ |
| 37 | $\{\psi^{④},\psi^{④}\}$ | boson | 0 | RR | 0 | $\bar{B}\bar{B}$ | gluon |
| 38 | $\{\psi^{④},\psi^{⑤}\}$ | boson | 0 | RR | 0 | $\bar{B}\bar{G}$ | multicolor |
| 39 | $\{\psi^{④},\psi^{⑥}\}$ | boson | 0 | RR | 0 | $\bar{B}\bar{R}$ | multicolor |
| 40 | $\{\psi^{④},\psi^{⑦}\}$ | fermion | -1 | RL | -1/3 e | $\bar{B}$ | down quark |
| 41 | $\{\psi^{⑤},\psi^{⓪}\}$ | fermion | 2 | RR | 0 | $\bar{G}$ | Anti-neutrino |
| 42 | $\{\psi^{⑤},\psi^{①}\}$ | boson | 1 | RL | 1/3 e | $\bar{G}R$ | Anti ZZ |
| 43 | $\{\psi^{⑤},\psi^{②}\}$ | boson | 3 | RL | e | $G$ | $W_{+}$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 44 | $\{\psi^{⑤},\psi^{③}\}$ | boson | 1 | RL | 1/3 e | $\bar{G}B$ | Anti ZZ |
| 45 | $\{\psi^{⑤},\psi^{④}\}$ | boson | 0 | RR | 0 | $\bar{R}\vec{G}$ | multicolor |
| 46 | $\{\psi^{⑤},\psi^{⑤}\}$ | boson | 0 | RR | 0 | $\bar{G}\vec{G}$ | gluon |
| 47 | $\{\psi^{⑤},\psi^{⑥}\}$ | boson | 0 | RR | 0 | $\bar{G}\vec{R}$ | multicolor |
| 48 | $\{\psi^{⑤},\psi^{⑦}\}$ | fermion | -1 | RL | -1/3 | $\bar{G}$ | down quark |
| 49 | $\{\psi^{⑥},\psi^{⓪}\}$ | fermion | 2 | RR | 0 | $\bar{R}$ | Anti-neutrino |
| 50 | $\{\psi^{⑥},\psi^{①}\}$ | boson | 3 | RL | e | $\bar{R}R$ | $W_+$ |
| 51 | $\{\psi^{⑥},\psi^{②}\}$ | boson | 1 | RL | 1/3 e | $\bar{R}G$ | Anti YY |
| 52 | $\{\psi^{⑥},\psi^{③}\}$ | boson | 1 | RL | 1/3 e | $\bar{R}B$ | Anti YY |
| 53 | $\{\psi^{⑥},\psi^{④}\}$ | boson | 0 | RR | 0 | $\bar{R}\bar{B}$ | multicolor |
| 54 | $\{\psi^{⑥},\psi^{⑤}\}$ | boson | 0 | RR | 0 | $\bar{R}\vec{G}$ | multicolor |
| 55 | $\{\psi^{⑥},\psi^{⑥}\}$ | boson | 0 | RR | 0 | $\bar{B}\bar{B}$ | gluon |
| 56 | $\{\psi^{⑥},\psi^{⑦}\}$ | fermion | -1 | RL | -1/3 | $\bar{R}$ | down quark |
| 57 | $\{\psi^{⑦},\psi^{⓪}\}$ | fermion | 3 | LR | e | none | positron |
| 58 | $\{\psi^{⑦},\psi^{①}\}$ | boson | 2 | LL | 0 | $R$ | Z |
| 59 | $\{\psi^{⑦},\psi^{②}\}$ | boson | 2 | LL | 0 | $G$ | Z |
| 60 | $\{\psi^{⑦},\psi^{③}\}$ | boson | 2 | LL | 0 | $B$ | Z |
| 61 | $\{\psi^{⑦},\psi^{④}\}$ | boson | 1 | LR | 1/3 e | $\bar{B}$ | Anti XX |
| 62 | $\{\psi^{⑦},\psi^{⑤}\}$ | boson | 1 | LR | 1/3 e | $\bar{G}$ | Anti XX |
| 63 | $\{\psi^{⑦},\psi^{⑥}\}$ | boson | 1 | LR | 1/3 e | $\bar{R}$ | Anti XX |
| 64 | $\{\psi^{⑦},\psi^{⑦}\}$ | boson | 0 | LL | 0 | none | photon |

32 zero charge particles exist, of which 12 lose their charge due to missing handedness switch. These twelve are neutrino and Z (anti)particles. Otherwise they would have charge ± ⅔ e. Thus according to the HBM either neutrinos and Z bosons do not exist or up quarks are no elementary particles.

24 particles exist that have charge ± ⅓ e. 8 particles exist that have charge ± e. Eight waves exist that have zero mass. They cover two photons and six mono-color gluons.

## 4.10 Limits of the model

The restricted elementary coupling scheme does not distinguish between generations of elementary particle types.

It must be stated that the reason of being a fermion as it is applied here, differs strongly from the usual fermion/boson assignment. As a consequence also the notion of spin will differ. This approach is due to the fact that the elementary particles are defined as a pair of coupled QPAD's and not as a single state function QPAD. Only the ordered pair will define the value of the spin and the fact that the particle is a fermion.

We adopt the existing convention that fermions go together with half integer valued spin. Here it will not be explained why that relation exists. With other words, having half integer spin and being a fermion is related on the one hand to experimental results and on the other hand to the ordered pair of coupled QPAD's that represents the particle. In the HBM according to the current rules, a particle is a fermion when its state function is coupled to the general background QPAD. What spin actually is, is not explained in this model. We just accept the existing convention. The same holds for the electrical charges.

In short, this model does not explain why particles get their electric charge or spin. The model only explains the origin and the habits of the coupling factor and it explains how the values of the electric charge and spin relate with the ordered coupled QPAD pair that represents the particle.

Later, it will be explained how the coupling factor relates to the mass of the particle.

The scheme does not provide a way to construct up-quarks from coupled QPAD's. Their electric charge conflicts with the existence of anti-neutrinos. Instead the scheme provides some extra particles that do not appear in the standard model. It might be possible to compose up-quarks from a combination of XX particles and positrons or compose up-quarks from a combination of XX particles and $W_+$ particles.

The scheme provides two photons and six mono-colored waves, that we called gluons. In the standard model the massless gluons are multicolored. The HBM scheme provides twelve multi-colored particles that have no charge but have mass.

With this addition and by neglecting generations the scheme is capable to generate all known elementary particles that are contained in the standard model. In fact if generations are neglected, then the generated set is much larger than the standard model. This is only partly due to the fact that color charge is considered for all anisotropic particles.

The scheme comprises electrons, positrons, neutrinos, anti-neutrinos, down quarks, anti-down quarks, $W_+$ particles, $W_-$ particles, Z particles, anti-Z particles, photons and gluons. Where applicable these particles appear with color charge. Color charge is noticeable for down quarks when baryons of three down quarks are formed.

Extra particles are the multicolor particles, the Y-particles, the XX particles, the YY particles and the ZZ particles.

# 5 Origin of curvature

The primary QPAD's cause local pressure differences in the QPAD-sphere. This only occurs when the QPAD is coupled to another QPAD. For a restricted elementary coupling both QPAD's must be sign flavors of the same base QPAD. On its turn the local pressure causes a local space curvature.

## 5.1 Physical fields

In the HBM two categories of physical fields exist. The first category is formed by free primary QPAD's. These QPAD's oscillate. Depending on their sign flavor they are photons or gluons. Primary QPAD's are linked with an eigenvector of an operator in separable Hilbert space. For the free primary QPAD's this is the canonical conjugate of the locator operator.

The second category is formed by fields that represent the properties of the couplings of primary QPAD's that correspond to elementary particles. These properties are coupling factor, electric charge and spin. The state functions of massive elementary particles are primary QPAD's that are linked with an eigenvector of the particle locator operator in separable Hilbert space. The corresponding physical fields are gravitation fields, electrostatic fields and magnetostatic fields. These fields move with the particles, but they do not oscillate like the first category does. However, in atoms the elementary particles may oscillate.

It is assumed that in atoms only the state function QPAD part of the coupled QPAD pair oscillates and the other part; the background QPAD, stays stable. It implements inertia. Still it moves with the movement of the atom as a whole. The physical field that

represents the influence of a property of an elementary particle is assumed to be attached to the background QPAD of that elementary particle. Thus this second category physical field does not oscillate. A first category physical field is released or absorbed when the elementary particle switches its oscillation status.

With other words, photons and gluons are fundamentally different from the second category of physical fields.

This may be the reason that photons cannot pass the event horizon of black holes, while the information of the BH's mass, electric charge and spin are available to the environment of the BH.

## 5.2 Curvature and inertia

All primary couplings affect the local curvature

Only the state function QPAD's that couple to a background QPAD will experience inertia

This holds for particles as well as for antiparticles.

### 5.2.1 Inertia versus antiparticle

Besides of the fact that all particles possess corresponding antiparticles, each state function category seems to correspond with a corresponding background QPAD, which in some cases is the conjugate of the state function QPAD. That happens with W particles and leptons. Fermions and their antiparticles seem to prefer isotropic background QPAD's. Most fermions share the same (isotropic) background QPAD and anti-fermions do the same with the conjugate.

### 5.2.2 Inertia of W and Z Bosons

W and Z bosons use their "own" background QPAD, which is anisotropic.

The background QPAD of a $W_-$ boson has the form of the state function QPAD of a $W_+$ boson, which has the form of an anti-$W_-$ boson.

The background QPAD of a $W_+$ boson has the form of the state function QPAD of a $W_-$ boson, which has the form of an anti-$W_+$ boson.

## 5.3 Effect of primary coupling

The coupling may compress the local parameter space.

In the table above, 56 different kinds of primary coupling are discerned.

Together with the balance equations that differ for particles and antiparticles and shadow particles this defines a large number of ways of how the local parameter space is affected.

## 5.4 The HBM Palestra reviewed

In the Hilbert Book Model all QPAD's and the fields that are derived from them or from their couplings, share the same (affine) parameter space. For that reason this common parameter space has been given a special name; ***the HBM Palestra***[69]. This shared parameter space spreads universe wide. It is the place where universe is located.

The parameter space of the QPAD can be interpreted as a quaternionic distribution. It has itself a parameter space, which is formed by a 3D continuum. This continuum is taken from the eigenspace of a location operator that resides in the Gelfand triple of the separable Hilbert space. Only the imaginary part of the quaternionic distribution is used. It can be considered as a 3D Riemannian manifold. The local metric defines the local curvature. What occurs in this manifold is described by the QPAD's.

The Palestra is the playground of all what happens in fundamental physics. It is governed by a special kind of fluid dynamics[70]. Things like charge density distributions, current density distributions, sources, drains, compressed regions and decompressed regions occur in this space[71]. The QPAD's are not the transporters. They only describe the transport process. The action takes place in their shared parameter space. That's how these QPADS's can influence each other.

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[69] The name Palestra is suggested by Henning Dekant's wive Sarah. It is a name from Greek antiquity. It is a public place for training or exercise in wrestling or athletics

[70] See: Quantum fluid dynamics

[71] Vortexes may also appear in the Palestra. The HBM does not yet treat them.

The sources and drains are controlled by Poisson processes[72]. The generated quanta take a probable track. The most probable track has preference. Thus the currents described by the QPAD's are corridors rather than actual paths. Similarly the locations of the charges in the density distributions are places where charges may exist with corresponding probability. It need not be the place where the charges are. The phrase "is governed by a special kind of fluid dynamics" must be interpreted in this view. The transported quanta are sparsely present in the HBM Palestra.

Nevertheless this special space may be characterized by notions such as temperature and entropy. Both indicate a relation with information.

Two kinds of processes determine the dynamics in the parameter space. The first kind is formed by quantum generating or quantum annihilating Poisson processes. The second kind is caused by the coupling of QPAD's. It means that the dynamics in the parameter space can be described by a combination of fluid dynamics and statistical mechanics.

Since the HBM implements dynamics via a sequence of pages, each HBM page only shows the static status quo of the parameter space. The real part of the quaternionic distributions that define the HBM Palestra can be used to store the progression parameter.

The Palestra is characterized by a local metric tensor. This tensor and the continuity of the quaternionic distribution that defines the Palestra cause that what happens in the Palestra is controlled by a special Lie group.

When the inertial frame of the observer moves relative to the inertial frame of the observed item, then $|ds|^2$ becomes the squared spacetime interval and this Lie group is the Einstein group[73],

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[72] See: What image intensifiers reveal

[73] http://en.wikipedia.org/wiki/Einstein_group#The_Einstein_group

The elementary coupling equation is a continuity equation. It describes primary couplings. The coupling can be characterized by a small set of properties. These properties are conserved. Most of these properties can be considered as sources of second category physical fields. These second category physical fields correspond to physical fields that are attached to the corresponding elementary particle.

The elementary coupling equation appears in 64 forms. Eight of these correspond to a zero coupling factor, thus these situations describe free QPAD's. It concerns photons and gluons. The 56 resulting cases concern specific elementary particles. Thus, there are 56 ways to constitute a primary coupling. This count neglects the anti-particle equation. Thus, a very large number of different primary couplings are possible.

After a coupling the resulting currents need not be exhausted completely. The properties of the coupling are conserved and the corresponding physical fields and the resulting currents can be exploited in higher order coupling. So what happens at these scales can become very complicated.

At some distance from the elementary particle the dynamics can be described as incompressible flow. However, close to the particle, thus close to the coupling the dynamics must be treated as compressible flow. It means that features like compression, entropy and temperature will start to play a role.

In short; the HBM Palestra is the sparsely occupied internally moving space that is used as parameter space by all QPAD's that play a role in the HBM.

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See: Symmetry in electrodynamics; M. Sachs

## 5.5 What image intensifiers reveal

The author spent eighteen years in the development of image intensifier tubes. These devices range from goggles via driver scopes to fourteen inch wide X-ray image intensifiers.

The image intensifiers had one feature in common. They were all capable of turning the impingement of a quantum at their input screen into a visible light spot at their output screen. A hail storm of impinging quanta at the input resulted in a noisy film at the output[74][75][76].

The starlight scopes enable visibility of very low dose scenes under starlight conditions. They turned infrared and visible light quanta into light spots on a luminescent phosphor screen.

The X-ray image intensifiers were designed to deliver a perceptible image of an X-ray shadow picture at the lowest possible X-ray dose for the diagnosed patient.

What still astonishes me is that I never saw any indication of a wave entering the input of the image intensifiers. I only saw clouds of quanta. That does not say that these clouds cannot have the shape of waves, but the detected quanta did not show that relation.

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[74] See: http://en.wikipedia.org/wiki/File:Moon_in_x-rays.gif . Low dose X-ray image of the moon.

[75] See: http://www.youtube.com/watch?v=U7qZd2dG8uI ; Hail storm. Warning, this is NOT a video of an external object.

[76] Also see: http://en.wikipedia.org/wiki/Shot_noise

## Example

A short film of the output of an X-ray image intensifier made at a very low dose rate.

Provided by Philips Healthcare

- The pixel size is about 200µm
- The number of pixels is about 500 * 600
- The average number of X-ray quanta per pixel per frame in the mid gray area is circa 1
- The range inside these picture is about 20
- The direct radiance is about 5 quanta per pixel per frame
- The dark regions get <<1 quanta per frame
- The number of pictures is 33

http://www.crypts-of-physics/QuantumLimitedLlowDoseImaging.avi

With other words, what we can observe are the quanta. We cannot observe the envelop of the quantum clouds. We cannot observe the waves!

This is a significant experience. Light as a wave does not exist as a physically observable object. It only exists as a bunch of quanta. That quantum cloud may have the shape of a wave, but we cannot discern that wave. We can only detect interferences of these waves because the patterns of the detected quanta take that shape, but we cannot observe the interferences themselves.

What are then these quanta that impinge on the inputs of our detectors? They are tiny patches of something.

The Hilbert Book Model makes a very particular assumption about these quanta.

The HBM interprets the quanta as tiny patches of the parameter space of the state function. Further the HBM uses state functions that are quaternionic probability amplitude distributions. This makes it possible to interpret the state functions as combinations of charge density distributions and current density distributions. The state functions may also contain sources, drains, compressed regions and decompressed regions. These features all occur in the parameter space of the state function.

This daring interpretation explains why in the neighborhood of particles the parameter space appears to be compressed.

When measuring the properties of image intensifiers it appeared that the distribution of detected quanta can be characterized as a Poisson distribution. That indicated that the generators of the quanta can be considered as Poisson processes. It is well known that Poisson processes that are attenuated by subsequent binomial processes can be treated as a new less efficient Poisson process. Spatial and temporal spread can be interpreted as a binomial processes. It is also known that very efficient Poisson processes produce distributions that are close to Gaussian distributions. When looked upon as a charge distribution the Gaussian distribution produces a potential that approaches an Error function. At some distance from the generator the Error function takes the form of a 1/r function. On its turn this corresponds with a single charge source. Thus from a distance the generator looks as a (singular) single charge source. Of course the actual generator is not singular!

This gives a particular insight in radiation sources!

## 5.6 Quantum Fluid Dynamics

What happens in the HBM Palestra is controlled by Quantum Fluid Dynamics (QFD). QFD differs from conventional fluid dynamics in that in QFD the charge density distributions and current density distributions describe probable locations and paths in their own parameter space, while in conventional fluid dynamics these distributions describe actual locations and paths that occur in a considered medium such as a gas or liquid. That is why in QFD the charge density distributions and current density distributions are combined in quaternionic probability amplitude distributions, while in conventional fluid dynamics they are located in scalar and vector fields. These fields can also be combined in quaternionic distributions, but they are not probability amplitude distributions.

In the Palestra $\nabla_0$ represents the derivative with respect to progression. It is not the derivative with respect to coordinate time.

Already in 1927 Erwin Madelung[77] published a set of equations that treat quantum physics in the sense of QFD.

[77] See: http://en.wikipedia.org/wiki/Madelung_equations

# 6 Higher level coupling

The primary QPAD's cause a local pressure in the QPAD-sphere. On its turn that local pressure causes the local space curvature.

The streams of space patches that result after the primary couplings will be used in higher level interactions. It means that these resulting currents may still influence higher level coupling.

Hypothesis:
In these interactions the properties of the primary couplings are conserved

All elementary particles can emit or absorb photons or gluons via gauge transformations. The gauge boson must have the same sign flavor as the quantum state function has.
A down quark can become an up-quark by absorbing a $W_+$ boson of the proper color.

## 6.1 Interaction

The conventional way of treating interactions in quantum field theory is to apply a methodology called covariant derivation. This methodology works well in a complex representation, but fails in a quaternionic approach. With other words it works in special one-dimensional cases but it is not well suited for multidimensional cases. So, the HBM must find another approach in order to implement interactions. The HBM also does not consider the existence of virtual particles. However, this is compensated by the availability of primary QPAD's.

Free QPAD's are waves. If these waves have the same sign flavor as the quantum state function of an elementary particle, then they will couple to the elementary particle. In this way it can also couple particles together. Thus photons and gluons can glue appropriate particles together.

A similar mechanism can be implemented by elementary particles that have the proper coupling capabilities. These are multicolor, W and Z bosons.

## 6.2 A quaternionic theory of general relativity

A special quaternion based equivalent of general relativity theory (GRT) applied to the shared affine parameter space of QPAD's Palestra can be put together. Some of the ingredients for that theory are present in the appendix chapters 13 and 14. A quaternionic version of GRT will offer similar results as conventional GRT does. In the HBM this is not worked out in detail because the quaternionic GRT only gives information on curvature and does not give detailed information on how higher level coupling works.

## 6.3 The Kerr-Newman equation

The Kerr-Newman equation is NOT part of the HBM. The Kerr-Newman equation is a solution of conventional General Relativity Theory. It uses spherical coordinates as a reference frame. The Kerr-Newman equation describes the effects of physical fields on curvature for elementary particles as well as for black holes

The Kerr-Newman metric equation gives only a rough impression on how the primary couplings influence curvature. This equation uses parameters that are comparable to the properties of primary couplings. The equation neglects the geometry of these

couplings and the resulting flow of space patches. For that reason it can only deliver a coarse approximation. For that reason the HBM only uses the Kerr-Newman equation in order to get a coarse idea of how GRT curvature relates to the curvature that is caused by elementary couplings.

## 6.4 Role of second category physical fields

The properties that characterize primary couplings act as sources/drains of second category physical fields

The primary couplings are responsible for affecting the local curvature, but it looks as if the second category physical fields have this role

**This is a false impression!**

## 6.5 Higgs

This HBM does not include a Higgs particle or a Higgs field. However, the Hilbert Book model uses a background field that acts as a partner in the coupling to the quantum state function of an elementary particle. That background field takes the role of the Higgs field. It implements inertia. However, the coupling not only explains the existence of inertia, it also explains the existence of curvature.

### 6.5.1 Inertia versus curvature

The paper of Dennis Sciama with title “On the origin of inertia” is far better understandable when the fields that he considers are taken to be members of QPAD’s. In that case the fields relate directly to quantum state functions of (massive) particles. The background field is then better understandable as local superposition of the tails of the quantum state functions of distant particles.

In this way the explanation of inertia uses an integral balance equation. In contrast, the explanation of curvature in the HBM uses a differential balance equation. It is the elementary coupling equation.

### 6.5.2 Background field versions

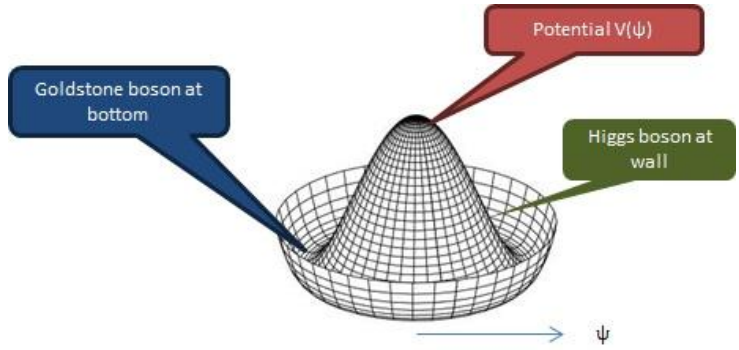
In the HBM, the background field exists in several versions. The version depends on the categories of quantum state functions that are allowed to contribute to the local superposition. Fermions all couple to the same version of background field. This version is isotropic. Anti-fermions couple to the conjugate. Bosons couple to anisotropic versions of the background field.

### 6.5.3 The Higgs mechanism

In complex quantum physics the Higgs mechanism is required in in order to explain inertia. It uses the fact that the complex quantum state function leaves the freedom of an arbitrary complex phase factor that has the form $e^{i\alpha}$. A covariant derivative in a gauge transformation may use that phase factor.

The Higgs mechanism differs considerable from the HBM explanation of inertia. It uses a potential that poses a spontaneous symmetry breaking[78]. The considered system can continuously move in a region that corresponds to its ground energy. This movement does not consume energy. Outside that region the movement costs energy and the system is forced back to its ground state. As an example, a potential $V(\psi)$ in the form of a Mexican hat can do this. It represents a U(1) symmetry. Be aware, the parameter is $\psi$ and not a spatial location.

[78] http://en.wikipedia.org/wiki/Spontaneous_symmetry_breaking




This situation gives rise to massless particles that are called Goldstone bosons. They can perform movements that cross low gradients and therefore use low energy. In complex number based quantum physics the corresponding quantum state function can be described by

$$\psi(\rho, \theta) = \rho(f)\, e^{i\theta} \tag{1}$$

In the example $f$ represents the values of $\psi$ that represent the ground energy in $V(\psi)$. At the same time $\rho(f)$ represents a Higgs boson and when $\theta$ is varied uniformly over space then $e^{i\theta}$ represents a Goldstone boson. The symmetry that is represented by potential $V(\psi)$ is spontaneously broken.

If $\theta$ is varied non-uniformly, then the derivative of the quantum state function equals

$$\partial_\mu \psi = \left(\partial_\mu \rho + i\rho\partial_\mu \theta\right) e^{i\theta} \tag{2}$$

$$\partial_\mu \psi^* = \left(\partial_\mu \rho^* - i\rho^* \partial_\mu \theta\right) e^{-i\theta} \tag{3}$$

The corresponding Lagrangian depends on $\partial_\mu \theta$.

$$\mathcal{L} = \partial_\mu \psi \partial_\mu \psi^* \tag{4}$$

$$= \left(\partial_\mu \rho\right)^2 + i\partial_\mu \theta \left[\rho \partial_\mu \rho^* - \rho^* \partial_\mu \rho\right] + \rho^2 \left(\partial_\mu \theta\right)^2$$

This means that $e^{i\theta}$ does not represent a symmetry. The varying Goldstone boson can be made the subject of a covariant derivative $D_\mu \psi$. This introduces a new vector potential $A_\mu$ that can be interpreted as a gauge boson.

$$D_\mu \psi = \partial_\mu \psi - i\, A_\mu \psi \tag{5}$$

$$= (\partial_\mu \theta + A_\mu) i\rho e^{i\theta}$$

Via a gauge transformation the term between brackets can be transformed in another vector field:

$$A'_\mu = \partial_\mu \theta + A_\mu \tag{6}$$

The term $\rho(f)$ is nearly equal to $f$. The new Lagrangian is

$$\mathcal{L} = D_\mu \psi D_\mu \psi^* = f^2 (\partial_\mu \theta + A_\mu)^2 = f^2 {A'_\mu}^2 \tag{7}$$

This reveals that an extra term $f^2 {A'_\mu}^2$ adds mass to the Lagrangian. This replacement removes $\theta$ from the Lagrangian and exchanges the Goldstone boson against a vector potential $A'_\mu$ and a representative of mass $f$.

The Lagrangian $\mathcal{L}_A$ of $A_\mu$ is not affected by the gauge transformation.

$$\mathcal{L}_A = F^{\mu\nu} F_{\mu\nu} \tag{8}$$

$$F^{\mu\nu} = \partial_\mu A^\nu - \partial^\nu A_\mu$$

In quaternionic quantum physics the complex number based strategy that uses a gauge transformation does not work properly. In quaternion based quantum physics the imaginary base number $i$ has a spatial direction and in general for quaternionic distributions $f$ and $g$ holds[79]

$$\nabla(f\ g) \neq\ f\ \nabla\ g\ +\ (\nabla f)\ g \tag{9}$$

Thus in quaternionic quantum physics the Higgs mechanism only works in one dimensional situations. However, in quaternionic quantum physics the quantum state function already contains a (real) scalar field and an extra vector field in the form of its real and imaginary parts. As a consequence the included fields can be superposed. Further, it is possible to add a free QPAD to the balance equation. Via these fields, quaternion quantum physics has its own method for applying mass to the particles. That method explains inertia as well as the influence of particles on the local curvature.

[79] Appendix; Covariant derivative

# 7 HADRONS

The HBM cannot discern generations of elementary particles. For that reason the elementary fermions are treated per category.

Symbol e or $e_-$ means electron

Symbol p or $e_+$ means positron

Symbol n means neutrino

Symbol d means down-quark

Symbol u means up-quark. But in the HBM up-quarks are no elementary particles.

## 7.1 Second level coupling

This chapter treats the second level of coupling. It treats couplings between elementary particles. This coupling uses the second category physical fields that are generated by the properties of the first level coupling. The Hilbert Book Model delivers the reason of existence of these properties; coupling factor, electric charge, angular momentum (spin) and possibly color charge of elementary particles. In higher level couplings these properties are conserved. Higher level coupling might also use what is left of primary QPAD's after primary coupling.

The primary couplings influence the local curvature. However, the primary QPAD's are not observable. Only the second category physical fields become observable. Thus, it looks as if the second category physical fields or their sources/drains are responsible for affecting curvature. This, of course, is a false impression.

The currents in the coupled primary QPAD's that remain after coupling may still influence higher level coupling.

The curvature is also a binding ingredient for the next levels of coupling. The formulas that describe the influence of the conserved properties on the curvature inform what will happen. Currently the best available formula is the Kerr-Newman metric formula, which is taken from GRT. The Kerr-Newman equation works on the base of locally existent properties (the sources or drains of the second category physical fields). It must be reckoned that the Kerr-newman equation gives only a rough indication of what happens. In order to know what happens in detail the full Quantum Fluid Dynamic situation must be analyzed.

At a fixed instant of proper time, thus inside a single page of the Hilbert Book Model, the Kerr-Newman equation is a static equation.

If in a subsequent HBM page the controlling properties have changed, then those new properties define the new configuration.

The first level coupling that constitutes elementary fermions uses a background field that is the partner in the coupling of that background field to the state function of the elementary fermion.

The background field represents the superposition of the tails of the state functions of all massive particles that exist in the universe. In this way inertia gets its implementation.

Free QPAD's and fitting bosons act in interactions. They glue particles together.

If particles couple, then their background QPAD's will superpose. Thus their inertia will combine.

Hadrons are the first of the next levels of binding products.

The Kerr-Newman equation shows an abnormality at the place where black holes get their horizon. Whether or not a hadron possesses a horizon is in this respect unimportant. The properties of the elementary particles that are bound together in order to form the hadron are sources of second category physical fields. The static versions of these second category physical fields reach beyond a possible horizon. To the outside world the superposition of these fields signal the properties of the hadron.

The primary QPAD's that constitute the elementary particles and that on their turn constitute the hadron also reach beyond the potential horizon.

## 7.2 Rules

The second level of coupling has its own set of rules.

- The total color of the composite must be white[80].
- The properties of the constituting particles will be conserved.
- However, mass (coupling factor) may be exchanged against field energy.
- Field energy is transported via oscillating QPAD's.
- Like the fermions, hadrons exist in generations.

[80] In observable particles, which are particles that may be detected in measuring machines like the LHC, the color is always white.

The properties of the elementary particles, including color, play a significant role in the Pauli principle. (*The HBM does not explain the Pauli principle*).

Coupling factor, electric charge, angular momentum (spin) and may be color charge are sources of second category physical fields.

Location (position or momentum) are not sources of second category physical fields.

## 7.3 Up-quarks

In the HBM the up-quarks cannot be constructed from a primary coupling of QPAD sign flavors. Therefore the HBM does not consider them to be elementary particles. Instead up-quarks are composed of down-quarks or XX particles and W+ particles or from down-quarks or XX particles, positrons and neutrinos. May be YY particles or ZZ particles are also involved. Further, the up-quarks annihilate into these constituents.

Like down-quarks the up-quarks have color. This attribute relates to the direction of the reflection of the constituting down-quark.

Apart from up-quarks also anti-up-quarks exist.

## 7.4 Mesons

Mesons are composed out of quarks and anti-quarks. The following meson categories exist.

- $(d\underline{d}+u\underline{u})/\sqrt{2}$
- $(d\underline{d}-u\underline{u})/\sqrt{2}$
- $u\underline{d}$
- $d\underline{u}$

The mixed color elementary particles resemble mesons.

## 7.5 Baryons

Baryons are composed out of triples of quarks. The following baryons exist[81].

| Symbol | configuration | name | Isospin $I_3$ |
|---|---|---|---|
| $\Delta^{--}$ | ddd | delta | $-3/2$ |
| $\Delta^{-}$ | ddu | neutron | $-1/2$ |
| $\Delta^{+}$ | uud | proton | $1/2$ |
| $\Delta^{++}$ | uuu | delta | $3/2$ |

For the anti-baryons the possibilities are:

- $\underline{ddd}$
- $\underline{ddu}$
- $\underline{uud}$
- $\underline{uuu}$

I relation to the Pauli principle the versions with multiple u or multiple d can only exist due to the fact that the constituting quarks have different spin and/or different color charge.

[81] The generations are ignored!

# 8 THE BUILDING

The building consists of everything that can be assembled from elementary particles, hadrons and fields.

## 8.1 Natures Music

In atoms electrons oscillate around a central point that acts as the location of the atom[82]. In molecules the same features occur in an even more complicated configuration.

The oscillations are harmonic. The most basic harmonic oscillations are, apart from a scalar factor, invariant under Fourier transformation[83]. These oscillations have modes and usually several of these modes exist in parallel. These modes can be generated and annihilated. Generation goes together with the absorption of a more elementary particle or elementary wave and annihilation goes together with a corresponding emission.

## 8.2 Hydrogen atom

In the hydrogen atom[84] one electron encircles the nucleus. The oscillation of the electron can be described as a quantum spherical harmonic oscillation. It can have different oscillation modes. These modes are characterized by quantum numbers. Mode switching is activated by creation and annihilation operators. The speed of the electrons is high enough such that relativity effects must be considered. Also the spin of the electron plays a role and causes magnetic effects.

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[82] Appendix; Oscillations

[83] Appendix; Functions invariant under Fourier transformation.

[84] http://en.wikipedia.org/wiki/Hydrogen_atom

## 8.3 Helium atom

In the helium atom[85] two electrons encircle the nucleus. In principle the electrons behave similarly as in the hydrogen atom, however due to the Pauli principle they cannot both occupy the same oscillation mode. The electrons not only interact with the nucleus, but they also interact with each other.

## 8.4 Modularization

Hadrons, atoms and molecules are products of a modularization process.

Modularization[86] encapsulates properties in a higher order individual and renders the resulting individual a specific behavior. Its main purpose is that the number of relations to the outside world is reduced. Usually the module can be accessed via a series of well-defined interfaces. The whole keeps the integrity of the individual intact.

Modularized systems are far simpler than their monolithic equivalents. Modularization exploits reuse. The modularization can have far reaching consequences. That is especially the case when modules can be used to create a new kind of modules. In this way nature is capable of constructing very complicated systems. On earth nature achieved the stage to be able to generate intelligent species. This tendency goes straightly against the tendency that is set by the second law of thermodynamics. These laws prescribe that disorganization, randomness and chaos will increase continuously.

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[85] http://en.wikipedia.org/wiki/Helium_atom

[86] http://vixra.org/abs/1101.0064

## 8.5 Black hole

### 8.5.1 Classical black hole

According to classical mechanics the no-hair theorem[87] states that, once a black hole achieves a stable condition after formation, it has only three independent physical properties:

- mass,
- charge, and
- angular momentum.

The surface gravity[88] $\kappa$ may be calculated directly from Newton's Law of Gravitation[89], which gives the formula

$$\kappa = \frac{Gm}{r^2} \tag{2}$$

where m is the mass of the object, r is its radius, and G is the gravitational constant[90]. If we let $\rho = m/V$ denote the mean density of the object, we can also write this as

$$\kappa = \frac{4\pi}{3} G\rho r \tag{3}$$

*For fixed mean density $\rho$, the surface gravity $\kappa$ is proportional to the radius $r$.*

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[87] http://en.wikipedia.org/wiki/No-hair_theorem

[88] http://en.wikipedia.org/wiki/Surface_gravity

[89] http://en.wikipedia.org/wiki/Newton%27s_Law_of_Gravitation

[90] http://en.wikipedia.org/wiki/Gravitational_constant

Sciama[91] relates $G$ to the potential that is raised by the community of particles. For fixed mean density $\rho$ this is shown by

$$\Phi = -\int_V \frac{\rho}{r} dV = -\rho \int_V \frac{dV}{r} = \rho 2\pi R^2 \tag{4}$$

$$G \approx \frac{-c^2}{\Phi} = \frac{-c^2}{\rho 2\pi R^2} \tag{5}$$

Here $R$ is the current radius of the universe.

### 8.5.2 Simple black hole

The Schwarzschild radius $r_S$ for a non-rotating spherical black hole is

$$r_S = \frac{2Gm}{c^2} \tag{1}$$

### 8.5.3 General black hole

More generally holds

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \phi dQ \tag{1}$$

where

- $M$ is the mass/energy,
- $A$ is the horizon area,
- $\Omega$ is the angular velocity,

[91] Influence;Inertia

- $J$ is the angular momentum,
- $\phi$ is the electrostatic potential,
- $\kappa$ is the surface gravity,
- $Q$ is the electric charge.

For a stationary black hole, the horizon has constant surface gravity.

It is not possible to form a black hole with surface gravity. $\kappa = 0$.

### 8.5.4 Quantum black hole

When quantum mechanical effects are taken into account, one finds that black holes emit thermal radiation (Hawking radiation) at temperature

$$T_H = \frac{\kappa}{2\pi} \tag{1}$$

A quantum black hole is characterized by an entropy $S$ and an area $A$.

The entropy of a black hole is given by the equation:

$$S = \frac{c^3 kA}{4\hbar G} \tag{2}$$

The Bekenstein-Hawking Entropy of three-dimensional black holes exactly saturates the bound

$$S = \frac{kA_P}{4} \tag{3}$$

where $A_P$ is the two-dimensional area of the black hole's event horizon in units of the Planck area,

$$A_P = l_p^2 = \frac{\hbar G}{c^3}. \tag{4}$$

In the Hilbert book model this equals the number of granules that covers the horizon of the black hole.

The horizon of the black hole is an event horizon because information cannot pass this horizon. (Near the horizon the speed of light goes to zero.)

### 8.5.5 Holographic principle

The holographic principle[92] states that the entropy contained in a closed surface in space equals the entropy of a black hole that has absorbed everything that is contained in this enclosure.

In the Hilbert book model it means that if the surface is considered as a sparsely covered horizon, then that sparse horizon contains as many granules as the densely covered horizon of the corresponding black hole.

It also means that the maximum entropy that can be contained inside a surface corresponds to a dense coverage with granules of that surface.

In this model, any dense or sparse horizon reflects via its contained entropy the number of granules that are contained in the corresponding volume.

We might extend this picture by stating that the number of granules in a volume corresponds with the entropy in the volume. In the Hilbert book model the number of granules corresponds to the

[92] http://en.wikipedia.org/wiki/Holographic_principle

number of Hilbert vectors that are attached to a QPAD. It also corresponds to the number of anchor points of the primary physical fields.

The eigenvectors of the particle locator operator $\mathfrak{S}$ correspond to quantum logical propositions that represent the location of physical particles. These propositions have a binary yes/no value. In the extended model these propositions get extra content via the attached QPAD's.

### 8.5.6 Black hole as a subspace of the Hilbert space

The coupling between the eigenvectors of the particle locator operator and the Palestra is not precise. It is stochastic and its minimal size is of the order of the Planck-length. This granularity is due to the fact that the Hilbert space is separable and as a consequence the set of eigenvalues is countable. That is why the granules in the surface of a black hole have this size. The surface of a black hole is an image of the unit sphere of a subspace of the Hilbert space where the eigenvectors of the particle location operator form an orthogonal base and all attached QPAD's pack the eigenvalues together in an optimal way.

The eigenvectors of the particle locator operator all touch such a granule. The relation with quantum logic means that the Hilbert vector stands for a proposition that has a yes/no value. In case of the Hilbert vectors that are attached to the granules the yes/no value represents group membership. Thus each granule represents a bit of information.

For the eigenvectors of the particle locator operator a densest packaging exists. It means that in that condition the QPAD's have shrunk to their smallest possible location difference.

Assumption 1: *In that condition, due to the properties of the QPAD's, the mutual tension works asymmetrically*.

This asymmetry means that in a surface that is formed by a set of densely packed granules the tension on one side is stronger than the surface tension at the other side. As a consequence the final configuration of a densest packaging becomes an empty bubble.

### 8.5.7 HBM interpretation of black hole

Three dimensional granular structures do not occur in nature other than in composites that have a regular structure. A regular granular structure of space would immediately cause anisotropy of that space. Still space includes two dimensional regular granular structures. They form the surfaces of black holes. They also represent the most densely form of packaging of entropy.

The surface patches can be seen as to represent Hilbert eigenvectors of the particle location operator. For these eigenvectors the corresponding eigenvalues are packed densely together into a surface. The packed eigenvectors all carry the properties of their mutual coupling. (mass, electric charge, spin). Together these data form the properties of the BH.

The coupling of the QPAD's that are attached to the packed eigenvectors is a special type that is not treated before. Its characteristics are such that the coupled particles form a stable bubble shaped body. This body can still absorb extra particles.

Black holes feature seemingly contradictory properties. On the one hand information cannot pass their event horizon. On the other hand it is possible to follow their history from occurrences in their environment. This history follows from the values of their properties. These properties are represented by second category physical

fields. Thus the second category physical fields CAN pass event horizons, while first category physical fields are blocked. This indicates that the two categories are fundamentally different.

### 8.5.8 Chandrasekhar limit

The Chandrasekhar limit[93] is an upper bound on the mass of a stable white dwarf star:

$$M_{limit} = \frac{\omega_3^0 \sqrt{3\pi}}{2} \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{(\mu_e m_H)^2} \quad (1)$$

where:

- $\hbar$ is the reduced Planck constant
- $c$ is the speed of light
- $G$ is the gravitational constant
- $\mu_e$ is the average molecular weight per electron, which depends upon the chemical composition of the star.
- $m_H$ is the mass of the hydrogen atom.
- $\omega_3^0 \approx 2.018236$ is a constant connected with the solution to the Lane-Emden equation.

Approximately:

$$M_{limit} \propto \frac{M_P^3}{m_H^2}. \quad (2)$$

[93] http://en.wikipedia.org/wiki/Chandrasekhar_limit

Where

$$M_P = \sqrt{\hbar c / G} \text{ is the Planck mass} \tag{3}$$

### 8.5.9 Similarity between black hole and massive fermion

According to the no hair theorem a black hole shows only a few properties to the outside world. These properties are sources of second category physical fields.

Massive fermions have a similar behavior.

Apart from the exposed features it is impossible to observe what goes on inside the black hole.

It is possible that the internals of a black hole are to a certain extent similar to the construction of a massive fermion. It would mean that the BH is based on a set of QPAD's that each are coupled to a background QPAD.

## 8.6 Birth of the universe

For the eigenvectors of the particle locator operator a densest packaging exists. It means that in that condition the QPAD's have shrunk to their smallest possible location difference.

Assumption 1: *In that condition, due to the properties of the QPAD's, the mutual tension works asymmetrically*.

This asymmetry means that in a surface that is formed by a set of densely packed granules the tension on one side is stronger than the surface tension at the other side. As a consequence the final configuration of a densest packaging becomes an empty bubble.

In the starting condition of the universe all eigenvectors of the particle locator operator are densely packed in one assembly.

Assumption 2: *At an instant after that moment the packaging density suddenly relaxes*.

Even when the chance is very low, which means that the start bubble is very stable, given enough time the occasion will certainly happen. (It means that black holes may collapse.)

The number of granules does not change. Thus, during this spreading the total entropy does not change.

The package may fall apart in several separated subassemblies and a large series of single or more loosely packed granules. For the single and the more loosely packed granules the corresponding QPAD's fold out. The densely packed subassemblies take again a bubble shape.

This process may occur instantly or gradually, but most probably it will be done in a sequence of these two possibilities.

First occurs a sudden change of scale between the particle locator operator in the separable Hilbert space **Ң** and the GPS operator that delivers the background coordinate system and that resides in the rigged Hilbert space **Ħ**. It is possible that originally the bubble covered the whole of the unit sphere of the Hilbert space **Ң,** or it may just cover a finite dimensional subspace of **Ң**. This means that the bubble contains an infinite or a finite amount of granules, which suddenly get diffused in a much larger space. That space is affine like the unit sphere of the Hilbert space **Ң**. The diffusion takes

place at every occupied location in the background coordinate system.

This kind of universe has no spatial origin or it must be the center of the outer horizon. With the aid of the background coordinate system, it will be possible to indicate a center of that universe. Each item in this universe has its own private information horizon. This horizon is set by the reach of the light that has been travelling since the birth of the universe. As long as this light does not reach the outer horizon that sub-universe looks isotropic. A multitude of such sub-universes exist that need not overlap. However, they all look at their border at an image of part of the start horizon. Such, sub-universes obey the cosmological principle[94].

In the next phase the further expansion occurs gradually. Because the QPAD's that are attached to the granules install a tendency for the granules to stay together, a different motor must be present behind this expansion. This motor can be found in the fact that with increasing radius the number of pulling granules grows faster than the decrease of the forces that are executed by the fields of these granules that is caused by the increasing distance. In an affine space this is always and everywhere true. This effect is also the source of inertia.

Due to local attraction, loosely packed and single granules may reassemble in bubble shaped subassemblies. These subassemblies are known as black holes. Single granules and small aggregates of granules are known as elementary particles, nuclei or atoms.

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[94] http://en.wikipedia.org/wiki/Cosmological_principle

Much larger aggregates may be formed as well but these are not densely packed. Elementary particles can be categorized according to the configuration of their constituting QPAD's. The coupled QPAD's determine how massive the particle is.

Inside the bubble the fact that the granule represents matter is not recognizable. It is only recognizable when the attached QPAD gets the chance to unfold. That condition is true when the granule is not part of a densely packed subassembly.

The requirements for the birth of the universe are:

1. The existence of a particle locator operator
2. The existence of QPAD's that install the tendency to keep these eigenvectors of the particle locator operator together
3. When the large numbers of eigenvectors are densely packed, then the assembly forms a bubble, because due to the properties of the QPAD's, the mutual tension works asymmetrically
4. In advance the eigenvectors of the particle locator operator are densely packed in one bubble.
5. A non-zero probability exists that the package density will be relaxed and the package falls apart. This may happen in a two stage process
    a. A sudden reduction of scale occurs
    b. Next a force that pulls the granules further away from each other exists

In the first episode of the universe the sudden scale change took place. This ripped the original bubble apart. Next a gradual further expansion took place.

The granules that move freely can at the utmost take one space step at every progression step. When the ratio of the space step and the progression step is fixed, then this determines a maximum speed of granules. A certain type of granules takes a space step at every progression step. That type transports information at the maximum possible speed.

When the path of these information transmitting particles is a straight line, then after a while, the other types of granules no longer get messages from the birth episode of the universe. But this need not be the case.

Since the messenger has a finite speed, it brings information from the past. First of all the speedy messenger and the slow addressee may have started from different locations. Further, due to curvature of space the path of the speedy messenger may take much longer than the duration of the much straighter path that the much slower addressee has taken. The information about the past that is included in the message might be close to the episode in which the granules were combined in one large bubble.

Thus despite the fact that most of the information that is generated during the birth of the universe is long gone, still some of that information may reach particles long after the instance of birth. When this information is interpreted it gives the impression of a metric expansion of the universe[95].

[95] http://en.wikipedia.org/wiki/Metric_expansion_of_space

# 9 COSMOLOGY

Cosmology concerns all particles with a mass above the limiting mass $\mathfrak{M}$.

## 9.1 Higher order couplings

It is assumed that during higher order couplings the constituting elementary particles keep their basic properties;

- coupling factor,
- electric charge
- angular momentum
- color charge.

The properties that characterize the coupling event in elementary particles are sources of second category physical fields. These fields are known as physical fields. For example the electric charge is a source for electromagnetic fields. The coupling factor is a source for the gravitation field. The spin also causes a field.

A secondary field is a specialized QPAD that has one of the properties of the elementary particle as its isolated source. The second category physical fields may still overlap (superpose).

These second category physical fields play a major role in the higher order couplings. The reason for this fact is that the sources of the second category physical fields influence the curvature of the parameter space. In fact this is a misconception, because the real cause of curvature is formed by the primary couplings.

In this respect a more complicated effect also plays a role. The currents of the primary QPAD's that result after the primary cou-

pling may also influence the higher order coupling. This effect is not covered by formulas like the Kerr-Newman metric formula.

The composite particles can be considered to have state functions that are formed by the superposition of the state functions of the constituting particles. However, at least part of these particles consist of coupled pairs of QPAD's in which one is a state function QPAD and the other is a background QPAD. With other words, a composite particle is a coupling between a superposition of a number of state function QPAD's and a superposition of a number of background QPAD's. Not only the charge density distributions superpose, but also the current density distributions superpose. Thus the flows intermix.

It means that the state function of the composite is a superposition of a set of QPAD's that have different sign flavors. However, the same holds for the superposition of the background QPAD's.

This fact would mean that higher order coupling is not well described by simple wave equations as those that describe elementary particles. Instead it may be better described by an equation that describes the dependence of the local curvature on the locally existing coupling properties.

The Kerr-Newman equation produces an abnormality when the limit[96]

$$m \geq \sqrt{r_Q^2 + (S/cm)^2}$$

[96] Appendix; Kerr-Newton limit

is passed. Above that limit the particle is encapsulated and guarded by a skin in the form of a horizon. Below that limit the enclosed particles are naked.

## 9.2 Curvature

### 9.2.1 Hilbert Book Model ingredients

Each page of the Hilbert Book Model consists of three quite independent ingredients.

> Ingredient 1: The quantum logic, or equivalently, its lattice isomorphic companion; the set of closed subspaces of an infinite dimensional separable Hilbert space
>
> Ingredient 2: A background coordinate system, called Palestra that is taken from the continuum eigenspace of an operator that resides in the Gelfand triple of the separable Hilbert space.
>
> Ingredient 3: A set of QPAD's that each couple an eigenvector of a particle locator operator that resides in the Hilbert space to a value in the Palestra.

Couplings between QPAD's that lead to elementary particles are characterized by four categories of properties:

- Coupling factor
- Electric charge
- Angular momentum
- Color charge

These primary couplings influence the curvature in the second ingredient that affects the third ingredient. It looks as if physical fields in the form of second category physical fields that relate to the primary couplings influence the curvature. This is a misconception. However, this view can deliver a coarse indication of what happens. The way that these properties influence curvature is described by metric equations, such as the Kerr-Newman metric formula.

The first category physical fields are restricted by the maximum speed at which information can be transported. The influence of primary couplings on curvature seems not to be restricted by this limit. Second category physical fields are not restricted by the maximum speed at which information can be transported

For example the properties (mass, electric charge, spin and surface area) of a black hole change in an observable way when matter is absorbed while in the neighborhood of the event horizon the transport speed of information by first category physical fields is known to reduce to zero.

### 9.2.2 Coordinate system

The coordinate system that is taken from the eigenspace of an operator that resides in the Gelfand triple is not applied directly. Instead a quaternionic distribution that uses the values of the flat coordinate system that is taken from the Gelfand triple as its parameters is used as the observed coordinate system.

Curvature can be described by the combination of a preselected coordinate system that defines location in a non-curved space and a local metric that describes the curvature in terms of that coordinate system. As is described above, both the flat and the curved coordinate system are based on eigenspaces of corresponding operators that reside in the Gelfand triple.

Several coordinate systems are possible. The most common coordinate systems for a non-curved three dimensional space are:

- Cartesian coordinates
- Spherical coordinates

Alternatives for spherical coordinates are:

- Schwarzschild coordinates[97]
- Kruskal-Szekeres coordinates[98]
- Lemaitre coordinates[99]
- Eddington–Finkelstein coordinates[100]

The advantage of the alternative coordinates is that they avoid unnecessary singularities. However, these alternatives are only relevant for situations in which the Schwarzschild radius plays a significant role. This is certainly the case for black holes and their environment, but it becomes irrelevant in the realm of some elementary particles.

### 9.2.3 Metric

The currently best suitable local metric equation for our purposes is the Kerr-Newman metric[101]. It uses three local properties. These properties are:

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[97] http://en.wikipedia.org/wiki/Schwarzschild_coordinates

[98] http://en.wikipedia.org/wiki/Kruskal-Szekeres_coordinates

[99] http://en.wikipedia.org/wiki/Lemaitre_coordinates

[100] http://en.wikipedia.org/wiki/Eddington%E2%80%93Finkelstein_coordinates

[101] Appendix;Metric tensor field;Local metric equation

- The coupling factor $m$
- The electric charge $Q$
- The angular momentum $J$

The angular momentum $J$ includes the spin $s$.

As stated before the Kerr-Newman can only give a coarse approximation because in fact the primary couplings determine the local situation and the properties represent only abstractions of these primary couplings.

This Kerr-Newman metric uses the sum of a category of properties that are collected within the observed sphere. However, in principle the summation produces different centers of activity for different property categories. Thus, these centers need not be at the same location. However, for large enough selected radius $r$ and applied to black holes or single particles, these centers coincide.

The simplest interpretation of the Kerr-Newman metric can be taken on the surface of a sphere that has a selected radius $r$.

The formula uses three characteristic radii. The largest characteristic radius plays the most prominent role.

This fact introduces the notion of geo-cavity.

### 9.2.4 Scales

The charge-to-mass ratio $Q/M$ is typically larger in smaller systems[102]. For most astrophysical systems,

$$Q/M \ll 1, \tag{1}$$

while for a Millikan oil drop,

$$Q/M \approx 10^{6}. \tag{2}$$

Going all the way down to elementary particles, the value for the electron is

$$Q/M \approx 10^{21}. \tag{3}$$

To achieve balance we require that Newton's gravitational force $f_N$ has the same magnitude as Coulomb's force $f_C$, that is

$$|f_N| = |f_C| \tag{4}$$

To be more specific, let us assume that $Q = e$ where $e$ is the elementary charge. We then adjust the mass $M$ to the value for which the forces are balanced. This gives the Stoney mass[103]

$$M = m_S = \frac{e}{\sqrt{G4\pi\epsilon_0}} = 1.85921 \times 10^{-9}\ \text{kg} \approx 2\ \mu g \tag{5}$$

It is only one order of magnitude lower than the Planck mass

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[102] For deeper investigation, see: http://arxiv.org/abs/0802.2914

[103] http://en.wikipedia.org/wiki/Natural_units

$$m_P = \sqrt{\hbar c/G} = 2.17644(11) \times 10^{-8}\ \text{kg} \approx 20\ \mu g \tag{6}$$

The ratio between them is given by the square root of the fine structure constant,

$$\alpha = \frac{e^2}{c\hbar 4\pi\epsilon_0} = (137.035999679)^{-1} \tag{7}$$

$$\frac{m_S}{m_P} = \sqrt{\alpha} \approx 0.1 \tag{8}$$

Thus, in case of electric charges, the Coulomb forces are nearly in balance with the gravitational forces at the Planck scale. However, at subatomic scale this picture is disturbed by the spin.

For subatomic systems there is an additional phenomenon which comes into play. In fact, according to general relativity, the gravitational field tends to become dominated by the spin at distances of the order of the Compton wavelength. The relevant quantity which governs this behavior is the ratio $S/M^2$ where $S$ is the (spin) angular momentum. For an electron,

$$S/M^2 \approx 10^{44}. \tag{9}$$

As a consequence, the gravitational field becomes dominated by gravitomagnetic effects in the subatomic domain. This fact has important consequences for the electromagnetic fields of spinning charged particles.

The four known gravitational and electromagnetic multi-pole moments of the electron are:

- the mass $m_e$,
- the spin $S_e = \hbar/2$,
- the charge $e$
- the magnetic moment $\mu = \frac{eS_e}{m_e}$

The spin is a gravitomagnetic dipole moment, i.e. a gravitational analogue of the magnetic dipole moment.

$$\frac{S_e}{m_e} \gg e \gg m_e \tag{10}$$

The corresponding Kerr-Newman field is therefore dominated by the spin in the subatomic domain. In particular, it has no event horizon and it has no ergo-region. (The ergo-region is a region of space-time located outside the event horizon of a rotating black hole where no object even if traveling at the speed of light, can remain stationary.)

An important conclusion is that gravity tends to become spin dominated in the subatomic domain.

The Kerr-Newman metric formula indicates that small particles that are encapsulated by a horizon are restricted by the limit:

$$m \geq \sqrt{e^2 + (S_e/m)^2} \tag{11}$$

Where m is the particle mass, $e$ is the elementary charge and $S_e$ is the elementary spin.

## 9.3 Inside black holes

Objects that fulfill the rules for the existence of a BH horizon hide their internals. Their virtual construction is similar to that of a massive elementary particle. That means that nothing is inside that horizon than a set of coupled QPAD's. One of these is a background QPAD. The others form a superposition. This superposition is a kind of super QPAD. It lets the BH act as one particle that has the properties of the combination of the gathered fields.

## 9.4 Hadrons

The Hilbert Book Model delivers the reason of existence of the properties; coupling factor, electric charge and angular momentum (spin) of elementary particles. In higher level couplings these properties are conserved. These properties influence the local curvature. The curvature is the binding ingredient for this next level. The formulas that describe the influence of the conserved properties on the curvature control what is happening. Currently the best available formula is the Kerr-Newman metric formula.

This story does not include a Higgs particle or Higgs fields. However, the Hilbert Book model uses a background field that is one partner in the coupling of that background field to the state function of an elementary fermion. The background field represents the superposition of the tails of the state functions of all massive particles that exist in the universe. In this way inertia gets its implementation. The coupling event is characterized by a set of properties. These are the mentioned properties of the elementary particles.

Hadrons are the first of the next levels of binding products.

The Kerr-Newman equation shows an abnormality at the place where black holes get their horizon. Whether or not a hadron pos-

sesses a horizon is in this respect unimportant. The properties of the elementary particles that are bound together are sources of second category physical fields. These fields reach beyond a possible horizon. To the outside world the superposition of these fields signal the properties of the hadron.

# 10 CONCLUSION

It is quite possible to build a model of physics on a solid axiomatic foundation. It has the advantage that from the beginning the model stays consistent and trustworthy. For the Hilbert Book Model this inroad has brought some rather revolutionary deviations from contemporary physics. The way that fields are treated and how dynamics is implemented differs strongly from the ordinary course of physics. Through the switch from complex Hilbert spaces to quaternionic Hilbert spaces and the attention that is given to sign flavors of quaternionic probability amplitude distributions it becomes possible to derive unique continuity equations rather than equations of motion. The elementary coupling equations reveal the properties and habits of all known elementary particles.

This step only reaches to the first level of binding. The properties of the coupling that occurs inside elementary particles form the factors that influence the local curvature. The current status of the model already indicates that the next level of particle binding will use the effects of the coupling properties on the curvature of the local geometry besides the coupling of sign flavors of quaternionic probability amplitude distributions. It means that in the higher level binding for a coarse view the role of the metric equation will be greater than the role of the wave equation. This step is rather deterministic, while the first level of coupling is afflicted with indeterminism.

# PART III

# Appendix

The appendix is a toolbox and a grab bag that contains everything that the author has collected that can be used to build or analyze the Hilbert Book Model

# 1 Logic

## 1.1 History of quantum logic

Around 1930 John von Neumann and Garrett Birkhoff were searching for an acceptable explanation of the results of experiments that showed that the execution of an observation of a very small object can completely destroy the validity of an earlier observation of another observable of that object. The Schrödinger equation that agreed with the dynamic behaviour of the particles already existed. Not much later Heisenberg's matrix formulation became popular as well. Quite soon the conclusion was made that something was fundamentally wrong with the logic behind the behaviour of small particles. These small objects show particle behaviour as well as wave behaviour and they show quantization effects. It was found that the distribution axiom of classical logic had to be changed. Soon it became apparent that the lattice structure of classical logic must be weakened from an ortho-complementary modular form to an ortho-complementary weakly modular lattice. The quantum logic was born. The next step was to find a useful mathematical presentation of this new logic. A historic review of what happened can be found in: "Quantum Theory: von Neumann" vs. Dirac; http://www.illc.uva.nl/~seop/entries/qt-nvd/. It includes extensions of the concept of Hilbert space and application of these concepts to quantum field theory. Another source is: http://www.quantonics.com/Foulis_On_Quantum_Logic.html.

## 1.2 Quantum logic

Elementary particles behave non-classical. They can present themselves either as a particle or as a wave. A measurement of the particle properties of the object destroys the information that was

obtained from an earlier measurement of the wave properties of that object.

With elementary particles it becomes clear that that nature obeys a different logic than our old trusted classical logic. The difference resides in the modularity axiom. That axiom is weakened. The classical logic is congruent to an orthocomplemented modular lattice. The quantum logic is congruent to an orthocomplemented weakly modular lattice. Another name for that lattice is orthomodular lattice.

### 1.2.1 Lattices

A subset of the axioms of the logic characterizes it as a half ordered set. A larger subset defines it as a lattice.

A lattice is a set of elements $a, b, c,$ ...that is closed for the connections $\cap$ and $\cup$. These connections obey:

- The set is partially ordered. With each pair of elements $a, b$ belongs an element $c$, such that $a \subset c$ and $b \subset c$.
- The set is a $\cap$ half lattice if with each pair of elements $a, b$ an element $c$ exists, such that $c = a \cap b$.
- The set is a $\cup$ half lattice if with each pair of elements $a, b$ an element $c$ exists, such that $c = a \cup b$.
- The set is a lattice if it is both a $\cap$ half lattice and a $\cup$ half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a \tag{1}$$

$$(a \cap b) \cap c = a \cap (b \cap c) \tag{2}$$

$$a \cap (a \cup b) = a \tag{3}$$

$$a \cup b = b \cup a \tag{4}$$

$$(a \cup b) \cup c = a \cup (b \cup c) \tag{5}$$

$$a \cup (a \cap b) = a \tag{6}$$

The lattice has a partial order inclusion $\subset$:

$$a \subset b \Leftrightarrow a \subset b = a \tag{7}$$

A complementary lattice contains two elements $n$ and $e$ with each element a an complementary element a' such that:

$$a \cap a' = n \tag{8}$$

$$a \cap n = n \tag{9}$$

$$a \cap e = a \tag{10}$$

$$a \cup a' = e \tag{11}$$

$$a \cup e = e \tag{12}$$

(13)

$$a \cup n = a$$

An orthocomplemented lattice contains two elements $n$ and $e$ and with each element $a$ an element $a''$ such that:

$$a \cup a'' = e \tag{14}$$

$$a \cap a'' = n \tag{15}$$

$$(a'')'' = a \tag{16}$$

$$a \subset b \Leftrightarrow b'' \subset a'' \tag{17}$$

$e$ is the unity element; $n$ is the null element of the lattice

A distributive lattice supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{18}$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \tag{19}$$

A modular lattice supports:

$$(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c)) \tag{20}$$

A weak modular lattice supports instead:

There exists an element $d$ such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c = a \cup (b \cap c) \cup (d \cap c) \tag{21}$$

where $d$ obeys:

$$(a \cup b) \cap d = d \tag{22}$$

$$a \cap d = n \tag{23}$$

$$b \cap d = n \tag{24}$$

$$[(a \subset g)\, and\, (b \subset g) \Leftrightarrow d \subset g \tag{25}$$

In an atomic lattice holds

$$\exists_{p \in L} \forall_{x \in L} \{x \subset p \Rightarrow x = n\} \tag{26}$$

$$\forall_{a \in L} \forall_{x \in L} \{(a < x < a \cap p) \Rightarrow (x = a\, or\, x = a \cap p)\} \tag{27}$$

$p$ is an atom

Both the set of propositions of quantum logic and the set of subspaces of a separable Hilbert space **H** have the structure of an orthomodular lattice. In this respect these sets are congruent.

In Hilbert space, an atom is a pure state (a ray spanned by a single vector).

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.

Quantum logic has the structure of an orthomodular lattice. That is an orthocomplented weakly modular and atomic lattice. The set of closed subspaces of a Hilbert space also has that structure.

### 1.2.2 Proposition

In Aristotelian logic a proposition is a particular kind of sentence, one which affirms or denies a predicate of a subject. Propositions have binary values. They are either true or they are false.

Propositions take forms like "*This is a particle or a wave*". In quantum logic "*This is a particle.*" is not a proposition.

In mathematical logic, propositions, also called "propositional formulas" or "statement forms", are statements that do not contain quantifiers. They are composed of well-formed formulas consisting entirely of atomic formulas, the five logical connectives[104], and symbols of grouping (parentheses etc.). Propositional logic is one of the few areas of math-

[104] http://en.wikipedia.org/wiki/Logical_connective

ematics that is totally solved, in the sense that it has been proven internally consistent, every theorem is true, and every true statement can be proved. Predicate logic is an extension of propositional logic, which adds variables and quantifiers.

In Hilbert space a vector is either inside or not inside a closed subspace. A proper quantum logical proposition is "*Vector |f> is inside state s*".

In Hilbert space, an atomic predicate corresponds with a subspace that is spanned be a single vector.

Predicates may accept attributes and quantifiers. The predicate logic is also called first order logic. A dynamic logic can handle the fact that predicates may influence each other when atomic predicates are exchanged.

### 1.2.3 Observation

In physics, particularly in quantum physics, a system **observable** is a property of the system state that can be determined by some sequence of physical operations. This paper distinguishes between measurements and observations.

- With an observation the state is considered as a linear combination of eigenvectors of the observable. An observation returns the statistical expectation value of the eigenvalue of the observable.
- A measurement transforms the observed state to one of the eigenvectors of the observable. What happens depends on the characteristics of the measuring equipment. The measurement can be seen as a combination of a transformation and an observation.

Depending on the characteristics of the measuring equipment a measurement and a clean observation can give the same result.

With this interpretation of the concept of observation it is possible to let states observe other states. A state might do a transformation before doing an observation but in general it fails the equipment to arrange that transformation. In nature observations are far more common than measurements.

# 2 Numbers

## 2.1 Cayley-Dickson onstruction

The Cayley-Dickson construction formula enables the generation of a quaternion from two complex numbers:

$$p = a_0 + a_1\mathbf{k} + \mathbf{i}(b_0 + b_1\mathbf{k}) \tag{1}$$

$$q = c_0 + c_1\mathbf{k} + \mathbf{i}(d_0 + d_1\mathbf{k}) \tag{2}$$

$$(a, b)\,(c, d) = (ac - db^{*}; a^{*}d + cb) \tag{3}$$

$$r = pq \tag{4}$$

$$r_0 = a_0c_0 - a_1c_1 - b_0d_0 - b_1d_1 \tag{5}$$

$$r_k = a_0c_1 - a_1c_0 - b_0d_1 + b_1d_0 \tag{6}$$

$$r_i = a_0d_0 + a_1d_1 + b_0c_0 - b_1c_1 \tag{7}$$

$$r_j = -a_1d_0 + a_0d_1 + b_0c_1 + b_1c_0 \tag{8}$$

## 2.2 Warren Smith's numbers

All hyper-complex numbers are based on real numbers. Two main construction formulas for hyper-complex numbers exist. The Cayley-Dickson construction is the most widely known. The Warren-Smith construction gives best algorithmic properties at higher dimensions. Until the octonions both construction formulas deliver the same results.

The quaternions are the highest dimensional hyper-complex numbers that deliver a division ring.

### 2.2.1 $2^n$-on construction

The $2^n$-ons use the following doubling formula

$$\begin{aligned}(a,b)(c,d) &= (a\,c - (b\,d^*)^*, (b^*c^*)^* \\ &+ (b^*(a^*((b^{-1})^*d^*)^*)^*)^*)\end{aligned} \tag{1}$$

Up until the 16-ons the formula can be simplified to

$$(a,b)(c,d) = (a\,c - b\,d^*, c\,b + (a^*\,b^{-1})\,(b\,d)) \quad (2)$$

Up to the octonions the Cayley Dickson construction delivers the same as the $2^n$-on construction. From n>3 the $2^n$-ons are 'nicer'.

#### 2.2.1.1 $2^n$-ons

Table of properties of the $2^n$-ons.

See scorevoting.net/WarrenSmithPages/homepage/nce2.ps.

| Type | *name* | Lose |
|---|---|---|
| 1-ons | *Reals*. | |
| 2-ons | *Complex numbers* | $z^* = z$ (the * denotes conjugating); the ordering properties that both {z > 0, -z > 0, or z = 0} and {w > 0, z > 0 implies w + z > 0, wz > 0}. |
| 4-ons | *Quaternions* | commutativity ab = ba; the algebraic closedness property that every univariate polynomial equation has a root. |
| 8-ons | *Octonions* | associativity ab · c = a · bc. |
| 16-ons | (*not Sedenions!*) | right-alternativity x · yy = xy · y; right-cancellation $x = xy \cdot y^{-1}$; flexibility x · yx = xy · x; left-linearity (b + c)a = ba + ca; anti-automorphism ab = ba, $(ab)^{-1} = b^{-1} a^{-1}$; left-linearity (b + c)a = ba + ca; continuity of the map x → xy; Moufang and Bol identities; diassociativity |
| 32-ons | | generalized-smoothness of the map x → |

| | | |
|---|---|---|
| | | xy;<br>right-division properties that xa = b has (generically) a solution x, and the uniqueness of such an x;<br>the "fundamental theorem of algebra" that every polynomial having a unique "asymptotically dominant monomial" must have a root; Trotter's formula:<br>$\lim_{n\to\infty}\left[e^{x/n}e^{y/n}\right]^n = \lim_{n\to\infty}\left(1 + \frac{x+y}{n}\right)^n = e^{x+y}$ |

| Type | Retain |
|---|---|
| $2^n$-ons | Unique 2-sided multiplicative & additive identity elements 1 & 0;<br>Norm-multiplicativity $\|xy\|^2 = \|x\|^2 \cdot \|y\|^2$ ;<br>Norm-subadditivity $\|a + b\| \leq \|a\| + \|b\|$;<br>2-sided inverse $a^{-1} = a^*/\|a\|^2$ (a # 0);<br>$a^{**} = a$;<br>$(x \pm y)^* = x^* \pm y^*$;<br>$(a^{-1})^{-1} = a$;<br>$(a^*)^{-1} = (a^{-1})^*$ ;<br>$\|a\|^2 = \|a\|^2 = a^*a$;<br>Left-alternativity yy · x = y · yx;<br>Left-cancellation $x = y^{-1} \cdot yx$;<br>Right-linearity a(b + c) = ab + ac;<br>$r^{th}$ power-associativity $a^n a^m = a^{n+m}$;<br>Scaling s · ab = sa · b = as · b = a · sb = a · bs = ab · s (s real); Power-distributivity $(ra^n + sa^m)b = ra^n b + sa^m b$ (r, s real);<br>Vector product properties of the imaginary part: **ab** - re(**ab**) of the product for pure-imaginary $2^n$-ons **a**,**b** regarded as ($2^n$ - |

| | |
|---|---|
| | 1)-vectors;<br>$\langle xa,b\rangle = \langle a,x^*b\rangle$, $\langle xa,xb\rangle = \|x\|^2\cdot\langle a,b\rangle$ and $\langle x,y\rangle = \langle x^*,y^*\rangle$<br>Numerous weakened associativity, commutativity, distributivity, antiautomorphism, and Moufang and Bol properties including 9-coordinate ``niner'' versions of most of those properties; contains $2^{n-1}$-ons as subalgebra. |

### *2.2.1.1.1 The most important properties of $2^n$-ons*

If a,b,x,y are $2^n$-ons, $n \geq 0$, and s and t are scalars (i.e. all coordinates are 0 except the real coordinate) then

**unit**: A unique $2^n$-on 1 exists, with $1\cdot x = x\cdot 1 = x$.

**zero:** A unique $2^n$-on 0 exists, with $0 + x = x + 0 = x$ and $0\cdot x = x\cdot 0 = 0$.

**additive properties**: $x+y = y+x$, $(x+y)+z = x+(y+z)$;

$-x$ exists with $x + (-x) = x - x = 0$.

**norm:** $|x|^2 = xx^* = x^*x$.

**norm-multiplicativity**: $|x|^2\cdot|y|^2 = |x\cdot y|^2$.

**scaling**: $s\cdot x\cdot y = s\cdot x\cdot y = x\cdot s\cdot y = x\cdot s\cdot y = x\cdot y\cdot s$.

**weak-linearity**: $(x + s)\cdot y = x\cdot y + s\cdot y$ and $x\cdot(y + s) = x\cdot y + x\cdot s$.

**right-linearity**: $x\cdot(y + z) = x\cdot y + x\cdot z$.

**inversion**: If $x \neq 0$ then a unique $x^{-1}$ exists, obeying $x^{-1}\cdot x = x\cdot x^{-1} = 1$. It is $x^{-1} = x\cdot|x|^{-2}$.

**left-alternativity**: $x\cdot xy = x^2\cdot y$.

**left-cancellation**: $x\cdot x^{-1}\cdot y = y$.

**effect on inner products**: $\langle x\cdot a,b\rangle = \langle a, x^*\cdot b\rangle$, $\langle x,y\rangle = \langle x^*, y^*\rangle$, $\langle x^*\cdot a, x^{-1}\cdot b\rangle = \langle a,b\rangle$,

and $\langle x\cdot a,x\cdot b\rangle = |x|^2\cdot\langle a,b\rangle$.

**Conjugate of inverse:** $(x^{-1})^* = (x^*)^{-1}$.

**Near-anticommutativity of unequal basis elements:** $e_k^2 = -1$ and $e_k \cdot e_l^* = -e_l \cdot e_k^*$ if $k \neq l$.

(Note: the case $k$; $l > 0$ shows that unequal pure-imaginary basis elements anticommute.)

**Alternative basis elements:** $e_k \cdot e_l \cdot e_k = e_k \cdot e_l \cdot e_k$, $e_l \cdot e_k \cdot e_k = e_l \cdot e_k \cdot e_k$, and $e_k \cdot e_k \cdot e_l = e_k \cdot e_k \cdot e_l$. (However, when $n \geq 4$ the $2^n$-ons are not flexible i.e. it is not generally true that $x \cdot y \cdot x = x \cdot y \cdot x$ if x and y are 16-ons that are not basis elements. They also are not right-alternative.)

**Quadratic identity:** If x is a $2^n$-on (over any field F with charF $\neq 2$), then $x^2 + |x|^2 = 2 \cdot x \text{ re } x$

**Squares of imaginaries:** If x is a $2^n$-on with re x = 0 ("pure imaginary") then $x^2 = -|x|^2$ is nonpositive pure-real.

**Powering preserves im*x* direction**

## *2.2.1.1.2 Niners*

Niners are 2n-ons whose coordinates with index > 8 are zero. The index starts with 0.

9-**flexibility** $xp \cdot x = x \cdot px$, $px \cdot p = p \cdot xp$.

9-**similitude unambiguity** $xp \cdot x^{-1} = x \cdot px^{-1}$, $px \cdot p^{-1} = p \cdot xp^{-1}$.

9-**right-alternativity** $xp \cdot p = x \cdot p^2$, $px \cdot x = p \cdot x^2$.

9-**right-cancellation** $xp^{-1} \cdot p = x$, $px^{-1} \cdot x = p$.

9-**effect on inner products** $\langle x, yp\rangle = \langle xp, y\rangle$, $\langle xp, yp\rangle = |p|^2\langle x, y\rangle$.

9-**left-linearity** $(x + y)p = xp + yp$, $(p + q)x = px + qx$.

9-**Jordan-identity** $xp \cdot xx = x(p \cdot xx)$, $py \cdot pp = p(y \cdot pp)$.

9-**coordinate-distributivity** $([x + y]z)_{0;...;8} = (xz + yz)_{0;...;8}$.

9-**coordinate-Jordan-identity** $[xy \cdot xx]_{0;...;8} = [x(y \cdot xx)]_{0;...;8}$.

9-**anticommutativity for orthogonal imaginary $2^n$-ons**

If $\langle p, x\rangle = \text{re } p = \text{re } x = 0$ then $px = -xp$.

9-**reflection** If |a| = 1 and the geometric reflection operator is defined below then $-(\text{refl}[a](y))_{0;:::;8} = (a \cdot y^{*}a)_{0;:::;8}$, and $-\{\text{refl}[a](y)\}^{*}_{0;:::;8} = (a^{*}y \cdot a^{*})_{0;:::;8}$, and

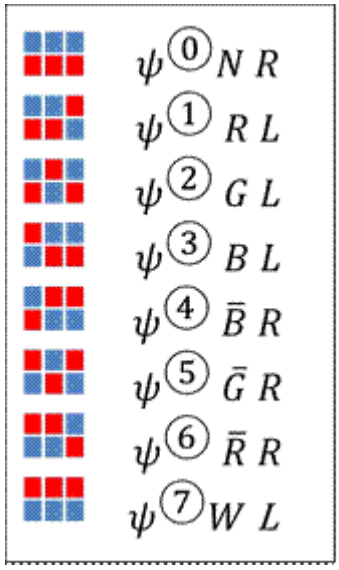
if either a or y is a niner then $-\text{refl}[a](y) = a \cdot y^{*}a$ and $-\text{refl}[a](y) = a^{*}y \cdot a^{*}$.

$$\text{refl}[\vec{x}](\vec{t}) \stackrel{\text{def}}{=} \vec{t} - \frac{2\langle \vec{x}, \vec{t} \rangle}{|\vec{x}|^2}\vec{x} \tag{1}$$

What holds for the niners, also holds for the octonions.

## 2.3 Quaternions

### 2.3.1 Sign flavors




The quaternions that form the values of a continuous quaternionic distribution must all feature the same set of sign selections. This fact attaches a sign flavor to each quaternionic distribution. Quaternionic distributions come in eight sign flavors[11]. We indicate color by an extra index: $i = R, G, B$. $\bar{R}, \bar{G}, \bar{B}$ are anti-colors. See figure.

In the right column the symbols *R* and *L* stand for the right or left handedness of the quaternion product. We will use the symbol $\psi$ or $\psi^{0}$ for the sign flavor of the quaternionic distribution that

has the same sign flavor as its parameter space. However, selecting $\psi^{⑦}$ as base instead of $\psi^{⓪}$ may work as well. Since $\psi^{⑦}$ acts as background QPAD for fermions, it might be a better choice.

We will use

$$\psi^{⑦} = \psi^*$$
$$\psi^{⓪} = \psi$$

Often the symbols $\psi$ and $\psi^*$ will be used instead of the symbols $\psi^{⓪}$ and $\psi^{⑦}$.

#### 2.3.1.1 Sign selections and quaternionic distributions

Quaternionic distributions are supposed to obey a distribution wide sign selection. Thus, the distribution is characterized by one of the eight quaternionic sign flavors.

$$\psi^{⓪}, \psi^{①}, \psi^{②}, \psi^{③}, \psi^{④}, \psi^{⑤}, \psi^{⑥}, \psi^{⑦} \tag{1}$$

Many of the elementary particles are characterized by an ordered pair of two field sign flavors. These fields are coupled with a coupling strength that is typical for the particle type. These particles obey a characteristic continuity equation[105].

#### 2.3.1.2 Product rule

We use the quaternionic product rule.

$$ab = a_0 b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b} \tag{1}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{2}$$

---

[105] Hilbert field equations; Continuity equation for charges

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{i}(a_2 b_3 - a_3 b_2) + \boldsymbol{j}(a_3 b_1 - a_1 b_3) + \boldsymbol{k}(a_1 b_2 - a_2 b_1) \tag{3}$$

#### 2.3.1.3 Operators

The sign selections of operator $\nabla = (\nabla_0, \boldsymbol{\nabla})$ depend on the sign selections of position operator Q, which determines the sign selections for its eigenvalues $q = (q_0, \boldsymbol{q})$.

Normally we consider the sign selection for operators $Q$ and $\nabla$ fixed to operators $Q^{⓪}$ and $\nabla^{⓪}$. Sometimes we chose $\nabla^*$ instead of operator $\nabla$.

Quaternionic sign selection are directly connected with the concepts of **parity** and **spin**.

For quaternionic functions symmetry reduces the differences that are produced by conjugation and anti-symmetry stresses the differences. The same holds for operators.

#### 2.3.1.4 Matrices

Another possibility is to present sign selections by matrices[106].

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

The $\sigma_1$ matrix switches the complex fields that together form the quaternion field.

(2)

[106] http://www.vttoth.com/qt.htm

$$\begin{bmatrix} \varphi_b \\ \varphi_a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_a \\ \varphi_b \end{bmatrix}$$

The $\sigma_2$ matrix switches the real parts and the imaginary parts of the complex fields that together form the quaternion field and it switches both fields.

$$i \begin{bmatrix} -\varphi_b \\ \varphi_a \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \varphi_a \\ \varphi_b \end{bmatrix} \tag{3}$$

The $\sigma_3$ matrix switches the sign of the first complex field.

$$\begin{bmatrix} -\varphi_a \\ \varphi_b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_a \\ \varphi_b \end{bmatrix} \tag{4}$$

$$\sigma_k^2 = -i\sigma_1\sigma_1\sigma_1 = I \tag{5}$$

The Pauli matrices are involutory.
The determinants[107] and traces[108] of the Pauli matrices are:

$$\det(\sigma_k) = -1 \tag{6}$$

$$\mathrm{Tr}(\sigma_k) = 0 \tag{7}$$

$$\alpha_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix} \tag{8}$$

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{9}$$

[107] http://en.wikipedia.org/wiki/Determinant
[108] http://en.wikipedia.org/wiki/Trace_of_a_matrix

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{10}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{11}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

The $\alpha_k$ matrices together select the imaginary base vectors. The $\beta$ matrix exchanges the sign of all imaginary base vectors. Thus the $\beta$ matrix implements the quaternionic conjugate. The conjugation also exchanges right handedness against left handedness.

Another way of exchanging right handedness against left handedness is the exchange of the sign of one of the imaginary base vectors.

$$\begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix} \tag{13}$$

$$\psi^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi \tag{14}$$

The gamma matrices[109] translate directly from complex fields to fully featured quaternionic fields. In this way four sign flavors of quaternionic fields are constructed from four complex fields. This is represented by four dimensional matrices and four dimensional spinors. The equivalent of the $\beta$ matrix is the $\gamma_\beta$ matrix.

(15)

[109] Appendix; Gamma matrices

$$\begin{bmatrix} \varphi_{La} \\ \varphi_{Lb} \\ \varphi_{Ra} \\ \varphi_{Rb} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_{Ra} \\ \varphi_{Rb} \\ \varphi_{La} \\ \varphi_{Lb} \end{bmatrix}$$

It is false to interpret the matrices as vectors. They form a shorthand for handling spinors.

The Pauli matrix $\sigma_1$ represents the sign selection $a \rightarrow a^{\otimes}$ ($\otimes =$ ①, ②, ③, ④, ⑤, ⑥), while the $\beta$ matrix represents the sign selection $a \rightarrow a^{⑦}$. The other Pauli matrices and the $\alpha$ matrices implement the resulting part of the quaternion behavior for spinors.

### 2.3.2 Waltz details

The 16-ons lose the continuity of the map $x \Rightarrow xy$. Also, in general holds $(x\,y)x \neq x\,(y\,x)$ for 16-ons. However, for all $2^n$-ons the base numbers fulfill $(e_i\,e_j)\,e_i = e_i\,(e_j\,e_i)$. All $2^n$-ons feature a conjugate and an inverse. The inverse only exists for non-zero numbers. The $2^n$-ons support the **number waltz**

$$c = a\,b/a. \tag{1}$$

Often the number waltz appears as a unitary number waltz

$$c = u^* b\,u \tag{2}$$

where $u$ is a unit size number and $u^*$ is its conjugate $u\,u^* = 1$.

In quaternion space the **quaternion waltz** $a\,b/a$ can be written as

$$a\,b\,/\,a = exp(2\,\pi\,\tilde{\imath}\,\varphi)\,b\,exp(-2\,\pi\,\tilde{\imath}\,\varphi) \tag{3}$$

$$= b - \boldsymbol{b}_{\perp} + exp(2\,\pi\,\tilde{\imath}\,\varphi)\,\boldsymbol{b}_{\perp}\,exp(-2\,\pi\,\tilde{\imath}\,\varphi)$$

$$= b - \boldsymbol{b}_{\perp} + exp(4\,\pi\,\tilde{\imath}\,\varphi)\boldsymbol{b}_{\perp}$$

$$\Delta b = (exp(4\,\pi\,\tilde{\imath}\,\varphi) - 1)\boldsymbol{b}_{\perp} \tag{4}$$

$$= (cos(4\,\pi\,\varphi) + \tilde{\imath}\,sin(4\,\pi\,\varphi) - 1)\,\boldsymbol{b}_{\perp}$$

$$= exp(2\,\pi\,\tilde{\imath}\,\varphi)\,2\,\tilde{\imath}\,sin\,(2\,\pi\,\varphi)\,\boldsymbol{b}_{\perp}$$

$$\|\Delta b\| = \|2\,sin(2\,\pi\,\varphi)\,\boldsymbol{b}_{\perp}\| \tag{5}$$

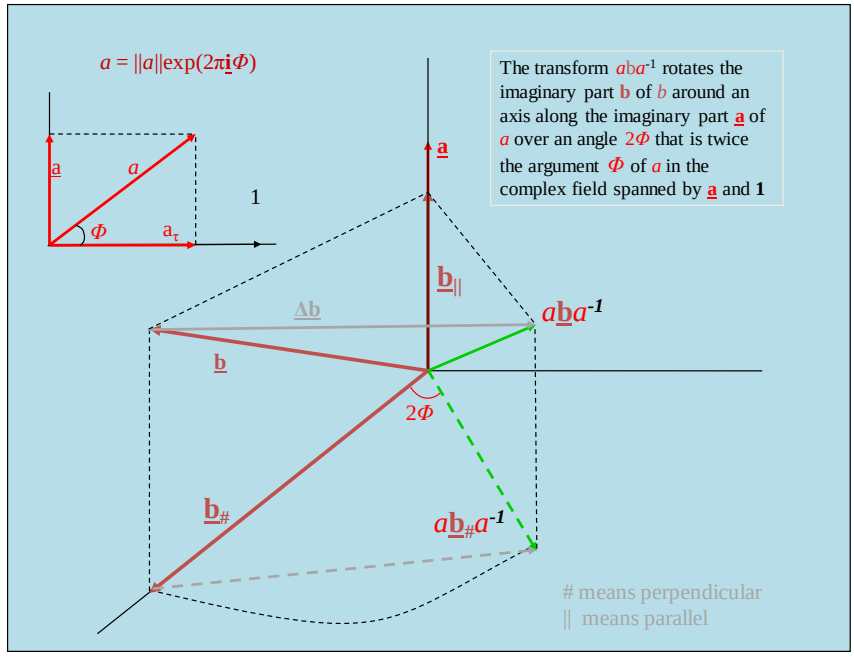



**Figure 6. The rotation of a quaternion by a second quaternion.**

Another way of specifying the difference is:

$$\Delta b \;=\; (a \cdot b - b \cdot a)/a \;=\; 2 \cdot (\boldsymbol{a} \times \boldsymbol{b})/a \tag{6}$$

$$\|\Delta b\| \;= 2\, \|\boldsymbol{a} \times \boldsymbol{b}\| / \, \|a\| \tag{7}$$

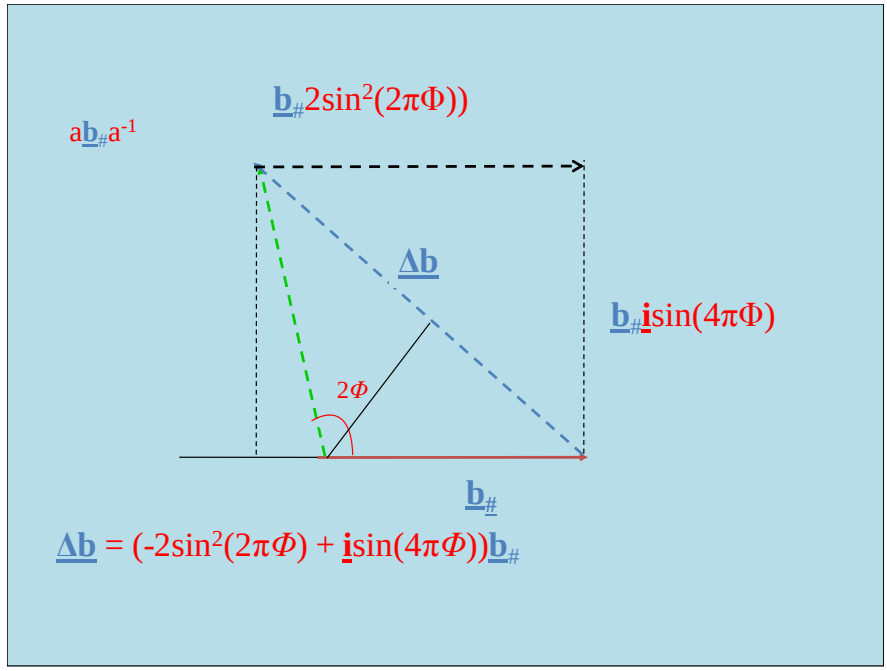



**Figure 7: The difference after rotation**

### *2.3.2.1.1 Infinitesimal number transformation*

The number $v$ is close to 1. Thus $v = 1 + \Delta s$. Let us investigate the transform $c = v^*\, b\, v$.

$$c = (1 + \Delta s^*)\, b\, (1 + \Delta s) \tag{1}$$

$$= b + \Delta s^* \cdot b + b\,\Delta s + \Delta s^*\, b\, \Delta s$$

$$\approx b + \Delta s *\cdot b + b \cdot \Delta s$$

$$= b + \Delta s_0\, b + 2\, \boldsymbol{b} \times \boldsymbol{\Delta s}$$

$$\Delta b = \Delta s_0\, b + 2\, \boldsymbol{b} \times \boldsymbol{\Delta s} \tag{2}$$

This comes close to the effect of an infinitesimal number waltz, especially when $\Delta s_0 = 0$ In that case $\Delta b_0 = 0$ and $\boldsymbol{\Delta b}$ is perpendicular $to\ \boldsymbol{\Delta s}$.

For $2^n$-ons with $n > 1, a\, b \neq a$ in general does not equal $b$. This effect stays unnoticed when quantum mechanics sticks to a complex Hilbert space.

## 2.4 Quaternion coordinates

This part of the appendix describes candidates for the coordinates on the coordinate sphere.

### 2.4.1 Polar coordinates

The equivalent to rectangular coordinates in quaternion space is $(a_\tau, a_x, a_y, a_z)$

$$a = a_\tau + \boldsymbol{i}\, a_x + \boldsymbol{j}\, a_y \pm \boldsymbol{i}\boldsymbol{j}\, a_z \tag{1}$$

The equivalent to polar coordinates in quaternion space is

$$a_\tau = \|a\| \cos(\psi) \tag{2}$$

$$a_x = \|a\| \sin(\psi) \sin(\theta) \cos(\varphi) \tag{3}$$

$$a_y = \|a\| \sin(\psi) \sin(\theta) \sin(\varphi) \tag{4}$$

$$a_z = \|a\| \sin(\psi) \cos(\theta) \tag{5}$$

$sin(\psi)$, where $\psi = (0, \pi)$, is known as the (imaginary) amplitude of the quaternion.

Angle $\theta = (0, \pi)$ is the (co-)latitude and angle $\varphi = (0, 2\pi)$ is the longitude.

For any fixed value of $\psi, \theta$ and $\varphi$ parameterize a 2-sphere of radius $sin(\psi)$, except for the degenerate cases, when $\psi$ equals 0 or $\pi$, in which case they describe a point.

This suggests the following structure of the argument $\boldsymbol{\Lambda}$

$$a = \|a\| \, exp(\tilde{\imath} \cdot \psi) \tag{6}$$

$$= \|a\| \, (cos(\psi) + \tilde{\imath} \, sin(\psi)) \tag{7}$$

$$= a_\tau + \|a\| \, \tilde{\imath} \, sin(\psi) = a_\tau + \boldsymbol{a} \tag{8}$$

The imaginary number ĩ may take any direction.

### 2.4.2 3 sphere

A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply-connected. What this means, loosely speaking, is that any loop, or circular path, on the 3-sphere can be continuously shrunk to a point without leaving the 3-sphere. The Poincaré conjecture[110] proposes that the 3-sphere is the only three dimensional manifold with these properties (up to homeomorphism)[111].

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\psi^2 + sin^2(\psi) \, (d\theta^2 + sin^2(\theta) d\varphi^2) \tag{1}$$

The volume form is given by

$$dV = sin^2(\psi) \, sin(\theta) \, d\psi \wedge d\theta \wedge d\varphi \tag{2}$$

The 3-dimensional volume (or **hyperarea**) of a 3-sphere of radius $r$ is

$$2 \, \pi^2 \, r^3 \tag{3}$$

The 4-dimensional **hypervolume** (the volume of the 4-dimensional region bounded by the 3-sphere) is

---

[110] http://en.wikipedia.org/wiki/Poincar%C3%A9_conjecture

[111] http://en.wikipedia.org/wiki/3-sphere

$$½\,\pi^2\,r^4 \tag{4}$$

The 3-sphere has constant positive sectional curvature equal to $1/r^2$.

The 3-sphere has a natural Lie group structure SU(2) given by quaternion multiplication.

The 3-sphere admits non-vanishing vector fields (sections of its tangent bundle). One can even find three linearly-independent and non-vanishing vector fields. These may be taken to be any left-invariant vector fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial.

There is an interesting action of the circle group $\mathbb{T}$ on $\mathbb{S}^3$ giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of $\mathbb{S}^3$ as a subset of $\boldsymbol{C}^2$, the action is given by

$$(z_1, z_2)\,\lambda \;=\; (z_1\,\lambda, z_2\,\lambda)\;\forall_{\lambda\,\in\,\mathbb{T}}. \tag{5}$$

The orbit space of this action is homeomorphic to the two-sphere $\mathbb{S}^2$. Since $\mathbb{S}^3$ is not homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, the Hopf bundle is nontrivial.

### 2.4.3 Hopf coordinates

Another choice of hyperspherical coordinates, $(\eta, \xi_1, \xi_2)$, makes use of the embedding of $\mathbb{S}^3$ in $\boldsymbol{C}^2$. In complex coordinates $(z_1, z_2) \in \boldsymbol{C}^2$ we write

$$z_1 = exp(ĩ\,\xi_1)\,sin(\eta) \tag{1}$$

$$z_2 \;=\; exp(ĩ\,\xi_2)\,cos(\eta) \tag{2}$$

Here $\eta$ runs over the range 0 to $\pi/2$, and $\xi_1$ and $\xi_2$ can take any values between 0 and $2\pi$. These coordinates are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2 \tag{3}$$

For any fixed value of η between 0 and $\pi/2$, the coordinates $(\xi_1, \xi_2)$ parameterize a 2-dimensional torus. In the degenerate cases, when $\eta$ equals 0 or $\pi/2$, these coordinates describe a circle.

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\eta^2 + sin^2(\eta)\ (d\zeta_1^2 + cos^2(\eta)\ d\ \zeta_2^2) \tag{4}$$

and the volume form by

$$dV\ =\ sin(\eta)\ cos(\eta)\ d\eta \hat{}\ d\zeta_1 \hat{}\ d\zeta_2 \tag{5}$$

### 2.4.4 Group structure

Because the set of unit quaternions is closed under multiplication, $\mathbb{S}^3$ takes on the structure of a group. Moreover, since quaternionic multiplication is smooth, $\mathbb{S}^3$ can be regarded as a real Lie group. It is a non-abelian, compact Lie group of dimension 3. When thought of as a Lie group $\mathbb{S}^3$ is often denoted $Sp(1)$ or $\mathrm{U}(1, \mathbb{H})$.

It turns out that the only spheres which admit a Lie group structure are $\mathbb{S}^1$, thought of as the set of unit complex numbers, and $\mathbb{S}^3$, the set of unit quaternions. One might think that $\mathbb{S}^7$, the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is non-associative. The octonionic structure does give $\mathbb{S}^7$ one important property: *parallelizability*[112]. It turns out that the only spheres which are parallelizable are $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$.

[112] http://en.wikipedia.org/wiki/Parallelizability

By using a matrix representation of the quaternions, $\mathbb{H}$, one obtains a matrix representation of $\mathbb{S}^3$. One convenient choice is given by the Pauli matrices:

$$\left(a_\tau + a_x \cdot \mathbf{i} + a_\mathrm{y} \cdot \mathbf{j} + a_z \cdot \mathbf{k}\right) = \begin{bmatrix} a_\tau + \tilde{\imath} \cdot a_x & a_\mathrm{y} + \tilde{\imath} \cdot a_z \\ -a_\mathrm{y} + \tilde{\imath} \cdot a_z & a_\tau - \tilde{\imath} \cdot a_x \end{bmatrix} \tag{1}$$

This map gives an injective algebra homomorphism from $\mathbb{H}$ to the set of 2×2 complex matrices. It has the property that the absolute value of a quaternion $q$ is equal to the square root of the determinant of the matrix image of $q$.

The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group SU(2). Thus, $\mathbb{S}^3$ as a Lie group is isomorphic to SU(2).

Using our hyperspherical coordinates $(\eta, \xi_1, \xi_2)$ we can then write any element of SU(2) in the form

$$\begin{bmatrix} \exp(\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) & \exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) \\ -\exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) & \exp(-\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) \end{bmatrix} \tag{2}$$

Another way to state this result is if we express the matrix representation of an element of SU(2) as a linear combination of the Pauli matrices. It is seen that an arbitrary element U ∈ SU(2) can be written as

$$U = \alpha_\tau \cdot 1 + \sum_{n=x,y,z} \alpha_n \boldsymbol{I_n} \tag{3}$$

The condition that the determinant of U is +1 implies that the coefficients $\alpha_n$ are constrained to lie on a 3-sphere.

### 2.4.5 Versor

Any **unit quaternion** $q$ can be written as a **versor**:

$$u = exp(\tilde{\imath}\ \psi) = cos(\psi) + \tilde{\imath}\ sin(\psi) \tag{1}$$

This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in Im $\mathbb{H}$ so any such $\underline{\tilde{\mathbf{i}}}$ can be written:

$$\tilde{\imath} = \boldsymbol{i}\, cos(\varphi)\, sin(\theta) + \boldsymbol{j}\, sin(\varphi)\, sin(\theta) + \boldsymbol{k}\, cos(\theta) \tag{2}$$

### 2.4.6 Symplectic decomposition

Quaternions can be written as the combination of two complex numbers and an imaginary number k with unit length.

# 3 *The separable Hilbert space H*

## 3.1 Notations and naming conventions

$\{f_x\}_x$ means ordered set of $f_x$ . It is a way to define functions.

*The use of bras and kets differs slightly from the way Dirac uses them*.

|f> is a ket vector, f> is the same ket
<f| is a bra vector, <f is the same bra

A is an operator. |A is the same operator
A† is the adjoint operator of operator A. A| is the same operator
| on its own, is a nil operator
|A| is a self-adjoint (Hermitian) operator

We will use capitals for operators and lower case for quaternions, eigenvalues, ket vectors, bra vectors and eigenvectors. Quaternions and eigenvalues will be indicated with *italic* characters. Imaginary and anti-Hermitian objects are often underlined and/or indicated in **bold** text.

$\sum_k$ means: sum over all items with index k.
$\int_x$ means: integral over all items with parameter x.

## 3.2 Quaternionic Hilbert space

The Hilbert space is a **linear space**. That means for the elements |f>, |g> and |h> and numbers *a* and *b*:

### 3.2.1 Ket vectors

For **ket** vectors hold

$$|f\rangle + |g\rangle = |g\rangle + |f\rangle = |g + f\rangle \tag{1}$$

$$(|f\rangle + |g\rangle) + |h\rangle = |f\rangle + (|g\rangle + |h\rangle) \tag{2}$$

$$|(a + b)\, f\rangle = |f\rangle \cdot a + |f\rangle \cdot b \tag{3}$$

$$(|f\rangle + |g\rangle) \cdot a = |f\rangle \cdot a + |g\rangle \cdot a \tag{4}$$

$$|f\rangle \cdot 0 = |0\rangle \tag{5}$$

$$|f\rangle \cdot 1 = |f\rangle \tag{6}$$

Depending on the number field that the Hilbert space supports, $a$ and $b$ can be real numbers, complex numbers or (real) quaternions.

### 3.2.2 Bra vectors

The **bra** vectors form the dual Hilbert space $\mathbf{H}^{\dagger}$ of $\mathbf{H}$.

$$\langle f| + \langle g| = \langle g| + \langle f| = |g + f\rangle \tag{1}$$

$$(\langle f| + \langle g|) + \langle h| = \langle f| + (\langle g| + \langle h|) \tag{2}$$

$$\langle f\,(a + b)\rangle = \langle f| \cdot a + \langle f| \cdot b = a^{*} \cdot \langle f| + b^{*} \cdot \langle f| \tag{3}$$

$$(\langle f| + \langle g|) \cdot a = \langle f| \cdot a + \langle g| \cdot a = a^{*} \cdot \langle f| + a^{*} \cdot \langle g| \tag{4}$$

$$0 \cdot \langle f| = \langle 0| \tag{5}$$

$$1 \cdot \langle f| = \langle f| \tag{6}$$

### 3.2.3 Scalar product

The Hilbert space contains a **scalar product,** also called **inner product,** $\langle f|g\rangle$ that combines $\mathbf{H}$ and $\mathbf{H}^\dagger$ in a direct product that we also indicate with $\mathbf{H}$.

The scalar product $\langle f|g\rangle$ satisfies:

$$\langle f|g + h\rangle = \langle f|g\rangle + \langle f|h\rangle \tag{1}$$

$$\langle f|\{|g\rangle \cdot a\}_g = \{\langle f|g\rangle\}_g \cdot a \tag{2}$$

With each ket vector $|g\rangle$ in $\mathbf{H}$ belongs a bra vector $\langle g|$ in $\mathbf{H}^\dagger$ such that for all bra vectors $\langle f|$ in $\mathbf{H}^\dagger$

$$\langle f|g\rangle = \langle g|f\rangle^* \tag{3}$$

$$\langle f|f\rangle = 0 \text{ when } |f\rangle = |0\rangle \tag{4}$$

$$\langle f|a\ g\rangle = \langle f|g\rangle \cdot a = \langle g|f\rangle^* \cdot a = \langle g\ a|f\rangle^* = (a^* \cdot \langle g|f\rangle)^* = \langle f|g\rangle \cdot a \tag{5}$$

In general is $\langle f|a\ g\rangle \neq \langle f\ a|g\rangle$. However for real numbers r holds $\langle f|r\ g\rangle = \langle f\ r|g\rangle$

Remember that when the number field consists of quaternions, then also $\langle f|g\rangle$ is a quaternion and a quaternion q and $\langle f|g\rangle$ do in general not commute.

The scalar product defines a **norm**:

$$||f|| = \sqrt{(<f|f>)} \tag{6}$$

And a **distance**:

$$D(f,g) = ||f - g|| \tag{7}$$

The Hilbert space **H** is closed under its norm. Each converging row of elements of converges to an element of this space.

### 3.2.4 Separable

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Every continuous function on the separable space **H** is determined by its values on this countable dense subset.

### 3.2.5 *Base vectors*

The Hilbert space **H** is **separable**. That means that there exist a countable row of elements $\{f_n>\}$ that **spans** the whole space.

If $<f_n|f_m> = \delta(m,n) = [1$ when $n = m$; 0 otherwise$]$
then $\{|f_n>\}$ forms an **orthonormal base** of the Hilbert space.

A ket base $\{|k>\}$of **H** is a minimal set of ket vectors $|k>$ that together span the Hilbert space **H**.

Any ket vector $|f>$ in **H** can be written as a linear combination of elements of $\{|k>\}$.

$$|f> = \sum_k (|k>\cdot<k|f>) \tag{1}$$

A bra base {<b|}of $\mathbf{H}^{\dagger}$ is a minimal set of bra vectors <b| that together span the Hilbert space $\mathbf{H}^{\dagger}$.

Any bra vector <f| in $\mathbf{H}^{\dagger}$ can be written as a linear combination of elements of {<b|}.

$$<f| = \sum_b (<f|b>\cdot<b|) \tag{2}$$

Usually base vectors are taken such that their norm equals 1. Such a base is called an othonormal base.

### 3.2.6 *Operators*

Operators act on a subset of the elements of the Hilbert space.

#### 3.2.6.1 Linear operators

An operator Q is linear when for all vectors |f> and |g> for which Q is defined and for all quaternionic numbers a and b:

$$|Q\cdot a\ f> + |Q\cdot b\ g> = |a\cdot Q\ f> + |b\cdot Q\ g> = |Q\ f>\cdot a + |Q\ g>\cdot b = \tag{1}$$

$$Q\ (|f>\cdot a + |g>\cdot b) = Q\ (|a\ f> + |b\ g>) \tag{2}$$

B is **colinear** when for all vectors |f> for which B is defined and for all quaternionic numbers *a* there exists a quaternionic number *c* such that:

$$|B\cdot a\ f> = |a\cdot B\ f> = |B\ f>\ c\cdot a\cdot c^{-1} \tag{3}$$

If |f> is an eigenvector of operator A with quaternionic eigenvalue $a$, then is $|b$ f> an eigenvector of A with quaternionic eigenvalue $b \cdot a \cdot b^{-1}$.

$A| = A^{\dagger}$ is the **adjoint** of the **normal** operator A. |A is the same as A.

$$<f\,A|\,g> = <fA^{\dagger}|g>^{*} \tag{4}$$

$$A^{\dagger\,\dagger} = A \tag{5}$$

$$(A \cdot B)^{\dagger} = B^{\dagger} \cdot A^{\dagger} \tag{6}$$

|B| is a **self adjoint** operator.
| is a nil operator.

The construct |f><g| acts as a linear operator. |g><f| is its adjoint operator.

$$\sum_n \{|f_n> \cdot a_n \cdot <f_n|\}, \tag{7}$$

where a n is real and acts as a density function.

The set of eigenvectors of a normal operator form an orthonormal base of the Hilbert space.

A self adjoint operator has real numbers as eigenvalues.

$\{<q|f>\}_q$ is a function f(q) of parameter $q$.
$\{<g|q>\}_q$ is a function g(q) of parameter $q$.

When possible, we use the same letter for identifying eigenvalues, eigenvalues and the corresponding operator.

So, usually |q> is an eigenvector of a normal operator Q with eigenvalues $q$.

{q} is the set of eigenvalues of Q.
$\{q\}_q$ is the ordered field of eigenvalues of $q$.
$\{|q>\}_q$ is the ordered set of eigenvectors of Q.
$\{<q|f>\}_q$ is the **Q view** of |f>.

#### 3.2.6.2 Normal operators

The most common definition of continuous operators is:

> A **continuous** operator is an operator that creates images such that the inverse images of open sets are open.
>
> Similarly, a **continuous** operator creates images such that the inverse images of closed sets are closed.

A normal operator is a continuous linear operator.
A normal operator in **Ң** creates an image of **Ң** onto **Ң.** It transfers closed subspaces of **Ң** into closed subspaces of **Ң**.

Normal operators represent continuous quantum logical observables.

The normal operators N have the following property.

$$N: \mathbf{Ң} \Rightarrow \mathbf{Ң} \tag{1}$$

N commutes with its **(Hermitian) adjoint** $N^\dagger$

$$N \cdot N^\dagger = N^\dagger \cdot N \tag{2}$$

Normal operators are important because the spectral theorem holds for them.

Examples of normal operators are

- **unitary** operators: $U^{\dagger} = U^{-1}$ , unitary operators are bounded;
- **Hermitian** operators (i.e., self-adjoint operators): $N^{\dagger} = N$;
- **Anti-Hermitian** or anti-self-adjoint operators: $N^{\dagger} = -N$;
- **Anti-unitary** operators: $I^{\dagger} = -I = I^{-1}$ , anti-unitary operators are bounded;
- **positive operators**: $N = MM^{\dagger}$
- **orthogonal projection** operators: $N = N^{\dagger} = N^2$

#### 3.2.6.3 Spectral theorem

For every compact self-adjoint operator $T$ on a real, complex or quaternionic Hilbert space $\mathbf{H}$, there exists an orthonormal basis of $\mathbf{H}$ consisting of eigenvectors of T. More specifically, the orthogonal complement of the kernel (null space) of T admits, either a finite orthonormal basis of eigenvectors of T, or a countable infinite orthonormal basis {en} of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\} \subset R$, such that $\lambda_n \to 0$. Due to the fact that $\mathbf{H}$ is separable the set of eigenvectors of T can be extended with a base of the kernel in order to form a complete orthonormal base of $\mathbf{H}$.

If $T$ is compact on an infinite dimensional Hilbert space $\mathbf{H}$, then $T$ is not invertible, hence $\sigma(T)$, the spectrum of $T$, always contains 0. The spectral theorem shows that $\sigma(T)$ consists of the eigenvalues $\{\lambda_n\}$ of $T$, and of 0 (if 0 is not already an eigenvalue). The set $\sigma(T)$ is a compact subset of the real line, and the eigenvalues are dense in $\sigma(T)$.

A normal operator has a set of eigenvectors that spans the whole Hilbert space **Ḥ**.

In quaternionic Hilbert space a normal operator has quaternions as eigenvalues.

The set of eigenvalues of a normal operator is NOT compact. This is due to the fact that **Ḥ** is separable. Therefore the set of eigenvectors is countable. As a consequence the set of eigenvalues is countable. Further, the eigenspace of normal operators has no finite diameter.

A continuous bounded linear operator on **Ḥ** has a compact eigenspace. The set of eigenvalues has a closure and it has a finite diameter.

#### 3.2.6.4 Eigenspace

The set of eigenvalues $\{q\}$ of the operator Q form the eigenspace of Q

#### 3.2.6.5 Eigenvectors and eigenvalues

For the eigenvector |q> of normal operator Q holds

$$|Q\ q> = |q\ q> = |q> \cdot q \tag{1}$$

$$<q\ Q^{\dagger}| = <q\ q^{*}| = q^{*} \cdot <q| \tag{2}$$

$$\forall_{|f> \in Ḥ} [\{< f|Q\ q >\}_q = \{< f|q > q\}_q = \{< q\ Q^{\dagger}|f >^{*}\}_q = \{q^{*} < q|f >^{*}\}_q] \tag{3}$$

The eigenvalues of $2^n$-on normal operator are $2^n$-ons

$$Q = \sum_{j=0}^{n-1} \mathrm{I}_j Q_i \tag{4}$$

The $Q_j$ are self-adjoint operators.

### 3.2.6.6 Generalized Trotter formula

For bounded operators $\{A_j\}$ hold:

$$\lim_{n\to\infty}\left(\prod_{j=1}^{p} e^{A_j/n}\right)^n = \exp\left(\sum_{j=1}^{p} A_j\right) = \lim_{n\to\infty}\left(1+\frac{\sum_{j=1}^{p} A_j}{n}\right)^n \tag{1}$$

In general

$$\exp\left(\sum_{j=1}^{p} A_j\right) \neq \prod_{j=1}^{p} e^{A_j} \tag{2}$$

In the realm of quaternionic notion the Trotter formula is confusing.

### 3.2.6.7 Unitary operators

For unitary operators holds:

$$U^{\dagger} = U^{-1} \tag{1}$$

Thus

$$U\cdot U^{\dagger} = U^{\dagger}\cdot U = 1 \tag{2}$$

Suppose $U = I + C$ where $U$ is unitary and $C$ is compact. The equations $U\,U^* = U^*U = I$ and $C = U - I$ show that $C$ is normal. The spectrum of $C$ contains 0, and possibly, a finite set or a se-

quence tending to 0. Since $U = I + C$, the spectrum of $U$ is obtained by shifting the spectrum of $C$ by 1.

The unitary transform can be expressed as:

$$U = \exp(\underline{\tilde{\mathbf{I}}} \cdot \Phi / \hbar) \tag{3}$$

$$\hbar = h/(2 \cdot \pi) \tag{4}$$

$\Phi$ is Hermitian. The constant h refers to the granularity of the eigenspace.

Unitary operators have eigenvalues that are located in the unity sphere of the $2^n$-ons field.

The eigenvalues have the form:

$$u = \exp(\underline{\mathbf{i}} \cdot \varphi / \hbar) \tag{5}$$

$\varphi$ is real. $\underline{\mathbf{i}}$ is a unit length imaginary number in $2^n$-on space. It represents a direction.

$u$ spans a sphere in $2^n$-on space. For constant $\underline{\mathbf{i}}$, $u$ spans a circle in a complex subspace.

## *3.2.6.7.1* Polar decomposition

Normal operators N can be split into a real operator A and a unitary operator U. U and A have the same set of eigenvectors as N.

$$N = ||N|| \cdot U = A \cdot U \tag{1}$$

$$N = A \cdot U = U \cdot A \tag{2}$$

$$= A \cdot \exp(\underline{\tilde{\mathbf{I}}} \cdot \Phi)/\hbar)$$

$$= \exp\left(\Phi_r + \underline{\tilde{\mathbf{I}}} \cdot \Phi\right)/\hbar)$$

$\Phi_r$ is a positive normal operator.

#### 3.2.6.8 Ladder operator

### *3.2.6.8.1 General formulation*

Suppose that two operators $X$ and $N$ have the commutation relation:

$$[N, X] = c \cdot X \tag{1}$$

for some scalar $c$. If $|n\rangle$ is an eigenstate of $N$ with eigenvalue equation,

$$|N\, n\rangle = |n\rangle \cdot n \tag{2}$$

then the operator $X$ acts on $|n\rangle$ in such a way as to shift the eigenvalue by $c$:

$$\begin{aligned} |N \cdot X\, n\rangle &= |(X \cdot N + [N, X])\, n\rangle \\ &= |(X \cdot N + c \cdot X)\, n\rangle \\ &= |X \cdot N\, n\rangle + |X\, n\rangle \cdot c \\ &= |X\, n\rangle \cdot n + |X\, n\rangle \cdot c \\ &= |X\, n\rangle \cdot (n+c) \end{aligned} \tag{3}$$

In other words, if $|n\rangle$ is an eigenstate of $N$ with eigenvalue $n$ then $|X\, n\rangle$ is an eigenstate of $N$ with eigenvalue $n + c$.

The operator $X$ is a *raising operator* for $N$ if $c$ is real and positive, and a *lowering operator* for $N$ if $c$ is real and negative.

If $N$ is a Hermitian operator then $c$ must be real and the Hermitian adjoint of $X$ obeys the commutation relation:

$$[N, X^{\dagger}] = - c \cdot X^{\dagger} \tag{4}$$

In particular, if $X$ is a lowering operator for $N$ then $X^{\dagger}$ is a raising operator for $N$ and vice-versa.

### 3.2.7 Unit sphere of Ң

The ket vectors in **Ң** that have their norm equal to one form together the **unit sphere** Θ of **Ң**.

Base vectors are all member of the unit sphere. The eigenvectors of a normal operator are all member of the unit sphere.

The end points of the eigenvectors of a normal operator form a **grid** on the unit sphere Θof **Ң**.

### 3.2.8 Closure

The closure of **Ң** means that converging rows of vectors converge to a vector of **Ң**.

In general converging rows of eigenvalues of Q do not converge to an eigenvalue of Q.

Thus, the set of eigenvalues of Q is open.

At best the density of the coverage of the set of eigenvalues is comparable with the set of $2^n$-ons that have rational numbers as coordinate values.

With other words, compared to the set of real numbers the eigenvalue spectrum of Q has holes.

The set of eigenvalues of operator Q includes 0. This means that Q does not have an inverse.

The rigged Hilbert space **ℍ** can offer a solution, but then the direct relation with quantum logic is lost.

### 3.2.9 Canonical conjugate operator P

The existence of a canonical conjugate represents a stronger requirement on the continuity of the eigenvalues of canonical eigenvalues.

Q has eigenvectors $\{|q>\}_q$ and eigenvalues $q$.

P has eigenvectors $\{|p>\}_p$ and eigenvalues $p$.

For each eigenvector |q> of Q we define an eigenvector |p> and eigenvalues $p$ of P such that:

$$< q|p > = < p|q >^* = exp\ (\hat{\imath} \cdot p \cdot q/\hbar) \tag{1}$$

$\hbar = h/(2\pi)$ is a scaling factor. $< q|p >$ is a quaternion. $\hat{\imath}$ is a unit length imaginary quaternion.

### 3.2.10 Displacement generators

Variance of the scalar product gives:

$$\boldsymbol{i}\,\hbar\,\delta < q|p > = -p < q|p > \delta q \tag{1}$$

$$\boldsymbol{i}\,\hbar\,\delta < p|q > = \; -q < p|q > \delta p \tag{2}$$

In the rigged Hilbert space **Ħ** the variance can be replaced by differentiation.

Partial differentiation of the function <q|p> gives:

$$\boldsymbol{i}\,\hbar\,\partial/\partial q_s < q|p > = \; -p_s < q|p > \tag{3}$$

$$\boldsymbol{i}\,\hbar\frac{\partial}{\partial p_s} < p|q > = \; -q_s < p|q > \tag{4}$$

### 3.2.11 Quaternionic L² space

The space of quaternionic measurable functions is a quaternionic Hilbert space. The quaternionic probability amplitude distributions are measurable.[113]

This space is spanned by an orthonormal basis of quaternionic measurable functions. When the shared affine parameter space of these functions is non-curved, then this base has a canonical conjugate, which is the quaternionic Fourier transform of the original base.

As soon as curvature of the parameter space arises, this relation is disturbed.

With other words: “In advance the Palestra has a virgin state.”

[113] http://en.wikipedia.org/wiki/Lp_space#Lp_spaces

# 4 Gelfand triple

The separable Hilbert space only supports countable orthonormal bases and countable eigenspaces. The rigged Hilbert space $\mathbf{H}$ that belongs to a separable Hilbert space $\mathbf{Н}$ is a Gelfand triple. It supports non-countable orthonormal bases and continuum eigenspaces.

A rigged Hilbert space is a pair $(Н, \Phi)$ with $\mathbf{Н}$ a Hilbert space, $\Phi$ a dense subspace, such that $\Phi$ is given a topological vector space structure for which the inclusion map $i$ is continuous. Its name is not correct, because it is not a Hilbert space.

Identifying $\mathbf{Н}$ with its dual space $\mathbf{Н}^*$, the adjoint to $i$ is the map

$$i^*: Н = Н^* \to \Phi^* \tag{1}$$

The duality pairing between $\Phi$ and $\Phi^*$ has to be compatible with the inner product on $\mathbf{Н}$, in the sense that:

$$\langle u, v\rangle_{\Phi\times\Phi^*} = (u, v)_Н \tag{2}$$

whenever $u \in \Phi \subset Н$ and $v \in Н = Н^* \subset \Phi^*$.

The specific triple $(\Phi \subset Н \subset \Phi^*)$ is often named after the mathematician Israel Gelfand).

Note that even though $\Phi$ is isomorphic to $\Phi^*$ if $\Phi$ is a Hilbert space in its own right, this isomorphism is *not* the same as the composition of the inclusion $i$ with its adjoint $i^*$

$$i^* i : \Phi \subset \mathrm{H} = \mathrm{H}^* \to \Phi^* \tag{3}$$

# 5 Quaternionic distributions

## 5.1 Sign flavors

Quaternionic distributions are quaternion valued functions of a quaternionic parameter. If not otherwise stated, the quaternionic parameter space is not curved. Quaternions feature sign selections. Inside a quaternionic distribution the quaternionic sign selections of the values are all the same. Due to the four possible sign selections of quaternions, quaternionic distributions exist in four sign flavors.

## 5.2 Differentiation

A quaternionic distribution f(q) can be differentiated.

$$\begin{aligned} g(q) = \nabla f(q) \\ &= \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) \\ &+ \boldsymbol{\nabla} f_0(q) \pm \left( \pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \end{aligned} \tag{1}$$

The colored $\mp$ and $\pm$ signs refer to the influence of conjugation of $f(q)$ on quaternionic multiplication. The $\pm$sign refers to the influence of reflection of $f(q)$.

### 5.2.1 Extra freedom

The solution $f(q)$ of

$$g(q) = \nabla f(q) \tag{1}$$

Is determined apart from a gauge term $h(q)$

$$g(q) = \nabla(f(q) + h(q)) \tag{2}$$

Where

$$\nabla h(q) = 0 \tag{3}$$

This leads to three equations:

$$\nabla_0 h_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{h}(q) \rangle = \mathbf{0} \tag{4}$$

$$\pm \nabla_0 \boldsymbol{h}(q) + \boldsymbol{\nabla} h_0(q) = 0 \tag{5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{h}(q) = 0 \tag{6}$$

This leads to

$$\boldsymbol{h}(q) = \boldsymbol{\nabla} \Lambda_0(q) \tag{7}$$

## 5.3 Fourier transform

In order to simplify the discussion we restrict it to the case that the parameter spaces of the functions are not curved.

In Fourier space differentiation becomes multiplication with the canonical conjugate coordinate $k$ and therefore the equivalent equation becomes:

$$\tilde{\mathrm{g}}(k) = \mathrm{k}\tilde{f}(k) = \mathrm{k}_0\tilde{f}_0(k) \mp \langle \mathbf{k}, \tilde{\boldsymbol{f}}(k)\rangle \pm \mathrm{k}_0\tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm\mathbf{k}\times\tilde{\boldsymbol{f}}(k)\right) \tag{1}$$

For the imaginary parts holds:

$$\mathbf{g}(q) = \pm\nabla_0\boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left(\pm\boldsymbol{\nabla}\times\boldsymbol{f}(q)\right) \tag{2}$$

$$\tilde{\mathbf{g}}(k) = \pm\mathrm{k}_0\tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm\mathbf{k}\times\tilde{\boldsymbol{f}}(k)\right) \tag{3}$$

By using[114]

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla} f_0(q) = \mathbf{0} \tag{4}$$

and

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\times\boldsymbol{f}(q)\rangle = 0 \tag{5}$$

It can be seen that for the static part ($\nabla_0 f(q) = 0$) holds:

$$\mathbf{g}(q) = \boldsymbol{\nabla} f_0(q) \pm \left(\pm\boldsymbol{\nabla}\times\boldsymbol{f}(q)\right) \tag{6}$$

$$\tilde{\mathbf{g}}(k) = \mathbf{k}\tilde{f}_0(k) \pm \left(\pm\mathbf{k}\times\tilde{\boldsymbol{f}}(k)\right) \tag{7}$$

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[114] Bo Thidé: http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf ;Formulas:F.104, F.105

## 5.4 Helmholtz decomposition

Formula (7) of the last paragraph leads to the Helmholtz decomposition. The Helmholtz decomposition splits the **static** vector field $\boldsymbol{F}$ in a (transversal) divergence free part $\boldsymbol{F_t}$ and a (one dimensional longitudinal) rotation free part $\boldsymbol{F_l}$.

$$\boldsymbol{F} = \boldsymbol{F_t} + \boldsymbol{F_l} = \boldsymbol{\nabla} \times \boldsymbol{f} - \boldsymbol{\nabla} f_0 \tag{1}$$

Here $f_0$ is a scalar field and $\boldsymbol{f}$ is a vector field. In quaternionic terms $f_0$ and $\boldsymbol{f}$ are the real and the imaginary part of a quaternionic field $f$. $\boldsymbol{F}$ is an imaginary quaternion.[115]

The significance of the terms "longitudinal"and "transversal" can be understood by computing the local three-dimensional Fourier transform of the vector field $\boldsymbol{F}$, which we call $\widetilde{\boldsymbol{F}}$. Next decompose this field, at each point $\boldsymbol{k}$, into two components, one of which points longitudinally, i.e. parallel to $\boldsymbol{k}$, the other of which points in the transverse direction, i.e. perpendicular to $\boldsymbol{k}$.

$$\widetilde{\boldsymbol{F}}(\boldsymbol{k}) = \widetilde{\boldsymbol{F}}_{\boldsymbol{l}}(\boldsymbol{k}) + \widetilde{\boldsymbol{F}}_{\boldsymbol{t}}(\boldsymbol{k}) \tag{2}$$

$$\langle \boldsymbol{k}, \widetilde{\boldsymbol{F}}_{\boldsymbol{t}}(\boldsymbol{k}) \rangle = 0 \tag{3}$$

$$\boldsymbol{k} \times \widetilde{\boldsymbol{F}}_{\boldsymbol{l}}(\boldsymbol{k}) = \boldsymbol{0} \tag{4}$$

The Fourier transform converts gradient into multiplication and vice versa. Due to these properties the inverse Fourier transform gives:

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[115] See next paragraph

$$\boldsymbol{F} = \boldsymbol{F_l} + \boldsymbol{F_t} \tag{5}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{F_t} \rangle = 0 \tag{6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F_l} = \boldsymbol{0} \tag{7}$$

So, this split indeed conforms to the Helmholtz decomposition.

This interpretation relies on idealized circumstance in which the decomposition runs along straight lines. This idealized condition is in general not provided. In normal conditions the decomposition and the interpretation via Fourier transformation only work locally and with reduced accuracy.

Inside a single HBM page the QPAD's are static. It means that there the Helmholtz decomposition is valid.

# 6 Fields

## 6.1 The origin of physical fields.

The Hilbert Book Model is a simple Higgsless model of physics that is strictly based on traditional quantum logic and on the lattice isomorphic model; the set of subspaces of an infinite dimensional separable Hilbert space for which the inner product is specified by using quaternions[116].

This restriction results in the fact that all sets of variables are countable. At the same time most observations are taken from a continuum. As a result the set of potential observations overwhelms the set of variables[117]. The situation is comparable to the situation in which the number of equations is far larger than the number of variables that should form the result. Probably, the set of equations will appear to be inconsistent. In order to cure the situation, it is common to assume that the observations are inaccurate. The inaccuracy must be stochastic or with other words the observation result must be blurred[118].

Nature applies a similar solution, but instead of a simple spread function in the form of a probability density distribution, nature applies a quaternionic probability amplitude distribution (QPAD). This QPAD can be split into a real part that represents a "charge" density distribution and an imaginary part that represents a corresponding "current" density distribution. The "charge" represents the set of properties of the thing that is being observed. The parameter of the distribution represents the location at which the "charge"

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[116] See: http://www.crypts-of-physics.eu/HilbertBookModelEssentials.pdf

[117] A continuum has a higher cardinality than a countable set.

[118] The statistics must support the coherence of observations.

is observed. The squared modulus of the QPAD represents the probability density of the presence of the "charge" at the location that is specified by the parameter.

This approach transfers the dynamics of the observation into a streaming problem. The equation of motion of the "charge" becomes a continuity equation[119].

The properties of particles move according to the above principle. With each elementary particle belongs a state function QPAD that act as a private field of the particle and that determines its dynamic behavior when it moves freely. However, these fields overlap. In this way these fields and the corresponding particles interact.

Particles can interact in an extra way. A gauge field that acts as a solution of the coupling equation when the coupling factor equals zero implements the extra interaction.

A subset of the elementary particles is massless. These particles correspond to a single QPAD. That does not say that their fields cannot overlap.

All other elementary particles are identified by an ordered pair of QPAD's that are two field sign flavors of the same base field. The coordinate system, whose values are used as field parameter, has its own field sign flavor and acts as a sign flavor reference.

### 6.1.1 Categories of fields

Two categories of fields exist.

#### 6.1.1.1 Primary fields

The first category consists of quaternionic probability amplitude distributions (QPAD's). The QPAD's may overlap and through this

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[119] Another name for "continuity equation" is "balance equation".

superposition they may form covering fields. The QPAD's exist in eight sign flavors. The same holds for the covering fields. The QPAD's may interact. When different sign flavors interact the strength of the local interaction is characterized by a coupling factor. The members of this category will be called primary fields.

#### 6.1.1.2 Second category physical fields

The second category consists of administrator fields. These fields administer the effect of interactions on the local curvature of the positioning coordinate system. For all properties that characterize a coupling of sign flavors of primary fields an administrator field exist that registers the influence of that property during interactions on the local curvature.

One of these administrator fields is the gravitation field. It administers the influence of the strength of the coupling between sign flavors of primary fields on the local curvature.

The electromagnetic fields administer the influence of the electric charge on the local curvature.

The angular momentum including the spin also influences the local curvature. Also this effect is administered.

The members of this category will be called second category physical fields or administrator fields.

All these influences can be administered by using the local metric. This generates a metric tensor field.

## 6.2 Example potential

The influence of local properties is represented by charges. The charge carrier may contain an assembly of charges.

Spatial Harmonic functions[120] are suitable charge spread functions.

For a harmonic function $f(q)$ holds:

$$\Delta f(q) = \nabla\nabla^{①} f(q) = 0 \tag{1}$$

If there is a static spherically symmetric Gaussian charge density ρ (*r*):

$$\rho(q) = \frac{Q}{\sqrt{2\pi\sigma^2}^3} \exp(-|q|^2/(2\sigma^2)) \tag{2}$$

where $Q$ is the total charge, then the solution φ (*r*) of Poisson's equation[121],

$$\nabla^2\varphi(q) = -\frac{\rho(q)}{\varepsilon} \tag{3}$$

is given by

$$\varphi(q) = \frac{Q}{4\pi\varepsilon|q|} \operatorname{erf}\left(\frac{|q|}{\sqrt{2\sigma}}\right) \tag{4}$$

where erf(*x*) is the error function.

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[120] http://en.wikipedia.org/wiki/Harmonic_function

[121] http://en.wikipedia.org/wiki/Poisson%27s_equation

In fact the quaternionic Poisson's equation represents two separate equations:

$$(\nabla_0{}^2 - \boldsymbol{\nabla}^2)\varphi_0(q) = -\frac{\rho_0(q)}{\varepsilon} \tag{5}$$

$$(\nabla_0{}^2 - \boldsymbol{\nabla}^2)\boldsymbol{\varphi}(q) = -\frac{\boldsymbol{\rho}(q)}{\varepsilon} \tag{6}$$

Note that, for $|q|$ much greater than σ, the erf function approaches unity and the potential φ ($r$) approaches the point charge potential $\frac{Q}{4\pi\varepsilon|q|}$, as one would expect. Furthermore the erf function approaches 1 extremely quickly as its argument increases; in practice for $|q| > 3\sigma$ the relative error is smaller than one part in a thousand[122].

The definition of the quaternionic potential $\boldsymbol{\phi}(q)$ is based on the convolution of a quaternionic distribution $\boldsymbol{\rho}(q)$ with the real function $\varphi(q)$ See Newton potential and Bertrand's theorem in Wikipedia. The real part $\rho_0(q)$ of the distribution $\boldsymbol{\rho}(q)$ can be interpreted as a charge distribution. The imaginary part $\boldsymbol{\rho}(q)$ can be interpreted as a current distribution.

The convolution blurs the distribution such that the result becomes differentiable.

In configuration space holds:

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122 http://en.wikipedia.org/wiki/Poisson's_equation#Potential_of_a_Gaussian_charge_density

$$\phi(q) = \rho(q) \circ \frac{1}{|q|}. \tag{7}$$

Reversely, according to Poisson's equation:

$$\rho(q) = = -\Delta\phi(q) \tag{8}$$

The real part of $\phi(q)$ presents a scalar potential. The imaginary part presents a vector potential.

$$\phi(q) = \phi_0(q) + \boldsymbol{\phi}(q) \tag{9}$$

In the above section:

| |
|---|
| The scalar potential is a blurred charge distribution.<br>The vector potential is a blurred current distribution.<br>Current is moving charge. |
| Mass is a form of charge. |

(The selected blurring function has striking resemblance with the ground state of the quantum harmonic oscillator[123]).

In Fourier space holds:

[123] Functions and fields:Functions invariant under Fourier transformation:Ladder operator:Ground state

$$\tilde{\phi}(p) = \tilde{\rho}(p) \cdot \frac{1}{|p|} . = \tilde{\phi}_0(p) + \tilde{\boldsymbol{\phi}}(p) \tag{10}$$

In Fourier space the frequency spectrum of the Hilbert distribution is multiplied with the Fourier transform of the blurring function. When this falls off when the frequencies go to infinity, then as a consequence the frequency spectrum of the potential is bounded. This is valid independent of the fact that the frequency spectrum of the Hilbert distribution is unbounded.

# 7 Fourier transform

The Fourier transform mechanism cannot cope with curvature of the parameter space of the considered functions. Thus the scope of Fourier transforms must be restricted to regions where this curvature is negligible or a reduced accuracy must be accepted.

## 7.1 Quaternionic Fourier transform split

The longitudinal Fourier transform represents only part of the full quaternionic Fourier transform. It depends on the selection of a radial line $\boldsymbol{k}(q)$ in p space that under ideal conditions runs along a straight line.

$$\mathcal{F}_{\mathbf{k}}\big(g(q)\big) = \mathcal{F}(g(q), \boldsymbol{k}(q)) \tag{1}$$

Or

$$\mathcal{F}_{\parallel}\big(g(q)\big) \stackrel{\text{def}}{=} \mathcal{F}\left(g_{\parallel}(q)\right) \tag{2}$$

It relates to the full quaternionic Fourier transform ℱ

$$\mathcal{F}\big(g(q)\big) = \tilde{g}(p) \tag{3}$$

The inverse Fourier transform runs:

$$\mathcal{F}^{-1}(\tilde{g}(p)) = g(q) \tag{4}$$

The split in longitudinal and transverse Fourier transforms corresponds to a corresponding split in the multi-dimensional Dirac delta function.

We consider a field $g(q)$ that equals the quaternionic differentiation of another field $f$ with respect to a selected (ideal) coordinate system $Q$.

$$g(q) = \nabla_q f \tag{5}$$

We use the results of the paragraph on decomposition. We only use the static and imaginary version of field $g(q)$.

For the static imaginary part $\mathbf{g}(q)$ holds:

$$\mathbf{g}(q) = \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)\right) = \boldsymbol{g_l}(q) + \boldsymbol{g_t}(q) \tag{6}$$

In Fourier space differentiation becomes multiplication with the canonical conjugate coordinate $\mathbf{p}$ and therefore the equivalent equation becomes:

$$\tilde{\mathbf{g}}(p) = \mathbf{p}\tilde{f}_0(p) \pm \left(\pm \mathbf{p} \times \tilde{\boldsymbol{f}}(p)\right) = \tilde{\boldsymbol{g}}_{\boldsymbol{l}}(p) + \tilde{\boldsymbol{g}}_{\boldsymbol{t}}(p) \tag{7}$$

Since

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f_0(q) = \mathbf{0} \rightarrow \boldsymbol{\nabla} \times \boldsymbol{g_l}(q) = \mathbf{0} \tag{8}$$

and

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{f}(q) \rangle = 0 \rightarrow \langle \boldsymbol{\nabla}, \boldsymbol{g_t}(q) \rangle = 0 \tag{9}$$

Now we take

$$< q \left| \check{P} f > = \hbar \cdot \nabla_q < q \right| f > = \hbar \cdot \nabla_q f^*(q) = \mathrm{g}(q) \tag{10}$$

$$= \int_{\boldsymbol{p}} < q|p > \cdot < p|g >$$

The static imaginary part is

$$< q\big|\breve{\boldsymbol{P}}\, f > = \hbar \cdot \boldsymbol{\nabla}_q < q\big|f > = \hbar \cdot \boldsymbol{\nabla}_q \boldsymbol{f}^*(q) = \mathbf{g}(q) \tag{11}$$

$$= Im\left(\int_{\boldsymbol{p}} < q|p > \cdot < p|\boldsymbol{g} >\right)$$

$$= \int_{\boldsymbol{p}} Im(< q|p > \cdot < p|\boldsymbol{g} >)$$

$$= \int_{\boldsymbol{p}} Im(< q|p > \cdot < p|\boldsymbol{g_l} >) + \int_{\boldsymbol{p}} Im(< q|p > \cdot < p|\boldsymbol{g_t} >)$$

$$= \int_{\boldsymbol{p}} Im(< q|p > \cdot\, \tilde{\boldsymbol{g}}_{\boldsymbol{l}}(p)) + \int_{\boldsymbol{p}} Im(< q|p > \cdot\, \tilde{\boldsymbol{g}}_{\boldsymbol{t}}(p))$$

The left part is the longitudinal inverse Fourier transform of field $\tilde{\boldsymbol{g}}(p)$.

The right part is the transverse inverse Fourier transform of field $\tilde{\boldsymbol{g}}(p)$.

For the Fourier transform of $\mathbf{g}(q)$ holds the split:

$$\tilde{\boldsymbol{g}}(p) = \int_{\boldsymbol{q}} Im(< p|q >\cdot \boldsymbol{g_l}(q)) + \int_{\boldsymbol{p}} Im(< p|q >\cdot \boldsymbol{g_t}(q)) = \int_{\boldsymbol{q}} Im(< p|q >\cdot \boldsymbol{g}(q)) \tag{12}$$

The longitudinal direction is a one dimensional (radial) space. The corresponding transverse direction is tangent to a sphere in 3D. Its direction depends on the field $\mathbf{g}(q)$ or alternatively on the combination of field $f$ and the selected (ideal) coordinate system $\breve{Q}$.

For a weakly curved coordinate system $\breve{\Omega}$ the formulas hold with a restricted accuracy and within a restricted region.

## 7.2 Alternative transverse plane

The Cayley-Dickson construction, as well as Warren Smith's construction formula shows that the transverse part can be considered as a complex number space multiplied with a fixed imaginary quaternionic base number. The selection of the imaginary base number $\boldsymbol{i}$ is arbitrary as long as it is perpendicular to $\boldsymbol{k}$. The resulting plane is spanned by axes $\boldsymbol{i}$ and $\boldsymbol{ik}$. When base number $\boldsymbol{i}$ is divided away, then a normal complex number space results.

Also here a complex Fourier transform can be defined in a way that is similar to the longitudinal Fourier transform. It must be reckoned that the sign selections for these directions differ.

## 7.3 Alternative approach to Fourier transform

The following draws from the work of S. Thangavelu[124].

Let us take the non-abelian group $\mathbb{H}_1$ which is $\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}$ with the group law

$$(x, y, t)(x_0, y_0, t_0) = (x + x_0;\ y + y_0;\ t + t_0 + x\, y_0) \tag{1}$$

Then it is clear that $\mathbb{H}1$ is non-abelian and the Lebesgue measure d$x$ d$y$ d$t$ is both left and right invariant Haar measure on $\mathbb{H}_1$. With this measure we can form the Hilbert space $L^2(\mathbb{H}_1)$. Let $\Gamma = \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}$. Then it is easy to check that $\Gamma$ is a subgroup of $\mathbb{H}_1$ so that we can form the quotient $M = \Gamma/\mathbb{H}_1$ consisting of all right cosets of $\Gamma$. Functions on M are naturally identified with left $\Gamma$-invariant functions on $\mathbb{H}_1$. As the Lebesgue measure d$x$ d$y$ d$t$ is left $\Gamma$-invariant we can form $L_2(M)$ using the Lebesgue measure restricted to M. As a set we can identify M with $[0, 1)^3$ and we just think of $L^2(M)$ as $L^2([0, 1)^3)$.

Fourier expansion in the last variable allows us to decompose $L^2(M)$ into a direct sum of orthogonal subspaces. Simply define $\mathcal{H}_k$ to be the set of all $f \in L^2(M)$ which satisfy the condition

$$f(x, y, t + s) = \exp(2\pi i k s)\, f(x, y, t) \tag{2}$$

Then $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_j$ whenever $k \neq j$ and any $f \in L^2(M)$ has the unique expansion

[124] http://www.math.iitb.ac.in/atm/faha1/veluma.pdf

$$f = \sum_{k=-\infty}^{\infty} f_k \,;\; f_k \in \mathcal{H}_k \tag{3}$$

In quaternionic terms, the split sees *ik* as imaginary quaternion $\boldsymbol{k}$ and the quaternionic Hilbert space is split in components according to the imaginary direction of $\boldsymbol{k}$, where the choice is between three mutually perpendicular directions.

For the moment, we are mainly interested in $\mathcal{H}_1$ which is a Hilbert space in its own right. It is interesting to note that functions in $\mathcal{H}_1$ are also invariant under the left action of Γ.

Our next example of a unitary operator is the following. Consider the map $J : \mathcal{H}_1 \to \mathcal{H}_1$ given by

$$J(x,y,t) = (-x\,, y, t - xy) \tag{4}$$

$$J^{\dagger}(x,y,t) = (x, -y, t - xy) \tag{5}$$

$$J^{\dagger} = J^{-1} \tag{6}$$

$$J^2(x,y,t) = J(-x,\cdot\, y;\; t - xy) = (-x, -\,y; t) \tag{7}$$

$$J^4 = I \tag{8}$$

$$J(0,0,t) = (0,0,t) \tag{9}$$

$$J\,f(x,y,t) = f(J(x,y;\,t)) = f(-x,\cdot\, y, t - xy) \tag{10}$$

## 7.4 Weil-Brezin transform

Next consider the Weil-Brezin transform V:

$$V\,f(x,y,t) = exp(2\,\pi\,\boldsymbol{k}\,t)\sum_{n} f(x + n)\exp(2\,\pi\,\boldsymbol{k}\,n\,y) \tag{1}$$

$$\int_{y=0}^{1} |V\,f(x,y,t)|^2 dy = \int_{x=0}^{1}\sum_{n=-\infty}^{n=\infty} |f(x\,+\,n)|^2\,dx \tag{2}$$

$$\iiint_{0}^{1} |V\,f(x,y,t)|^2 dx\,dy\,dt = \int_{0}^{1} |\,f(x)|^2 dx \tag{3}$$

V is unitary.
See also Zak transform

## 7.5 Fourier transform

We define the Fourier transform $\mathcal{F}$ by:

$$\mathcal{F} = V^{\dagger}\,J\,V \tag{1}$$

- $\mathcal{F}^4\,f = f$; for every $f \in L^2(\mathbb{R})$
- $\mathcal{F}^2\,f(x) = f(-x)$; for almost every $x \in \mathbb{R}$
- $\|\mathcal{F}\,f\|^2 = \|f\|^2$

For $f \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$ the Fourier transform is given by

$$\mathcal{F}\mathrm{f}(\xi) = \int_{x \in \mathbb{R}} \mathrm{f}(\mathrm{x}) \cdot \exp(2\,\pi\, \boldsymbol{k}\, x) dx \tag{2}$$

If we further assume that $\mathcal{F}f \in L^1(\mathbb{R})$ then for almost every x we have

$$\mathrm{f}(\xi) = \int_{\xi \in \mathbb{R}} \mathcal{F}\mathrm{f}(\xi) \exp(2\,\pi\, \boldsymbol{k}\, x\, \xi) dx \tag{3}$$

## 7.6 Functions invariant under Fourier transform

In this section we confine to a complex part of the Hilbert space.
See http://en.wikipedia.org/wiki/Hermite_polynomials.
There exist two types of Hermite polynomials: (1, 2)

1. The probalist's Hermite polynomials:

$$H_n^{prob}(z) = (-1)^n \exp(½z^2)\, \frac{d^n}{dz^n} \exp(-½z^2).$$

2. The physicist's Hermite polynomials

$$H_n^{phys}(z) = (-1)^n \exp(z^2) \frac{d^n}{dx^n} \exp(-z^2)$$
$$= \exp(½z^2) \left(z - \frac{d}{dz}\right) \exp(-½z^2)$$

These two definitions are *not* exactly equivalent; either is a rescaling of the other:

$$H_n^{phys}(z) = 2^{n/2}\, H_n^{prob}\left(z\sqrt{2}\right) \tag{3}$$

In the following we focus on the physicist's Hermite polynomials.

The Gaussian function φ(z) defined by

$$\varphi(x) \;=\; exp(-\pi\, z^2) \tag{4}$$

is an eigenfunction of $\mathcal{F}$. It means that its Fourier transform has the same form.

As $\mathcal{F}^4 = \mathrm{I}$ any λ in its spectrum $\sigma\,(\mathcal{F})$ satisfies $\lambda^4 = 1$: Hence,

$$\sigma\,(\mathcal{F}) \;=\; \{1; -1;\; i; -i\}. \tag{5}$$

We take the Fourier transform of the expansion:

$$exp(-½\, z^2 +\; 2\, z\, c -\; c^2) \;=\; \sum_{n=0}^{\infty} exp(-½\, z^2)\, H_n(z)\, c^n /n! \tag{6}$$

First we take the Fourier transform of the left hand side:

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z)\, exp(-½\, z^2 \;+\; 2\, z\, c -\; c^2)\; dz \tag{7}$$

$$= exp(-½\, p_z^2 - 2\, \boldsymbol{k}\, p_z\, c + c^2)$$

$$= \sum_{n=0}^{\infty} exp(-½\, p_z^2)\, H_n(p_z)\, (-\boldsymbol{k}\, c)^n /n!$$

The Fourier transform of the right hand side is given by

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\, H_n(z)\, c^n/n!\; dz \tag{8}$$

Equating like powers of $c$ in the transformed versions of the left- and right-hand sides gives

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} \exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\, H_n(z)\, c^n/n!\; dz \tag{9}$$

$$= (-\boldsymbol{k})^n \cdot \exp(-½\, p_z^2)\; H_n(p_z)\; \frac{c^n}{n!}$$

Let us define the Hermite functions $\psi_n(z)$

$$\psi_n(z) \stackrel{\text{def}}{=} < z|\psi_n > = \mathrm{c}_n \exp(-½\, z^2)\; H_n(z) \tag{10}$$

$$|\mathcal{F}\, \psi_n > = |\psi_n > (-\boldsymbol{k})^n \tag{11}$$

with suitably chosen $c_n$ so as to make

$$\|\psi_n\|^2 = 1 \tag{12}$$

$$\mathrm{c}_n = \frac{1}{\sqrt{2^n n!\, \sqrt{\pi}}} \tag{13}$$

The importance of the Hermite functions lie in the following theorem.

"The Hermite functions $\psi_n$; $n \in \mathbb{N}$ form an orthonormal basis for $L^2(\mathbb{R})$"

Consider the operator

$$H = -½ \frac{d^2}{dz^2} + ½\, z^2 \tag{14}$$

Apply this to $\psi_n(z)$:

$$H \cdot \psi_n(z) = (½ + n)\, \psi_n(z) \tag{15}$$

Thus, $\psi_n$ is an eigenfunction of *H*.

Let $f = \psi_{4k+j}$ be any of the Hermite functions. Then we have

$$\sum_{n=-\infty}^{\infty} f(y + n) \cdot \exp\left(-2\pi \boldsymbol{k} x (y+n)\right) = (-\boldsymbol{k})^j \sum_{n=-\infty}^{\infty} f(x + n)\, exp(2\pi \boldsymbol{k} n y) \tag{16}$$

Proof: As

$$\mathcal{F} = V^{\dagger} J V \tag{17}$$

the equation

$$\mathcal{F} f = (-\boldsymbol{k})^j f \tag{18}$$

translates into

$$JVf(x;\, y;\, t) = (-\boldsymbol{k})^j\, Vf(x;\, y;\, t) \tag{19}$$

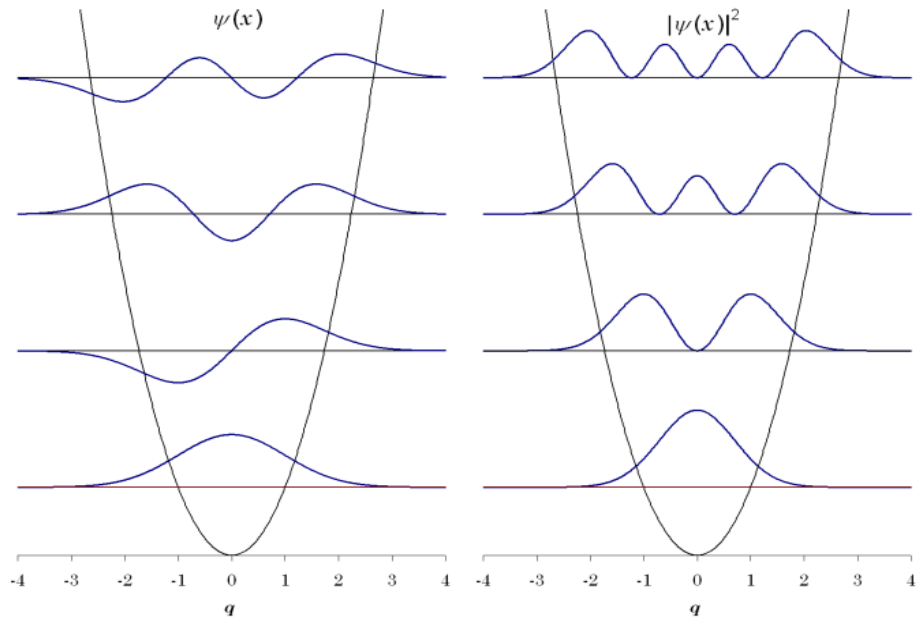
With the definition of *V* and $t = xy$:

$$V f(x,y,t) = exp(2\pi \boldsymbol{k} t) \sum_{n} f(x + n)\, exp(2\pi \boldsymbol{k} n y) \tag{20}$$

QED.

The vectors $|\psi_n\rangle$ are eigenvectors of the Fourier transform operator with eigenvalues $(-\boldsymbol{k})^n$. The eigenfunctions $\psi_n(x)$ represent eigenvectors $|\psi_n\rangle$ that span the complex Hilbert space $\mathbf{H}_{\boldsymbol{k}}$.

For higher $n$ the central parts of $\psi_n(x)$ and $|\psi_n(x)|^2$ become a sinusoidal form.



A **coherent state**[125] is a specific kind of state[126] of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system. The ground state is a squeezed coherent state[127].

The ground state here differs from the ground state of the QPAD. That ground state equals zero in the close neighborhood of the center. The size of that neighborhood is of the order of the Planck length. Thus in this region the QPAD has the form of a stretched turban mold. It has a form similar to the second state in the picture of $|\psi(x)|^2$, thus the lowest state where $\psi(x)$ is asymmetric. Asymmetric states are better localizable than symmetric states.

[125] http://en.wikipedia.org/wiki/Coherent_state
[126] States
[127] Canonical conjugate: Heisenberg's uncertainty

## 7.7 Special Fourier transform pairs

Functions that keep the same form through Fourier transformation are:

$$f(q) = \exp(-|q|^2) \tag{1}$$

$$f(q) = \frac{1}{|q|} \tag{2}$$

$$f(q) = comb(q) \tag{3}$$

The comb function consists of a set of equidistant Dirac delta functions.

Other examples of functions that are invariant under Fourier transformation are the linear and spherical harmonic oscillators and the solutions of the Laplace equation.

## 7.8 Complex Fourier transform invariance properties

Each even function $f(q) \iff \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) + \tilde{f}(q). \tag{1}$$

$$\tilde{h}(q) = \sqrt{2\pi}\, h(q) \tag{2}$$

Each odd function $f(q) \iff \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) - \tilde{f}(q). \tag{3}$$

A function $f(q)$ is invariant under Fourier transformation *if and only if* the function $f$ satisfies the differential equation

$$\frac{\partial^2 f(q)}{\partial q^2} - t^2 f(q) = \alpha\, f(q), \text{ for some scalar } \alpha \in C. \tag{4}$$

The Fourier transform invariant functions are fixed apart from a scale factor. That scale factor can be 1, $\boldsymbol{k}$, -1 or $-\boldsymbol{k}$. $\boldsymbol{k}$ is an imaginary base number in the longitudinal direction.

Fourier-invariant functions show iso-resolution, that is, $\Delta_p = \Delta_q$ in the Heisenberg's uncertainty relation.

For proves see: http://www2.ee.ufpe.br/codec/isoresolution_vf.pdf.

## 7.9 Fourier transform properties

### 7.9.1 Parseval's theorem

Parseval's theorem runs:

$$\int f^*(q) \cdot g(q) \cdot dV_q = \int \tilde{f}^*(p) \cdot \tilde{g}(p) \cdot dV_p \tag{1}$$

This leads to

$$\int |f(q)|^2 \cdot dV_q = \int \left|\tilde{f}(p)\right|^2 \cdot dV_p \tag{2}$$

### 7.9.2 Convolution

Through Fourier transformation a convolution changes into a simple product and vice versa.

$$\mathcal{F}(f(q) \circ g(q)) = \tilde{f}(p) \cdot \tilde{g}(p) \tag{1}$$

### 7.9.3 Differentiation

Fourier transformation converts differentiation into multiplication with the canonical conjugated coordinate.

$$\mathrm{g}(q) = \nabla f(q) \tag{1}$$

$$\tilde{\mathrm{g}}(p) = \mathrm{p}\tilde{f}(p) \tag{2}$$

# 8 Ladder operator

The Hermite functions $\psi_n$ represent Fock states[128].

Boson ladder operators are characterized by

$$\mathcal{A}|\psi_n> = \sqrt{n}\,|\psi_{n-1}> \tag{1}$$

$$\mathcal{A}^\dagger|\psi_n> = \sqrt{n+1}\,|\psi_{n+1}> \tag{2}$$

$$\mathcal{A} = \frac{1}{\sqrt{2}}\left(c_1\frac{d}{dq} + c_2 q\right) = -\boldsymbol{k}\,\check{P}\sqrt{\frac{1}{2\,\hbar\,m\,\omega}} + \check{Q}\sqrt{\frac{m\,\omega}{2\,\hbar}}$$

$$\mathcal{A}^\dagger = \frac{1}{\sqrt{2}}\left(-c_1\frac{d}{dq} + c_2 q\right) = \boldsymbol{k}\,\check{P}\sqrt{\frac{1}{2\,\hbar\,m\,\omega}} + \check{Q}\sqrt{\frac{m\,\omega}{2\,\hbar}}$$

In the Heisenberg picture, the operators have the following time dependence:

$$\mathcal{A}(t) = \mathcal{A}(t_0)\exp\left(-\boldsymbol{k}\,\omega\,(\mathrm{t} - t_0)\right) \tag{3}$$

$$\mathcal{A}^\dagger(t) = \mathcal{A}^\dagger(t_0)\exp\left(\boldsymbol{k}\,\omega\,(\mathrm{t} - t_0)\right) \tag{4}$$

We can also define an enumeration operator *N* which has the following property:

$$N = \mathcal{A}^\dagger\,\mathcal{A} \tag{5}$$

[128] http://en.wikipedia.org/wiki/Fock_state

$$N|\psi_n> = \ |\psi_n> \ n \tag{6}$$

In deriving the form of $\mathcal{A}^\dagger$, we have used the fact that the operators $X$ and $P_x$, which represent observables, are Hermitian. These observable operators can be expressed as a linear combination of the ladder operators as

$$\check{Q}(t) \ = \ \sqrt{\frac{\hbar}{2\,m\,\omega}} \ \ (\mathcal{A}^\dagger(t) + \mathcal{A}(t)) \tag{7}$$

$$\check{P}(t) \ = \ \boldsymbol{k}\sqrt{½\,\hbar\,m\,\omega} \ \ (\mathcal{A}^\dagger(t) - \mathcal{A}(t)) \tag{8}$$

The $\check{\boldsymbol{Q}}$ and $\check{\boldsymbol{P}}$ operators obey the following identity, known as the canonical commutation relation:

$$\left[\check{Q}, \check{P}\right] = \ \boldsymbol{k}\,\hbar \tag{9}$$

Using the above, we can prove the identities

$$H \ = \ \hbar\,\omega\,(\mathcal{A}^\dagger\,\mathcal{A} \ + \ ½) = \ \hbar\,\omega\,(N \ + \ ½) \tag{10}$$

$$[\mathcal{A}^\dagger, \mathcal{A}] \ = \ 1 \tag{11}$$

Now, let $|f_E>$denote an energy eigenstate with energy $E$. The inner product of any ket with itself must be non-negative, so

$$< \ \psi_E\ \mathcal{A}|\mathcal{A}\ \psi_E \ > = < \ \psi_E E|\ \mathcal{A}^\dagger\,\mathcal{A}\ \psi_E \ > \geq \ 0 \tag{12}$$

Expressing $\mathcal{A}^\dagger\mathcal{A}$ in terms of the Hamiltonian $H$:

$$< \psi_E|(H/(\hbar\,\omega)\ -\ ½)\,\psi_E >=\ (E/(\hbar\,\omega)\ -\ ½)\ \geq\ 0 \tag{13}$$

so that

$$E\ \geq\ ½\,\hbar\,\omega. \tag{14}$$

Note that when $|\mathcal{A}\ \psi_E >=\ |0>$ (is the zero ket i.e. a ket with length zero), the inequality is saturated, so that

$$E = ½\,\hbar\,\omega \tag{15}$$

It is straightforward to check that there exists a state satisfying this condition; it is the ground state

$$|\psi_{ground} >=\ |\psi_{E_n} >;\ (n\ =\ 0) \tag{16}$$

Using the above identities, we can now show that the commutation relations of $\mathcal{A}$ and $\mathcal{A}^\dagger$ with $H$ are:

$$[H, \mathcal{A}] =\ -\hbar\,\omega\,\mathcal{A} \tag{17}$$

$$[H, \mathcal{A}^\dagger] =\ \hbar\,\omega\,\mathcal{A}^\dagger \tag{18}$$

Thus, provided $|\ \mathcal{A}\ \psi_E >$ is not the zero ket,

$$\begin{aligned} |H\,\mathcal{A}\,\psi_E >=\ |[H, \mathcal{A}]\ +\ \mathcal{A}\,H\,\psi_E > \\ =\ |-\hbar\,\omega\,\mathcal{A}\ +\mathcal{A}\,E\,\psi_E > \\ =\ |-\hbar\,\omega\,\mathcal{A}\ +\ \mathcal{A}\,E\,\psi_E > \end{aligned} \tag{19}$$

$$= (E - \hbar\omega)\,|\mathcal{A}\,\psi_E>$$

Similarly, we can show that

$$|H\,\mathcal{A}^\dagger\,\psi_E> = (E + \hbar\omega)\,|\mathcal{A}^\dagger\,\psi_E> \tag{20}$$

In other words, $\mathcal{A}$ acts on an eigenstate of energy $E$ to produce, up to a multiplicative constant, another eigenstate of energy $E - \hbar\omega$, and $\mathcal{A}^\dagger$ acts on an eigenstate of energy $E$ to produce an eigenstate of energy $E + \hbar\omega$. For this reason, $a$ is called a "lowering operator", and $\mathcal{A}^\dagger\mathcal{A}$ "raising operator". The two operators together are called ladder operators. In quantum field theory, $\mathcal{A}$ and $\mathcal{A}^\dagger$ are alternatively called "annihilation" and "creation" operators because they destroy and create particles, which correspond to our quanta of energy.

Given any energy eigenstate, we can act on it with the lowering operator $\mathcal{A}$, to produce another eigenstate with $\hbar\omega$-less energy. By repeated application of the lowering operator, it seems that we can produce energy eigenstates down to $E = -\infty$. However, this would contradict our earlier requirement that $E \geq \hbar\omega/2$.

# 9 States

## 9.1 Ground state

Therefore, there must be a ground-state energy eigenstate, which we label $|f_{ground}>$, such that

$$|\mathcal{A}\,\psi_{ground} > = \; |0 >; \text{(zero ket)}. \tag{1}$$

In this case, subsequent applications of the lowering operator will just produce zero kets, instead of additional energy eigenstates. Furthermore, we have shown above that

$$|H\,\psi_{ground} > = \; (½\,\hbar\,\omega)\,|\psi_{ground} > \tag{2}$$

Finally, by acting on $|\psi_{ground} >$ with the raising operator and multiplying by suitable normalization factors, we can produce an infinite set of energy eigenstates

$$\{|\psi_{ground} >, |\psi_{E_1} >, \; |\psi_{E_2} >, \ldots . , |\psi_{E_n} >\}, \tag{3}$$

such that

$$|H\,\psi_{E_n} > = \; \hbar\,\omega\,(n \; + ½)\,|\psi_{E_n} > \tag{4}$$

which matches the energy spectrum.

This method can also be used to quickly find the ground state state function of the quantum harmonic oscillator.

Indeed

$$|\mathcal{A}\,\psi_{ground} > = \; |0 > \tag{5}$$

becomes

$$\psi_{ground}(x) = \psi_0(x) \stackrel{\text{def}}{=} < x | \psi_{ground} > = -\frac{\hbar}{m\,\omega}\frac{d}{dx}\psi_0(x) \tag{6}$$

so that

$$d\,\psi_0(x) = \psi_0(x)\;\frac{\hbar}{m\,\omega}\,x\,d\,x \;\Rightarrow \ln\left(\psi_0(x)\right) = \frac{m\,\omega}{2\hbar}\,x^2 \;+\; const \tag{7}$$

After normalization this leads to the following position space representation of the ground state state function.

$$\psi_0(x) \;=\; \sqrt[4]{\frac{m\,\omega}{\pi\hbar}}\;\; e^{-\frac{m\,\omega}{2\hbar}x^2} \tag{8}$$

## 9.2 Coherent state

A **coherent state** is a specific kind of state[129] of the quantum harmonic oscillator[130] whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system.

The coherent state |α> is defined to be the 'right' eigenstate of the annihilation operator $\mathcal{A}$. Formally, this reads:

---

[129]States

[130] Functions invariant under Fourier transform

$$|\mathcal{A}\ \alpha> = \alpha|\alpha> \tag{1}$$

Since $\mathcal{A}$ is not Hermitian, α is a hyper complex number that is not necessarily real, and can be represented as

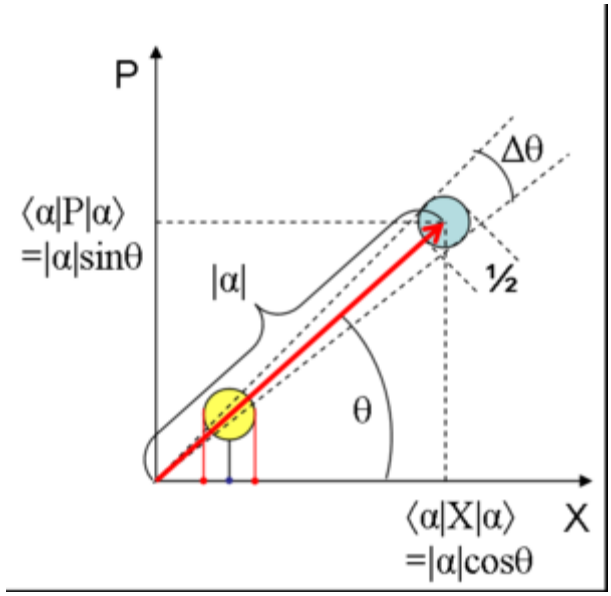
$$\alpha = |\alpha| \exp(\boldsymbol{k}\ \theta) \tag{2}$$

where $\theta$ is a real number. $|\alpha|$ is the amplitude and $\theta$ is the phase of state |α>.

This formula means that a coherent state is left unchanged by the annihilation or the creation of a particle. The eigenstate of the annihilation operator has a Poissonian[131] number distribution A Poisson distribution is a necessary and sufficient condition that all annihilations are statistically independent.

The coherent state's location in the complex plane (phase space[132]) is centered at the position and momentum of a classical oscillator of the same phase θ and amplitude. As the phase increases the coherent state circles the origin and the corresponding disk neither distorts nor spreads. The disc represents Heisenberg's uncertainty. This is the most similar a quantum state can be to a single point in phase space.

[131] http://en.wikipedia.org/wiki/Poissonian
[132] http://en.wikipedia.org/wiki/Phase_space




Phase space plot of a coherent state. This shows that the uncertainty (blur) in a coherent state is equally distributed in all directions. The horizontal and vertical axes are the X and P quadratures of the field, respectively. Oscillations that are said to be *in quadrature*, if they are separated in phase by $\pi/2$ radians. The red dots on the x-axis trace out the boundaries of the quantum noise. Further from the origin the relative contribution of the quantum noise becomes less important.

The representation of the coherent state in the basis of Fock states is:

$$|\alpha> = \exp(-½|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n> \\ = \exp(-½|\alpha|^2) \exp\left(\alpha \mathcal{A}^\dagger\right) |0> \tag{3}$$

where |n> are Hermite functions (eigenvectors of the Hamiltonian). This is a Poissonian distribution. The probability of detecting *n* photons is:

$$\mathcal{P}(n) = \exp(-\langle n \rangle) \frac{\langle n \rangle^n}{n!} \tag{4}$$

Similarly, the average photon number in a coherent state is

$$\langle n \rangle = \langle \mathcal{A}^\dagger \mathcal{A} \rangle = |\alpha|^2 \tag{5}$$

and the variance is

$$(\Delta n)^2 = Var\left(\mathcal{A}^\dagger \mathcal{A}\right) = |\alpha|^2 \tag{6}$$

## 9.3 Squeezing

The squeezing operator can squeeze a state more or less in the direction of either P or Q. The operator is defined as:

$$Sq(z) = \exp\left(½\left(z^* \mathcal{A} + z \mathcal{A}^\dagger\right)\right) \tag{1}$$

$$z = r \exp(\boldsymbol{k}\, \theta) \tag{2}$$

The ground state is a saturated squeezed coherent state where

$$\Delta p = \Delta q \text{ and } \Delta q \cdot \Delta p = \hbar/2 \tag{3}$$

# 10 Base transforms

Now we have discovered the following base transforms:
Position⇔momentum:

$$< q|p > = \frac{1}{\sqrt{2\pi\hbar}} \exp(\frac{\boldsymbol{k}\, q\, p}{\hbar}) \tag{1}$$

Position⇔Fock state:

$$< q|n > = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n\, n!}} \exp(-\frac{m\omega}{2\hbar}\, q^2) H_n(q\sqrt{\frac{m\omega}{\hbar}}) \tag{2}$$

Fock state⇔coherent state:

$$< n|z > = \frac{1}{\sqrt{n!}}\, z^n \exp(-½|z|^2) \tag{3}$$

# 11 Oscillations

## 11.1 Harmonic oscillating Quaternionic distribution

Take the ingredients of the complex harmonic oscillator and interpret these as similar ingredients of a harmonic oscillating Quaternionic distribution that is based on a Gaussian blur. The blur delivers the conditions of the ground state.

$$\psi_0(r) = \sqrt[4]{\frac{m\,\omega}{\pi\hbar}}\; e^{-\frac{m\,\omega}{2\hbar}r^2} \tag{1}$$

This means that the ground state corresponds with a Gaussian charge distribution. Higher states correspond to a blurred current. We indicate this current as vector potential $\boldsymbol{\varphi}$. Its time derivative $\dot{\boldsymbol{\varphi}}$ is perpendicular to $\boldsymbol{\varphi}$. The other ingredients are $P$, $Q$, $\mathcal{A}$ and $\mathcal{A}^\dagger$.

$$Q \Leftrightarrow \boldsymbol{\varphi} = \varphi_x = \sqrt{\frac{\hbar}{2m\omega}}(\mathcal{A}_x + \mathcal{A}_x^\dagger) \tag{2}$$

$$\mathcal{A}_x - \mathcal{A}_x^\dagger = 0 \tag{3}$$

$$P \Leftrightarrow m\dot{\boldsymbol{\varphi}} = m\dot{\varphi}_y = \sqrt{\frac{m\omega\hbar}{2}}(-\mathcal{A}_y + \mathcal{A}_y^\dagger) \tag{4}$$

$$\mathcal{A}_y + \mathcal{A}_y^\dagger = 0 \tag{5}$$

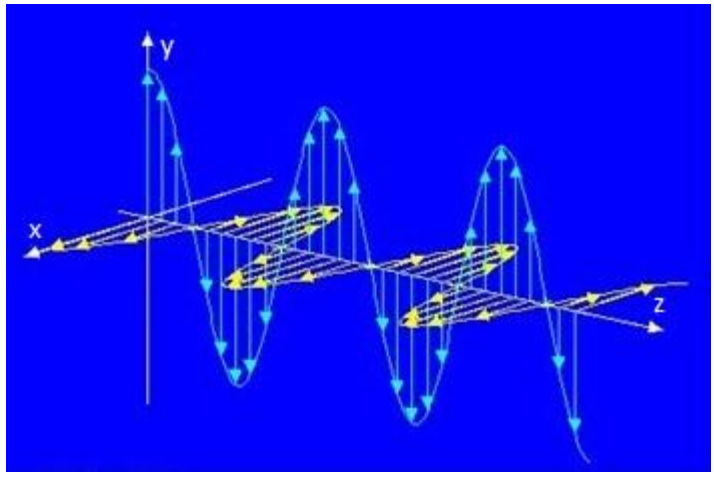
$$\begin{aligned}\mathcal{A} \Leftrightarrow \boldsymbol{\mathcal{A}} = \boldsymbol{i}\mathcal{A}_x - \boldsymbol{ik}\mathcal{A}_y &= \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{\varphi} + \frac{\dot{\boldsymbol{\varphi}}}{\omega}\right) \\ &= \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{i}\varphi_x + \boldsymbol{ik}\frac{\dot{\varphi}_y}{\omega}\right)\end{aligned} \tag{6}$$

$$\begin{aligned}\mathcal{A}^\dagger \Leftrightarrow \boldsymbol{\mathcal{A}}^\dagger = \boldsymbol{i}\mathcal{A}_x^\dagger + \boldsymbol{ik}\mathcal{A}_y^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{\varphi} - \frac{\dot{\boldsymbol{\varphi}}}{\omega}\right) \\ &= \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{i}\varphi_x - \boldsymbol{ik}\frac{\dot{\varphi}_y}{\omega}\right)\end{aligned} \tag{7}$$

The $\boldsymbol{\varphi}$ field and the $\dot{\boldsymbol{\varphi}}$ field are mutually perpendicular. If both fields are subjected to a synchronized quantum harmonic oscillation, then an oscillating wave results. We take the same ground state for each of the fields. These ground states correspond to a spherical symmetric Gaussian blur.

When bounds of the cavity are removed or relaxed, then the higher order modes may differ in a phase shift. The sign selections set the eigenvalues of the spin operator. The result is an elliptically polarized wave that moves in directions along $\boldsymbol{\varphi} \times \dot{\boldsymbol{\varphi}}$.

$\boldsymbol{\varphi}$ no longer stands for a single position, but instead for a Gaussian distribution of positions of virtual charges. Similarly $\dot{\boldsymbol{\varphi}}$ does not stand for a single moving particle, but for a moving Gaussian cloud of virtual charges.



## 11.2 Annihilator and creator

The annihilator $\mathcal{A}$ and the creator $\mathcal{A}^\dagger$ are examples of boson operators. This is a consequence of their commutation relations.

$$\mathcal{A} + \mathcal{A}^\dagger = \alpha\, \boldsymbol{\varphi} \tag{1}$$

$$\mathcal{A} - \mathcal{A}^\dagger = \beta\, \dot{\boldsymbol{\varphi}} \tag{2}$$

$$\mathcal{A} = \tfrac{1}{2}\alpha\, \boldsymbol{\varphi} + \tfrac{1}{2}\beta\, \dot{\boldsymbol{\varphi}} \tag{3}$$

$$\mathcal{A}^\dagger = \tfrac{1}{2}\alpha\, \boldsymbol{\varphi} - \tfrac{1}{2}\beta\, \dot{\boldsymbol{\varphi}} \tag{4}$$

$$\left[\mathcal{A}(f), \mathcal{A}^\dagger(g)\right] = < f | g > \tag{5}$$

$$[\mathcal{A}(f), \mathcal{A}(g)] = 0 \tag{6}$$

$$\left[\mathcal{A}^\dagger(f), \mathcal{A}^\dagger(g)\right] = 0 \tag{7}$$

The corresponding fermion operators are:

$$\left\{\mathcal{B}(f), \mathcal{B}^\dagger(g)\right\} = < f | g > \tag{8}$$

$$\{\mathcal{B}(f), \mathcal{B}(g)\} = 0 \tag{9}$$

$$\left\{\mathcal{B}^\dagger(f), \mathcal{B}^\dagger(g)\right\} = 0 \tag{10}$$

The fermion operators can be represented by imaginary quaternionic base numbers:

$$\mathcal{B} + \mathcal{B}^\dagger = \boldsymbol{i} \tag{11}$$

$$\mathcal{B} - \mathcal{B}^\dagger = \boldsymbol{j} \tag{12}$$

$$\mathcal{B} = \tfrac{1}{2}(\boldsymbol{i} + \boldsymbol{j}) \tag{13}$$

$$\mathcal{B}^\dagger = \tfrac{1}{2}(\boldsymbol{i} - \boldsymbol{j}) \tag{14}$$

$$\left(\mathcal{B} + \mathcal{B}^\dagger\right)\left(\mathcal{B} - \mathcal{B}^\dagger\right) = \mathcal{B}\mathcal{B} - \mathcal{B}\mathcal{B}^\dagger + \mathcal{B}^\dagger\mathcal{B} - \mathcal{B}^\dagger\mathcal{B}^\dagger \tag{15}$$

$$= \mathcal{B}^\dagger\mathcal{B} - \mathcal{B}\mathcal{B}^\dagger = \boldsymbol{i}\,\boldsymbol{j}$$

### 11.3 Rotational symmetry

In case of rotational symmetry in the imaginary part of quaternion space, the exponential function must be replaced by a Bessel function. The corresponding Fourier transform then becomes a Hankel transform[133].

The spherical harmonics are eigenfunctions of the square of the orbital angular momentum operator $-i\hbar\boldsymbol{r} \times \boldsymbol{\nabla}$ and therefore they represent the different quantized configurations of atomic orbitals.

[133] http://en.wikipedia.org/wiki/Hankel_transform

## 11.4 Spherical harmonics

The following draws from the work of S. Thangavelu[134].

In this subsection we look for eigenfunctions of the Fourier transform which have spherical symmetry. As in the one dimensional case we consider functions of the form

$$f(x) = p(x)\exp(-\pi|x|^2) \tag{1}$$

This will be an eigenfunction of $\mathcal{F}$ if and only $p$ satisfies

$$\int_{\mathbb{R}^n} p(x - iy)\exp(-\pi \cdot |x|^2)\, dx = \lambda\, p(y) \tag{2}$$

Here in quaternion terms $x$ and $iy$ represent two mutually perpendicular imaginary numbers while $x$ and $y$ are parallel. Thangavelu uses complex numbers. We keep as close as is possible to his text.

If (2) is true for all $y \in \mathbb{R}^n$ then we should also have

$$\int_{\mathbb{R}^n} p(x + y)\exp(-\pi \cdot |x|^2)\, dx = \lambda\, p(iy) \tag{3}$$

Integrating in polar coordinates the integral on the left is

$$\int_{r=0}^{\infty} |\mathbb{S}^{n-1}| \left( \int_{\mathbb{S}^{n-1}} p(y + r\,\omega)\, d\sigma(\omega) \right) exp(-\pi \cdot r^2) \cdot r^{n-1} dr \tag{4}$$

[134] http://www.math.iitb.ac.in/atm/faha1/veluma.pdf

where $d\sigma(\omega)$ is the normalised surface measure on the unit sphere $\mathbb{S}^{n-1}$.

If $p$ is homogeneous of degree $m$ then

$$p(i\,y) = i^m\,p(y) \tag{5}$$

and hence for such polynomials the equation

$$\int_{\mathbb{R}^n} p(x+y)\,\exp(-\pi\cdot|x|^2)\,dx = \lambda\,i^m\,p(iy) \tag{6}$$

will be satisfied for

$$\lambda = (-i)^m \tag{7}$$

if $p$ has the mean value property

$$\int_{\mathbb{S}^{n-1}} p(y + r\,\omega)d\sigma(\omega) = p(y) \tag{8}$$

Such functions are precisely the harmonic functions satisfying

$$\Delta u = 0 \tag{9}$$

Thus we have proved:

Let

$$f(x) = p(x)\,exp(-\pi\,|x|^2) \tag{10}$$

where $p$ is homogeneous of degree m and harmonic. Then

$$\mathcal{F} f = (-i)^m f \tag{11}$$

Let $\mathcal{P}^m$ stand for the finite dimensional space of homogeneous harmonic polynomials of degree $m$:

The above theorem says that the finite dimensional subspace of $L^2(\mathbb{R}^n)$ consisting of functions of the form

$$p(x) \exp(-\pi |x|^2);\ p \in \mathcal{P}^m \tag{12}$$

is invariant under the Fourier transform.
We claim that the following extension is true.
Let

$$f \in L^2(\mathbb{R}^n \tag{13}$$

be of the form

$$f(x) = p(x)g(|x|);\ p \in \mathcal{P}^m \tag{14}$$

Then

$$\mathcal{F} f(\xi) = p(\xi)g(|\xi|) \tag{15}$$

Thus the subspace of functions of the form

$$f(x) = p(x)g(|x|);\ p \in \mathcal{P}^m \tag{16}$$

is invariant under the Fourier transform.

Let

$$f \in L^2(\mathbb{R}^n) \tag{17}$$

be of the form

$$f(x) = p(x)g(|x|);\ p \in \mathcal{P}^m \tag{18}$$

Then

$$\mathcal{F}_n\,(\mathrm{f}) = (-\mathrm{i})^m\, p\, \mathcal{F}_{n+2m}\, g \tag{19}$$

The above result is known as the Hecke-Bochner formula for the Fourier transform.

We conclude our discussion on invariant subspaces with the following result which shows that the Fourier transform of a radial function reduces to an integral transform whose kernel is a Bessel function. This relates to the Hankel transform.

Let $J_\alpha$ stand for the Bessel function of type $\alpha > -1$
If

$$f(x) = g(|x|) \tag{20}$$

is radial and integrable then

$$\mathcal{F}_n\,(\mathrm{f})(\xi) = \mathrm{c}_n \int_0^\infty \mathrm{g(r)} \cdot \frac{\mathrm{J}_{\frac{n}{2}-1}(2\,\pi\,\mathrm{r}\,|\xi|)}{(2\,\pi\,\mathrm{r}\,|\xi|)^{\frac{\mathrm{n}}{2}-1}} \cdot \mathrm{r}^{\mathrm{n}-1} dr \tag{21}$$

## 11.5 Spherical harmonic transform

Next we like to decompose 2D and 3D functions into wave-like basic patterns that have simple radial and angular structures[135]. In that case, the base functions must take the separation-of-variable form:

$$R(r)\Phi(\varphi) = \frac{1}{\sqrt{2\pi}} R(r)\exp(im\varphi) \tag{1}$$

for 2D and

$$R(r)\Theta(\nu)\Phi(\varphi) = R(r)\Omega(\nu,\varphi) \tag{2}$$

$$\Omega(\nu,\varphi) = Y_{lm}(\nu,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}(\nu)\exp(im\varphi) \tag{3}$$

for 3D where $(r,\varphi)$ and $(r,\nu,\varphi)$ are the polar and spherical coordinates respectively. mand l are integers. $l \geq 0$ and $|m| \leq l$.

The base functions are eigenfunctions of the Laplacian. They represent wave-like patterns. The associated angular transform is closely related to the normal Fourier transform. For polar coordinates this reduces to a simple complex 1D Fourier transform.

The radial base function is a Bessel function $J_m(kr)$ for polar coordinates and a spherical Bessel function $j_l(kr)$ for spherical coordinates. The parameter $k$ can take either continuous or discrete

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135 http://lmb.informatik.uni-freiburg.de/papers/download/wa_report01_08.pdf

values, depending on whether the region is infinite or finite. For functions defined on $(0, \infty)$, the transform with $J_m(kr)$ as integral kernel and r as weight is known as the **Hankel transform**. For functions defined on a finite interval, with zero-value boundary condition for the base functions, one gets the **Fourier-Bessel series**. For the 3D case the transform is called **Spherical Harmonic (SH) transform**.

## 11.6 Polar coordinates

The Laplacian in polar coordinates is:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} \tag{1}$$

The Helmholtz differential equation is

$$\nabla^2\psi(r,\varphi) = -k^2\psi(r,\varphi) \tag{2}$$

$$\psi(r,\varphi) = R(r)\Phi(\varphi) \tag{3}$$

$$\Phi(\varphi) = -m^2\Phi(\varphi) \tag{4}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial R(r)}{\partial r}\right) = \left(\frac{m^2}{r^2} - k^2\right)R(r) \tag{5}$$

The solution is:

$$\Phi_m(\varphi) = \exp(im\varphi) \tag{6}$$

$$R(r) = aJ_m(kr) + bY_m(kr) \tag{7}$$

$J_m$ is the $m$-th order Bessel function. The Neumann function $Y_m$ is singular at $r = 0$. Therefore $a = 1$ and $b = 0$.

In finite solutions, the boundary conditions determine what set of functions can be used as base functions. The reference in the footnote shows which choices can be relevant.

## 11.7 Spherical coordinates

The Laplacian in polar coordinates is:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2 sin^2\nu}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{r^2 sin\nu}\frac{\partial}{\partial\nu}\left(sin\nu\frac{\partial\psi}{\partial\nu}\right) \tag{1}$$

The Helmholtz differential equation is

$$\nabla^2\psi(r,\varphi,\nu) = -k^2\psi(r,\varphi,\nu) \tag{2}$$

$$\psi(r,\varphi,\nu) = R(r)\Omega(\varphi,\nu) \tag{3}$$

$$\Omega(\varphi,\nu) = Y_{lm}(\varphi,\nu) \tag{4}$$

$$Y_{lm}(\nu,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\nu)\exp(im\varphi) \tag{5}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) = \left(\frac{l(l-1)}{r^2} - k^2\right)R(r) \tag{6}$$

A non-singular solution for $R(r)$ is:

$$R(r) = j_l(kr) \tag{7}$$

$j_l$ is the spherical Bessel function of order $l$.

$$j_l(r) = \sqrt{\frac{\pi}{2r}}\, J_{l+½}(r) \tag{8}$$

## 11.8 The spherical harmonic transform

The equivalent of the Fourier transform in terms of spherical harmonics depends on the boundary conditions. For example when the analysis is done over a limited region, then the zero boundary condition will give different results than the zero derivative boundary condition[136]. An infinite range will always request a zero value of contributions when the radius goes to infinity.

$$S_{klm} = \int_{r=0}^{\infty} \int_{\varphi=0}^{2\pi} \int_{\nu=0}^{\pi} f(r,\varphi,\nu)\ \psi^*_{klm}(\mathrm{r},\varphi,\nu)\ \mathrm{r}^2 \sin\nu\ \mathrm{dr}\ \mathrm{d}\varphi\ \mathrm{d}\nu \tag{1}$$

$$f(r,\varphi,\nu) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} S_{klm}\ \psi_{klm}(\mathrm{r},\varphi,\nu) \tag{2}$$

$$\psi_{klm}(r,\varphi,\nu) = j_l(kr)\, Y_{lm}(\varphi,\nu) \tag{3}$$

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136 http://lmb.informatik.uni-freiburg.de/papers/download/wa_report01_08.pdf

## 11.9 The Fourier transform of a black hole

In its simplest form a black hole is a bubble that is covered with a blanket of ground states.

The blanket is a comb function that is convoluted with a ground state. The Fourier transform of this blanket is the product of the Fourier transform of the comb function and the Fourier transform of the ground state. Apart from a factor, the ground state is invariant under Fourier transformation. Also the comb function is invariant. Thus the Fourier transform of the blanket is a modulated comb function. The modulation does not reach far.

The most complicated component is the bubble. In its simplest form this is a pulse on the radius. If we interpret this pulse as a Dirac delta function, then the Fourier coefficients have the form:

$$\psi_{k00}(r) = j_0(kr_0) = \sqrt{\frac{\pi}{2r}}\, J_{½}(kr_0)$$

If we sum these coefficients, then we get a sampled spherical Bessel function. These spheres are blurred with the transformed blanket.

## 11.10 Spherical harmonics eigenvalues

See: http://en.wikipedia.org/wiki/Spherical_harmonics for more details.

Spherical harmonics are best presented in polar coordinates. There exists a corresponding polar Fourier transform. This Fourier transform also has invariant functions. Like in the rectangular case, they form the basis for spherical harmonics.

Laplace's equation in spherical coordinates is:

$$\nabla^2 f = r^{-2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{df}{d\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial \varphi^2} = 0 \quad (1)$$

Try to find solutions in the form of the eigenfunctions of the Fourier transform.

By separation of variables, two differential equations result by imposing Laplace's equation:

$$f(r,\theta,\varphi) = R(r)\cdot Y(\theta,\varphi) \quad (2)$$

$$R^{-1}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda \quad (3)$$

$$\frac{1}{Y\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2(\theta)}\frac{\partial^2 Y}{\partial\varphi^2} = -\lambda \quad (4)$$

The second equation can be simplified under the assumption that $Y$ has the form

$$Y(\theta,\varphi) = \Theta(\theta)\Phi(\varphi) \quad (5)$$

Applying separation of variables again to the second equation gives way to the pair of differential equations

$$\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2 \quad (6)$$

$$\lambda\sin^2(\theta) + \frac{\sin(\theta)}{\Theta(\theta)}\frac{d}{d\theta}\left[\sin(\theta)\frac{d\Theta}{d\theta}\right] = m^2 \quad (7)$$

for some number $m$. A priori, $m$ is a complex constant, but because $\Phi$ must be a periodic function whose period evenly divides $2\pi, m$ is necessarily an integer and $\Phi$ is a linear combination of the complex exponentials $\exp(\pm i\, m\, \theta)$. The solution function $Y(\theta, \varphi)$ is regular at the poles of the sphere, where $\theta = 0, \pi$. Imposing this regularity in the solution $\Theta$ of the second equation at the boundary points of the domain is a Sturm–Liouville problem[137] that forces the parameter $\lambda$ to be of the form $\lambda = \ell(\ell + 1)$ for some non-negative integer with $\ell \geq |m|$; this is also explained below in terms of the orbital angular momentum. Furthermore, a change of variables $t = cos\theta$ transforms this equation into the Legendre equation, whose solution is a multiple of the associated Legendre function[138]. $P_l^m(cos(\theta))$. Finally, the equation for $R$ has solutions of the form $R(r) = A\, r^{\ell} + B\, r^{-\ell-1}$; requiring the solution to be regular throughout $\mathbb{R}^3$ forces $B = 0$.

Here the solution was assumed to have the special form

$$Y(\theta, \varphi) = \Theta(\theta)\, \Phi(\varphi) \tag{8}$$

For a given value of $\ell$, there are $2\ell + 1$ independent solutions of this form, one for each integer $m$ with $-\ell \leq m \leq \ell$. These angular solutions are a product of trigonometric functions, here represented as a complex exponential, and associated Legendre functions:

$$Y_l^m(\theta, \varphi) = N \exp(i\, m\, \varphi)\, P_l^m(cos(\theta)) \tag{9}$$

which fulfill

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137 http://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_problem

138 http://en.wikipedia.org/wiki/Associated_Legendre_function

$$r^2 \nabla^2 Y_l^m(\theta, \varphi) = -l\,(l+1)\, Y_l^m(\theta, \varphi) \tag{10}$$

Here $Y_l^m$ is called a spherical harmonic function of degree $\ell$ and order m, $P_l^m$ is an associated Legendre function, N is a normalization constant, θ represents the colatitude and φ represents the longitude. In particular, the colatitude[139] θ, or polar angle, ranges from 0 at the North Pole to π at the South Pole, assuming the value of π/2 at the Equator, and the longitude[140] $\varphi$, or azimuth[141], may assume all values with $0 \le \varphi < 2\pi$. For a fixed integer $\ell$, every solution $Y(\theta, \varphi)$ of the eigenvalue problem

$$r^2 \nabla^2 Y = -l\,(l+1)\, Y \tag{11}$$

is a linear combination of $Y_l^m$. In fact, for any such solution, $r^\ell\, Y(\theta, \varphi)$ is the expression in spherical coordinates of a homogeneous polynomial that is harmonic, and so counting dimensions shows that there are $2\ell + 1$ linearly independent of such polynomials.

The general solution to Laplace's equation in a ball centered at the origin is a linear combination of the spherical harmonic functions multiplied by the appropriate scale factor $r^l$,

$$f(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m r^l Y_l^m(\theta, \varphi) \tag{12}$$

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[139] http://en.wikipedia.org/wiki/Colatitude
[140] http://en.wikipedia.org/wiki/Longitude
[141] http://en.wikipedia.org/wiki/Azimuth

where the $f_l^m$ are constants and the factors $r^l Y_l^m$ are known as solid harmonics[142]. Such an expansion is valid in the ball

$$r < R = 1/\lim_{l\Rightarrow\infty} sup \ |f_l^m|^{1/l} \tag{13}$$

## 11.11 Orbital angular momentum

In quantum mechanics, Laplace's spherical harmonics are understood in terms of the orbital angular momentum[143]

$$\boldsymbol{L} = -i\,\hbar\,\boldsymbol{x} \times \boldsymbol{\nabla} = L_x\,\boldsymbol{i} + L_y\boldsymbol{j} + L_z\,\boldsymbol{k} \tag{1}$$

The spherical harmonics are eigenfunctions of the square of the orbital angular momentum

$$\boldsymbol{L}^2 = -r^2\,\nabla^2 + \left(r\,\frac{\partial}{\partial r} + 1\right) r\,\frac{\partial}{\partial r} \tag{2}$$

$$= \frac{1}{\sin(\theta)}\,\frac{\partial}{\partial\theta}\,\sin(\theta)\,\frac{\partial}{\partial\theta} - \frac{1}{\sin^2(\theta)}\cdot\frac{\partial^2}{\partial\varphi^2}$$

Laplace's spherical harmonics are the joint eigenfunctions of the square of the orbital angular momentum and the generator of rotations about the azimuthal axis:

$$L_z = -i\cdot\hbar\cdot\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z}\right) = i\,\hbar\,\frac{\partial}{\partial\varphi} \tag{3}$$

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[142] http://en.wikipedia.org/wiki/Solid_harmonics
[143] http://en.wikipedia.org/wiki/Orbital_angular_momentum

These operators commute, and are densely defined self-adjoint operators on the Hilbert space of functions $f$ square-integrable with respect to the normal distribution on $\mathbb{R}^3$:

$$(2\,\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} |f(x)|^2 \exp(-|x|^2/2) \;<\; \infty \tag{4}$$

Furthermore, $\boldsymbol{L}^2$ is a positive operator.

If $Y$ is a joint eigenfunction of $L^2$ and $L_z$, then by definition

$$\boldsymbol{L}^2\, Y \;=\; \lambda\, Y \tag{5}$$

$$L_z\, Y \;=\; m\, Y \tag{6}$$

for some real numbers $m$ and $\lambda$. Here $m$ must in fact be an integer, for $Y$ must be periodic in the coordinate $\varphi$ with period a number that evenly divides $2\,\pi$. Furthermore, since

$$\boldsymbol{L}^2 \;=\; L_x^2 \;+\; L_y^2 \;+\; L_z^2 \tag{7}$$

and each of $L_x\,, L_y, L_z$ are self-adjoint, it follows that $\lambda \geq m^2$.

Denote this joint eigenspace by $E_{\lambda,m}$, and define the raising and lowering operators by

$$L_+ \;=\; L_x \;+\; i\, L_y \tag{8}$$

$$L_- \;=\; L_x \;-\; i\, L_y \tag{9}$$

Then $L_+$ and $L_-$ commute with $\boldsymbol{L}^2$, and the Lie algebra generated by $L_+, L_-, L_z$ is the special linear Lie algebra, with commutation relations

$$[L_z, L_+] = L_+ \tag{10}$$

$$[L_z, L_-] = -L_- \tag{11}$$

$$[L_+, L_-] = 2\,L_z \tag{12}$$

Thus $L_+ : E_{\lambda,m} \to E_{\lambda,m+1}$ (it is a "raising operator") and $L_- : E_{\lambda,m} \to E_{\lambda,m-1}$ (it is a "lowering operator"). In particular, $L_+^k : E_{\lambda,m} \to E_{\lambda,m+k}$ must be zero for $k$ sufficiently large, because the inequality $\lambda \geq m^2$ must hold in each of the nontrivial joint eigenspaces. Let $Y \in E_{\lambda,m}$ be a nonzero joint eigenfunction, and let $k$ be the least integer such that

$$L_+^k \; Y = 0 \tag{13}$$

Then, since

$$L_- \, L_+ = \boldsymbol{L}^2 - L_z^2 - L_z \tag{14}$$

it follows that

$$0 = L_- \, L_+^k \; Y = \left(\lambda - (m + k)^2 - (m + k)\right) Y \tag{15}$$

Thus $\lambda = \ell(\ell + 1)$ for the positive integer $\ell = m + k$. (16)

## 11.12 Spherical harmonics expansion

The Laplace spherical harmonics form a complete set of orthonormal functions and thus form an orthonormal basis of the Hilbert space of square-integrable functions. On the unit sphere, any square-integrable function can thus be expanded as a linear combination of these:

$$f(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\varphi) \tag{1}$$

This expansion holds in the sense of mean-square convergence — convergence in $L^2$ of the sphere — which is to say that

$$\lim_{N\to\infty}\int_0^{2\pi}\int_0^{\pi}\left|f(\theta,\varphi)-\sum_{l=0}^{N}\sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\varphi)\right|^2 \sin(\theta)\,d\theta\,d\varphi=0 \tag{2}$$

The expansion coefficients are the analogs of Fourier coefficients, and can be obtained by multiplying the above equation by the complex conjugate of a spherical harmonic, integrating over the solid angle Ω, and utilizing the above orthogonality relationships. This is justified rigorously by basic Hilbert space theory. For the case of orthonormalized harmonics, this gives:

$$f_l^m=\int_\Omega f(\theta,\varphi)\,Y_l^{m*}(\theta,\varphi)d\Omega = \int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\sin(\theta)\;f(\theta,\varphi)\,Y_l^{m*}(\theta,\varphi) \tag{3}$$

If the coefficients decay in ℓ sufficiently rapidly — for instance, exponentially — then the series also converges uniformly to $f$.

A real square-integrable function $f$ can be expanded in terms of the real harmonics $Y_{\ell m}$ above as a sum

$$f(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\varphi) \tag{4}$$

Convergence of the series holds again in the same sense.

### 11.13 Spin weighted spherical harmonics

Regard the sphere $\mathbb{S}^2$ as embedded into the three-dimensional imaginary part of the quaternionic number field. At a point **x** on the sphere, a positively oriented orthonormal basis of tangent vectors at **x** is a pair **a**, **b** of vectors such that

$$(\boldsymbol{x},\boldsymbol{a}) = (\boldsymbol{x},\boldsymbol{b}) = (\boldsymbol{a},\boldsymbol{b}) = 0 \tag{1}$$

$$(\boldsymbol{a},\boldsymbol{a}) = (\boldsymbol{b},\boldsymbol{b}) = 1 \tag{2}$$

$$\langle \boldsymbol{x},\boldsymbol{a}\times\boldsymbol{b}\rangle > 0 \tag{3}$$

where the first pair of equations states that **a** and **b** are tangent at **x**, the second pair states that **a** and **b** are unit vectors, **a** and **b** are orthogonal, and the $\{\boldsymbol{x},\boldsymbol{a},\boldsymbol{b}\}$ is a right-handed basis of $\mathbb{R}^3$.

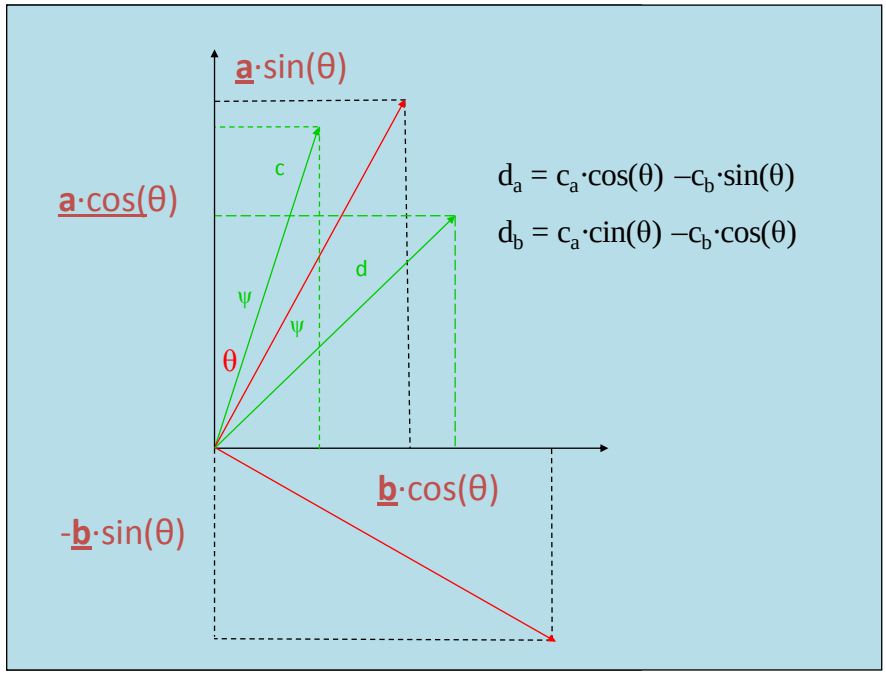


**Figure 8: θ and the parameters a and b of the spin-weight function f.**

A spin-weight *s* function *f* is a function accepting as input a point **x** of $S^2$ and a positively oriented orthonormal basis of tangent vectors at **x**, such that

(4)

$$f\big(\boldsymbol{x}, \boldsymbol{a}\, cos(\theta) - \boldsymbol{b}\, sin(\theta), \boldsymbol{a}\, sin(\theta) + \boldsymbol{b}\, cos(\theta)\big) = \exp(\boldsymbol{i}\, s\, \theta)\, f(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})$$

for every rotation angle $\theta$.

Following Eastwood & Tod (1982), denote the collection of all spin-weight *s* functions by **B**(*s*). Concretely, these are understood as functions $f$ on $\boldsymbol{C^2}\backslash\{0\}$ satisfying the following homogeneity law under complex scaling

$$f(\lambda\, \boldsymbol{z}, \bar{\lambda}\, \overline{\boldsymbol{z}}) = \left(\frac{\bar{\lambda}}{\lambda}\right)^{s} f(\boldsymbol{z}, \overline{\boldsymbol{z}}) \tag{5}$$

This makes sense provided *s* is a half-integer.

Abstractly, **B**(*s*) is isomorphic to the smooth vector bundle underlying the antiholomorphic vector bundle $O^{*}(2\cdot s)$ of the Serre twist on the complex projective line $\boldsymbol{CP^1}$. A section of the latter bundle is a function *g* on $\boldsymbol{C^2}\backslash\{0\}$ satisfying

$$g\big(\lambda\, \boldsymbol{z}, \bar{\lambda}\, \overline{\boldsymbol{z}}\big) = \big(\bar{\lambda}\big)^{2s}\, g(\boldsymbol{z}, \overline{\boldsymbol{z}}) \tag{6}$$

Given such a *g*, we may produce a spin-weight *s* function by multiplying by a suitable power of the Hermitian form

$$P(\boldsymbol{z}, \overline{\boldsymbol{z}}) = \boldsymbol{z}\, \overline{\boldsymbol{z}} \tag{7}$$

Specifically, $\mathfrak{f} = P^{-s} g$ is a spin-weight *s* function. The association of a spin-weighted function to an ordinary homogeneous function is an isomorphism.

## 11.14 Eth

The spin weight bundles **B**($s$) are equipped with a differential operator ð (eth). This operator is essentially the Dolbeault operator[144],

$$ð = \partial + \partial^* \tag{1}$$

Thus for $f \in \boldsymbol{B}(s)$,

$$ð\, f \equiv P^{-s+1}\, \partial(P^s\, f) \tag{2}$$

defines a function of spin-weight $s + 1$.

## 11.15 Spin-weighted harmonic functions

See http://en.wikipedia.org/wiki/Spin-weighted_spherical_harmonics for more details.

Just as conventional spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator on the sphere, the spin-weight *s* harmonics are the eigensections for the Laplace-Beltrami operator acting on the bundles $\mathcal{E}(s)$ of spin-weight *s* functions.

The spin-weighted harmonics can be represented as functions on a sphere once a point on the sphere has been selected to serve as the North Pole. By definition, a function η with *spin weight s* transforms under rotation about the pole via

$$\eta \rightarrow \exp(i\, s\, \psi)\, \eta \tag{1}$$

[144] http://en.wikipedia.org/wiki/Dolbeault_operator

Working in standard spherical coordinates, we can define a particular operator ð acting on a function η as:

$$ð\eta = -\sin^{s}(\theta)\{\frac{\partial}{\partial\theta} + \frac{i}{\sin(\theta)}\frac{\partial}{\partial\varphi}\}[\sin^{-s}(\theta)\,\eta] \tag{2}$$

This gives us another function of $\theta$ and $\varphi$. [The operator ð is effectively a covariant derivative operator in the sphere.]

An important property of the new function $ð\eta$ is that if η had spin weight $s, ð\eta$ has spin weight $s + 1$. Thus, the operator raises the spin weight of a function by 1. Similarly, we can define an operator which will lower the spin weight of a function by 1:

$$\bar{ð}\eta = -\sin^{-s}(\theta)\left\{\frac{\partial}{\partial\theta} - \frac{i}{\sin(\theta)}\frac{\partial}{\partial\varphi}\right\}[(\sin^{s}(\theta)\,\eta] \tag{3}$$

We extend the function $Y_l^m$ to ${}_sY_l^m$ according to

$${}_0Y_l^m(\theta,\varphi) = Y_l^m(\theta,\varphi) \tag{4}$$

$$l = 0, 1, 2, \ldots;\ m = -l, \ldots 0, \ldots l \tag{5}$$

The spin-weighted spherical harmonics are then defined in terms of the usual spherical harmonics as:

$${}_sY_l^m = \sqrt{\frac{(l-s)!}{(l+s)!}}\,ð^s\,Y_l^m;\ 0 \le s \le l \tag{6}$$

$${}_sY_l^m = \sqrt{\frac{(l+s)!}{(l-s)!}}\ (-1)^s\,ð^s\,Y_l^m;\ -l \le s \le 0 \tag{7}$$

$$ {}_sY_l^m = 0;\ l < |s|; \tag{8} $$

The functions ${}_sY_l^m$ then have the property of transforming with spin weight $s$.

Other important properties include the following:

$$ \eth({}_sY_l^m) = +\sqrt{(l-s)\,(l+s+1)}\ {}_{s+1}Y_l^m \tag{9} $$

$$ \eth({}_sY_l^m) = -\sqrt{(l+s)\,(l-s+1)}\ {}_{s-1}Y_l^m \tag{10} $$

# 12 Differentiation

A quaternionic distribution f(q) can be differentiated.

$$g(q) = \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left( \pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \tag{1}$$

The colored $\mp$ and $\pm$ signs refer to the influence of conjugation of $f(q)$ on quaternionic multiplication. The $\pm$sign refers to the influence of reflection of $f(q)$.

## 12.1 Continuity equation

When applied to a quaternionic probability amplitude distribution (QPAD), the equation for the differentiation leads to a continuity equation.

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of the corresponding charge is given by the continuity equation:

Total change within $V$ = flow into $V$ + production inside $V$ (1)

$$\frac{d}{dt} \int_V \rho_0 \, dV = \oint_S \hat{\boldsymbol{n}} \rho_0 \frac{\boldsymbol{v}}{c} \, dS + \int_V s_0 \, dV \tag{2}$$

$$\int_V \nabla_0 \rho_0 \, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho} \rangle \, dV + \int_V s_0 \, dV \tag{3}$$

For the imaginary parts hold.

$$\frac{d}{dt}\int_V \boldsymbol{\rho}\, dV = -\oint_S \hat{\boldsymbol{n}}\rho_0\, dS - \oint_S \hat{\boldsymbol{n}} \times \boldsymbol{\rho}\, dS + \int_V \boldsymbol{s}\, dV \tag{4}$$

$$\int_V \nabla_0\, \boldsymbol{\rho}\, dV = -\int_V \boldsymbol{\nabla}\rho_0\, dV - \int_V \boldsymbol{\nabla} \times \boldsymbol{\rho}\, dV + \int_V \boldsymbol{s}\, dV \tag{5}$$

For the full integral equation holds:

$$\frac{d}{dt}\int_V \rho\, dV + \oint_S \hat{\boldsymbol{n}}\rho\, dS = \int_V s\, dV \tag{6}$$

$$\int_V \nabla\, \rho\, dV = \int_V \boldsymbol{s}\, dV \tag{7}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface $S$, $\boldsymbol{v}(t, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(t, \boldsymbol{q})$ enters volume $V$ and $s_0$ is the source density inside $V$. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{8}$$

It is the flux (flow per unit area and unit time) of $\rho_0$ .
$t$ stands for progression (not coordinate time).

The combination of $\rho_0(t, \boldsymbol{q})$ and $\boldsymbol{\rho}(t, \boldsymbol{q})$ is a quaternionic skew field $\rho(t, \boldsymbol{q})$ and can be seen as a probability amplitude distribution (QPAD).

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{9}$$

$\rho(t,\boldsymbol{q})\rho^*(t,\boldsymbol{q})$ can be seen as an overall probability density distribution of the presence of the carrier of the charge. $\rho_0(t,\boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(t,\boldsymbol{q})$ is the current density distribution.

The conversion from formula (2) to formula (3) uses the Gauss theorem[145]. This results in the law of charge conservation:

$$\begin{aligned} s_0(t,\boldsymbol{q}) &= \nabla_0\rho_0(t,\boldsymbol{q}) \\ &\quad \mp \langle \boldsymbol{\nabla}, \left(\rho_0(t,\boldsymbol{q})\boldsymbol{v}(t,\boldsymbol{q}) + \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})\right)\rangle \\ &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q}) + \boldsymbol{A}(t,\boldsymbol{q})\rangle \\ &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}(t,\boldsymbol{q}), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \\ &\quad \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle\, \rho_0(t,\boldsymbol{q}) \\ &\quad \mp \langle \boldsymbol{\nabla}, \boldsymbol{A}(t,\boldsymbol{q})\rangle \end{aligned} \tag{10}$$

The blue colored ± indicates quaternionic sign selection through conjugation of the field $\rho(t,\boldsymbol{q})$. The field $\boldsymbol{a}(t,\boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})\rangle = 0 \tag{11}$$

$\boldsymbol{A}(t,\boldsymbol{q}) \stackrel{\text{def}}{=} \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(t,\boldsymbol{q})$.

In Fourier space the continuity equation becomes:

[145] http://en.wikipedia.org/wiki/Divergence_theorem

$$\tilde{s}_0(t,\boldsymbol{p}) = p_0\tilde{\rho}_0(t,\boldsymbol{p}) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(t,\boldsymbol{p})\rangle \tag{12}$$

Equation (6) represents a balance equation for charge density. What this charge is will be left in the middle. It can be one of the properties of the carrier or it can represent the full ensemble of the properties of the carrier.

This only treats the real part of the full equation. The full equation runs:

$$\begin{aligned} s(t,\boldsymbol{q}) &= \nabla\rho(t,\boldsymbol{q}) \\ &= s_0(t,\boldsymbol{q}) + \boldsymbol{s}(t,\boldsymbol{q}) \\ &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q})\rangle \pm \nabla_0\boldsymbol{\rho}(t,\boldsymbol{q}) \\ &\quad + \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \\ &\quad \pm \left(\pm\boldsymbol{\nabla} \times \boldsymbol{\rho}(t,\boldsymbol{q})\right) \\ &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}(t,\boldsymbol{q}), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \\ &\quad \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle \rho_0(t,\boldsymbol{q}) \\ &\quad \pm\nabla_0\boldsymbol{v}(t,\boldsymbol{q}) + \nabla_0\rho_0(t,\boldsymbol{q}) \\ &\quad + \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \\ &\quad \pm\left(\pm(\rho_0(t,\boldsymbol{q})\,\boldsymbol{\nabla} \times \boldsymbol{v}(t,\boldsymbol{q}) - \boldsymbol{v}(t,\boldsymbol{q})\right. \\ &\quad \left. \times \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\right) \end{aligned} \tag{13}$$

$$\begin{aligned} s_0(t,\boldsymbol{q}) &= 2\nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}(q), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \\ &\quad \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle \rho_0(t,\boldsymbol{q}) \end{aligned} \tag{14}$$

$$\boldsymbol{s}(t,\boldsymbol{q}) = \pm\nabla_0 \boldsymbol{v}(t,\boldsymbol{q}) \pm \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \pm \Big(\pm\big(\rho_0(t,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(t,\boldsymbol{q}) - \boldsymbol{v}(t,\boldsymbol{q}) \times \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\big)\Big) \tag{15}$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. Conjugation also causes a switch of handedness. It changes the sign of all three imaginary base vectors.

### 12.1.1 Continuity Equations

The equation for the conservation of charge:

$$s_0(q) = \nabla_0\rho_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle \tag{1}$$

We can define $\mathfrak{F}(\mathrm{q})$:

$$\mathfrak{F}(\mathrm{q}) \stackrel{\text{def}}{=} \nabla\boldsymbol{\rho}(q) \tag{2}$$

$$\mathfrak{F}_0(q) = \nabla_0\boldsymbol{\rho}_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle \tag{3}$$

$$\begin{aligned}\boldsymbol{\mathfrak{F}}(q) &= \boldsymbol{\nabla}\boldsymbol{\rho}_0(q) \pm \nabla_0\boldsymbol{\rho}(q) \pm \boldsymbol{\nabla}\times\boldsymbol{\rho}(q) \\ &= \boldsymbol{\mathfrak{E}}(q) + \boldsymbol{\mathfrak{B}}(q)\end{aligned} \tag{4}$$

$$\boldsymbol{\mathfrak{E}}(q) = -\boldsymbol{\nabla}\boldsymbol{\rho}_0(q) \mp \nabla_0\boldsymbol{\rho}(q) \tag{5}$$

$$\boldsymbol{\mathfrak{B}}(q) = \pm\boldsymbol{\nabla}\times\boldsymbol{\rho}(q) \tag{6}$$

The definition of $\mathfrak{B}(q)$ and $\mathfrak{E}(q)$ have the freedom of the gauge transform[146]

$$\boldsymbol{\rho}(q) \mapsto \boldsymbol{\rho}(q) + \boldsymbol{\nabla}\phi_0 \tag{7}$$

$$\mathfrak{E}(q) \mapsto \mathfrak{E}(q) - \boldsymbol{\nabla}\left(\nabla_0 \boldsymbol{\phi}_0(q)\right) \tag{8}$$

$$\boldsymbol{\nabla}^2\boldsymbol{\phi}_0 = \nabla_0^2\boldsymbol{\phi}_0 \tag{9}$$

This translates in the source free case $s_0(q) = 0$ into:

$$\nabla_0\rho_0(q) = \pm\langle\boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle \tag{10}$$

$$\mathfrak{F}_0(q) = \nabla_0\boldsymbol{\rho}_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle = 0 \tag{11}$$

In the source divergence free case $\boldsymbol{\nabla}s_0(q) = 0$ this means:

$$\nabla_0\boldsymbol{\nabla}\rho_0(q) = \pm\boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\rho}(q)\rangle \tag{12}$$

$$\nabla_0\boldsymbol{\nabla}\boldsymbol{\phi}_0(q) = \pm\boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \tag{13}$$

$$\boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\phi}(q) + \boldsymbol{\nabla}^2\boldsymbol{\phi}(q) \tag{14}$$

Due to the fact that there are other charges present, the divergence of the scalar potential need be in the direction of the current $\boldsymbol{\rho}(q)$, which for a spherical symmetric blur is also in the direction of the vector potential $\boldsymbol{\phi}(q)$. However, a tendency exists to

[146] http://en.wikipedia.org/wiki/Gauge_fixing

minimize that difference. Thus $\nabla_0 \boldsymbol{\nabla} \boldsymbol{\phi}_0(q)$ is parallel to $\boldsymbol{\phi}(q)$. With other words:

$$\boldsymbol{\phi}(q) \times \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = 0 \tag{15}$$

Reckoning the sign selections for the sign $\pm$ of the conjugation and the handedness $\pm$ of the cross product will provide four different sets of equations. This will provide eight different quaternionic distributions.

The continuity equation (or balance equation) is in fact quaternionic a differential equation. The operator $\nabla_0$ represents the Hamiltonian. It has a direct relation with the Lagrangian, which is derived from the path equation[147].

## 12.2 Discrete distribution

If $\rho(q)$ is discrete, such that

$$\rho(q) = \sum_i q_{E_i} \cdot \delta(q - q_i) \tag{1}$$

where $q'_E$ is a point charge at location q′, then the contribution to the field $\boldsymbol{E}(q)$ that is generated by a point charge at location $q_i$ is given by:

$$d\boldsymbol{E}(q) = q_{E_i} \cdot \frac{q_i - q}{|q_i - q|^3} = -q_{E_i} \cdot \nabla \cdot \frac{1}{|q_i - q|} \tag{2}$$

[147] Path charcteristics, Euler Lagrange equations

## 12.3 Differential potential equations

The gradient and curl of $\phi(q)$ are related. In configuration space holds:

$$\nabla\phi(q) = \nabla_0\phi_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \pm \nabla_0\boldsymbol{\phi}(q) \pm \boldsymbol{\nabla}\phi_0(q) \pm \left(\pm\boldsymbol{\nabla} \times \boldsymbol{\phi}(q)\right) \tag{1}$$

$$\mathfrak{E}(q) \stackrel{\text{def}}{=} -\boldsymbol{\nabla}\phi_0(q) \tag{2}$$

$$\mathfrak{B}(q) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \phi(q) \tag{3}$$

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla\phi(q) = \mathfrak{F}_0(q) \mp \mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\boldsymbol{\phi}(q) \tag{4}$$

$$\mathfrak{F}_0(q) = \nabla_0\phi_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \tag{5}$$

$$\mathfrak{F}(q) = \mp\mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\boldsymbol{\phi}(q) \tag{6}$$

When the field $\phi(q)$ is split into a private field $\phi_p(q)$ and a background field $\phi_b(q)$, then $\boldsymbol{\phi_p}(q)$ corresponds to the private field of the uniform moving item. When this item accelerates, then it goes together with an extra term $\nabla_0\boldsymbol{\phi_p}(q)$. This is the reason of existence of inertia[148].

$$\langle\boldsymbol{\nabla}, \mathfrak{E}(q)\rangle = -\boldsymbol{\nabla}^2\phi_0(q) = \rho_0(q) \tag{7}$$

$$\boldsymbol{\nabla} \times \mathfrak{E}(q) = 0; \text{ Rotation free field} \tag{8}$$

$$\langle\boldsymbol{\nabla}, \mathfrak{B}(q)\rangle = 0; \text{ Divergence free } B \text{ field} \tag{9}$$

[148] Influence; Inertia

$$\boldsymbol{\nabla} \times \mathfrak{B}(q) = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle - \boldsymbol{\nabla}^2\boldsymbol{\phi}(q) \\ = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle + \boldsymbol{\rho}(q) + {\nabla_0}^2\boldsymbol{\phi}(q) \tag{10}$$

$$\boldsymbol{\nabla} \times \mathfrak{B}(q) = \pm\nabla_0\boldsymbol{\nabla}\phi_0(q) + \boldsymbol{\rho}(q) + {\nabla_0}^2\boldsymbol{\phi}(q) \tag{11}$$

$$= \pm\nabla_0\mathfrak{E}(q) + \boldsymbol{\rho}(q) + {\nabla_0}^2\boldsymbol{\phi}(q)$$

Since $\nabla_0\boldsymbol{\phi}(q)$is supposed to be parallel to $\boldsymbol{\nabla}\phi_0(q)$, it is sensible to define $\boldsymbol{E}(q)$as the total field in longitudinal direction:

$$\boldsymbol{E}(q) = -\boldsymbol{\nabla}\phi_0(q) - \nabla_0\boldsymbol{\phi}(q) = \mathfrak{E}(q) - \nabla_0\boldsymbol{\phi}(q) \tag{12}$$

And

$$\boldsymbol{B}(q) = \mathfrak{B}(q) \tag{13}$$

With this definition:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(q) = -\nabla_0\boldsymbol{B}(q) \tag{14}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{B}(q)\rangle = 0 \tag{15}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(q) = \boldsymbol{\rho}(q) + \nabla_0\boldsymbol{E}(q) \tag{16}$$

### 12.3.1 In Fourier space

In Fourier space holds:

$$\tilde{\mathfrak{F}}(p) = p_0\tilde{\phi}_0(p) - \langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle \pm p_0\tilde{\boldsymbol{\phi}}(p) \pm \boldsymbol{p}\tilde{\phi}_0(p) \pm \boldsymbol{p} \times \tilde{\boldsymbol{\phi}}(p) \tag{1}$$

$$\tilde{\mathfrak{F}}(p) = p\tilde{\phi}(p) = \mathfrak{F}_0(p) \mp \tilde{\mathfrak{E}}(p) \pm \tilde{\mathfrak{B}}(p) \pm p_0\tilde{\boldsymbol{\phi}}(p) \tag{2}$$

$$\widetilde{\mathfrak{F}}_0(p) = p_0\tilde{\phi}_0(p) - \langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle \tag{3}$$

$$\widetilde{\mathfrak{E}}(p) = -\boldsymbol{p}\tilde{\phi}_0(p) \tag{4}$$

$$\widetilde{\boldsymbol{E}}(p) = -\boldsymbol{p}\tilde{\phi}_0(p) \pm p_0\tilde{\boldsymbol{\phi}}(p)$$

$$\widetilde{\mathfrak{B}}(p) = \boldsymbol{p} \times \tilde{\boldsymbol{\phi}}(p) \tag{5}$$

$$\widetilde{\mathfrak{F}}(p) = \mp\widetilde{\mathfrak{E}}(p) \pm \widetilde{\mathfrak{B}}(p) \pm p_0\tilde{\boldsymbol{\phi}}(p) \tag{6}$$

$$\langle \boldsymbol{p}, \widetilde{\mathfrak{E}}(p)\rangle = -\boldsymbol{p}^2\tilde{\phi}_0(p) = \tilde{\rho}_0(p) \tag{7}$$

$$\boldsymbol{p} \times \widetilde{\mathfrak{E}}(p) = 0; \text{ Rotation free field} \tag{8}$$

$$\langle \boldsymbol{p}, \widetilde{\mathfrak{B}}(p)\rangle = 0; \text{ Divergence free } B \text{ field} \tag{9}$$

$$\begin{aligned} \boldsymbol{p} \times \widetilde{\mathfrak{B}}(p) &= \boldsymbol{p}\langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(q)\rangle - \boldsymbol{p}^2\tilde{\boldsymbol{\phi}}(q) \\ &= \boldsymbol{p}\langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle + \tilde{\boldsymbol{\rho}}(p) \end{aligned} \tag{10}$$

$$\boldsymbol{p} \times \widetilde{\mathfrak{B}}(p) = \pm p_0\boldsymbol{p}\tilde{\phi}_0(p) + \tilde{\boldsymbol{\rho}}(p) = \pm p_0\widetilde{\mathfrak{E}}(p) + \tilde{\boldsymbol{\rho}}(p) \tag{11}$$

If the distribution $\rho(q)$ is differentiable, then the same equations that hold for fields $\phi(q)$ and $\tilde{\phi}(p)$ hold for the non-blurred distributions $\rho(q)$ and $\tilde{\rho}(q)$.

## 12.4 Maxwell equations

First it must be noted that the above derived field equations hold for general quaternionic fields.

The resemblance with physical fields holds for electromagnetic fields as well as for gravitational fields and for any fields whose blurring function approximates

$$f(q) \approx \frac{1}{|q|}.$$

In Maxwell equations, $\boldsymbol{E}(\mathbf{r})$ is defined as:

$$\boldsymbol{E}(\boldsymbol{r},t) \equiv -\boldsymbol{\nabla}\phi_0(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \mathfrak{E}(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t}$$

Further:

$$\langle \boldsymbol{\nabla}, \boldsymbol{E}(\boldsymbol{r},t)\rangle = -\boldsymbol{\nabla}^2\phi_0(\boldsymbol{r},t) - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{A}(\boldsymbol{r},t)\rangle}{\partial t}$$

$$= \frac{\rho_0(\boldsymbol{r},t)}{\varepsilon_0} - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{A}(\boldsymbol{r},t)\rangle}{\partial t}$$

In Maxwell equations, $\boldsymbol{B}(\boldsymbol{r})$ is defined as:

$$\boldsymbol{B}(\boldsymbol{r},t) \equiv \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r},t) = \mathfrak{B}(\boldsymbol{r},t)$$

Further:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{B}(\boldsymbol{r},t)\rangle = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{r},t) = \mu_0(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t})$$

### 12.4.1 Differentiable distribution

If the distribution $\rho(q)$ is differentiable, then the same equations that hold for fields $\boldsymbol{\phi}(q)$ and $\tilde{\boldsymbol{\phi}}(p)$ hold for the non-blurred distributions $\rho(q)$ and $\tilde{\rho}(q)$.

Using:

$$\begin{aligned}\boldsymbol{B} = \nabla \times \boldsymbol{\phi} \\ &= \boldsymbol{i}(\nabla_2\phi_\parallel - \nabla_\parallel\phi_2) + \boldsymbol{j}(\nabla_\parallel\phi_1 - \nabla_1\phi_\parallel) \\ &+ \boldsymbol{k}(\nabla_1\phi_2 - \nabla_2\phi_1)\end{aligned} \tag{1}$$

gives

$$\nabla_0\phi_\parallel(q) = \mp\nabla_\parallel\phi_0(q) \tag{2}$$

$$\nabla_0\phi_1(q) = \mp\left(\nabla_2\phi_\parallel(q) - \nabla_\parallel\phi_2(q)\right) \tag{3}$$

$$\nabla_0\phi_2(q) = \pm\left(\nabla_1\phi_\parallel(q) - \nabla_\parallel\phi_1(q)\right) \tag{4}$$

$$\nabla_0\phi_0(q) = \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle = \nabla_\parallel\phi_\parallel(q) + \nabla_1\phi_1(q) + \nabla_2\phi_2(q) \tag{5}$$

And correspondingly in Fourier space

$$p_0\tilde{\phi}_\parallel(p) = \mp p_\parallel\tilde{\phi}_0(p) \tag{6}$$

$$p_0\tilde{\phi}_1(p) = \pm\left(p_\parallel\tilde{\phi}_2(p) - p_2\tilde{\phi}_\parallel(p)\right) \tag{7}$$

$$p_0\tilde{\phi}_2(p) = \mp\left(p_\parallel\tilde{\phi}_1(p) - p_1\tilde{\phi}_\parallel(p)\right) \tag{8}$$

$$p_0\tilde{\phi}_0(p) = \langle \boldsymbol{p}, \tilde{\boldsymbol{\phi}}(p)\rangle = p_{\|}\tilde{\phi}_{\|}(p) + p_1\tilde{\phi}_1(p) + p_2\tilde{\phi}_2(p) \tag{9}$$

## 12.5 Covariant derivative

The covariant derivative plays a role in the Lagrangian and in the equation of motion. It plays an essential role in the Higgs mechanism.

The covariant derivative $D$ is defined as

$$Df(q) = \nabla f(q) - \boldsymbol{A}(q)\, f(q) \tag{1}$$

This is interesting with respect to a gauge transformation of the form

$$f'(q) = G(q)\, f(q) \tag{2}$$

$$G^*(q)\ G(q) = 1 \tag{3}$$

$$\nabla G(q) = \boldsymbol{H}(q)G(q) \tag{4}$$

where with a corresponding vector potential transformation

$$\boldsymbol{A}'(q) = \boldsymbol{A}(q) + \boldsymbol{H}(q) \tag{5}$$

$$D' = \nabla - \boldsymbol{A}(q) - \boldsymbol{H}(q) \tag{6}$$

The following step is in general not valid for quaternionic functions. However, we assume that it is valid for $G(q)$ and $f(q)$.

$$\nabla\big(G(q)f(q)\big) \neq \big(\nabla G(q)\big)f(q) + G(q)\nabla f(q) \tag{7}$$

$$D'f'(q) = \boldsymbol{H}(q)\, G(q)\, f(q) + G(q)\, \nabla f(q) \tag{8}$$

$$-\boldsymbol{A}(q)G(q)f(q) - \boldsymbol{H}(q)G(q)f(q)$$

$$= G(q)\big(\nabla f(q) - \boldsymbol{A}(q)\, f(q)\big)$$

$$D'f'(q) = G(q)Df(q) \tag{9}$$

Thus with that transformation pair not only the modulus of the function stays invariant but also the modulus of the covariant derivative stays invariant. Further

$$f'^{*}(q)D'f'(q) = f^{*}(q)\, G^{*}(q)\;\, G(q)\, Df(q) \tag{10}$$

$$= f^{*}(q)\, Df(q)$$

Above the right sided covariant derivative $D$ is defined

$$\vec{D}f(q) = \vec{\nabla}f(q) - \boldsymbol{A}(q)\, f(q) \tag{11}$$

The left sided covariant derivative is defined as:

$$f(q)\overleftarrow{D} = f(q)\overleftarrow{\nabla} - \; f(q)\, \boldsymbol{B}(q) \tag{12}$$

We will use $\overleftrightarrow{D}$ for both left sided and right sided covariant derivative:

$$\overleftrightarrow{D}f(q) = \frac{\vec{\nabla}f(q) + f(q)\overleftarrow{\nabla}}{2} - \boldsymbol{A}(q)\, f(q) - \; f(q)\, \boldsymbol{B}(q) \tag{13}$$

# 13 Conservation laws

## 13.1 Flux vector

The longitudinal direction **k** of $\boldsymbol{E}(q)$ and the direction **i** of $\boldsymbol{B}(q)$ fix two mutual perpendicular directions. This generates curiosity to the significance of the direction $\mathbf{k} \times \mathbf{i}$**.** With other words what happens with $\boldsymbol{E}(q) \times \boldsymbol{B}(q)$.

The **flux vector** $\mathfrak{S}(q)$ is defined as:

$$\mathfrak{S}(q) \stackrel{\text{def}}{=} \boldsymbol{E}(q) \times \boldsymbol{B}(q) \tag{1}$$

## 13.2 Conservation of energy

$$\begin{aligned} \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle &= \langle \boldsymbol{B}(q), \boldsymbol{\nabla} \times \boldsymbol{E}(q) \rangle - \langle \boldsymbol{E}(q), \boldsymbol{\nabla} \times \boldsymbol{B}(q) \rangle \\ &= -\langle \boldsymbol{B}(q), \nabla_0 \boldsymbol{B}(q) \rangle - \langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle \\ &\qquad - \langle \boldsymbol{E}(q), \nabla_0 \boldsymbol{E}(q) \rangle \\ &= -½\nabla_0 (\langle \boldsymbol{B}(q), \boldsymbol{B}(q) \rangle + \langle \boldsymbol{E}(q), \boldsymbol{E}(q) \rangle) \\ &\qquad - \langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle \end{aligned} \tag{1}$$

The **field energy density** is defined as:

$$\begin{aligned} u_{field}(q) &= ½(\langle \boldsymbol{B}(q), \boldsymbol{B}(q) \rangle + \langle \boldsymbol{E}(q), \boldsymbol{E}(q) \rangle) \\ &= u_B(q) + u_E(q) \end{aligned} \tag{2}$$

$\mathfrak{S}(q)$ can be interpreted as the **field energy current density**.
The continuity equation for field energy density is given by:

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle = -\phi_0(q) \langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle \tag{3}$$

This means that $\langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle$ can be interpreted as a source term.

$\phi_0(q)\boldsymbol{E}(q)$ represents **force** per unit volume.

$\phi_0(q)\langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle$ represents **work** per unit volume, or, in other words, the power density. It is known as the Lorentz power density and is equivalent to the time rate of change of the mechanical energy density of the charged particles that form the current $\boldsymbol{\phi}(q)$**.**

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \tag{4}$$

$$\nabla_0 u_{mechanical} = \langle \boldsymbol{E}(q), \boldsymbol{\phi}(q) \rangle = \phi_0(q) \langle \boldsymbol{E}(q), \boldsymbol{v}(q) \rangle \tag{5}$$

$$\nabla_0 \Big( u_{field}(q) + u_{mechanical}(q) \Big) = -\langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle \tag{6}$$

Total change within $V$ = flow into $V$ + production inside $V$ (7)

$$u(q) = u_{field}(q) + u_{mechanical}(q) = u_B(q) + u_E(q) + u_{mechanical}(q) \tag{8}$$

$$U = U_{field} + U_{mechanical} = U_B + U_E + U_{mechanical} = \int_V u \, dV \tag{9}$$

(10)

$$\frac{d}{dt}\int_V u\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{S}\rangle dS + \int_V s_0\, dV$$

Here the source $s_0$ is zero.

### 13.3 How to interprete $U_{mechanical}$

$U_{mechanical}$ is the energy of the private field (state function) of the involved particle(s).

### 13.4 Conservation of linear momentum

$\mathfrak{S}(q)$ can also be interpreted as the **field linear momentum density**. The time rate change of the field linear momentum density is:

$$\begin{aligned}\nabla_0 \mathfrak{S}(q) &= \boldsymbol{g}_{field}(q) = \nabla_0\, \boldsymbol{E}(q) \times \boldsymbol{B}(q) \\ &\quad + \boldsymbol{E}(q) \times \nabla_0 \boldsymbol{B}(q)\end{aligned} \tag{1}$$

$$= \big(\boldsymbol{\nabla} \times \boldsymbol{B}(q) - \boldsymbol{\rho}(q)\big) \times\ \boldsymbol{B}(q) - \boldsymbol{E}(q) \times \boldsymbol{\nabla} \times \boldsymbol{E}(q) \tag{2}$$

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{E}) = \boldsymbol{E} \times (\boldsymbol{\nabla}\ \times\ \boldsymbol{E}) &= \langle \boldsymbol{\nabla E}\, , \boldsymbol{E}\rangle - \langle \boldsymbol{E}, \boldsymbol{E}\rangle \\ &= ½\boldsymbol{\nabla}\langle \boldsymbol{E}\, , \boldsymbol{E}\rangle - \langle \boldsymbol{E}, \boldsymbol{E}\rangle\end{aligned} \tag{3}$$

$$= -\boldsymbol{\nabla}(\mathbf{EE}) + ½\boldsymbol{\nabla}\langle \boldsymbol{E}\, , \boldsymbol{E}\rangle + \langle \boldsymbol{\nabla}\, , \boldsymbol{E}\rangle \boldsymbol{E}$$

$$= -\boldsymbol{\nabla}(\mathbf{EE} + ½\mathbf{1_3}\langle \boldsymbol{E}\, , \boldsymbol{E}\rangle) + \langle \boldsymbol{\nabla}\, , \boldsymbol{E}\rangle \boldsymbol{E}$$

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{B}) = \boldsymbol{B} \times (\boldsymbol{\nabla}\ \times\ \boldsymbol{B}) \\ = -\boldsymbol{\nabla}(\mathbf{BB} + ½\mathbf{1_3}\langle \boldsymbol{B}\, , \boldsymbol{B}\rangle) + \langle \boldsymbol{\nabla}\, , \boldsymbol{B}\rangle \boldsymbol{B}\end{aligned} \tag{4}$$

$$\boldsymbol{H}(\boldsymbol{B}) = -\boldsymbol{\nabla}(\mathbf{BB} + \tfrac{1}{2}\mathbf{1_3}\langle \boldsymbol{B}, \boldsymbol{B}\rangle) \tag{5}$$

$$\begin{aligned}\nabla_0 \mathfrak{S}(q) &= \boldsymbol{G}(\boldsymbol{B}) + \boldsymbol{G}(\boldsymbol{E}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) + \langle \boldsymbol{\nabla}, \boldsymbol{B}\rangle \boldsymbol{B} \\ &\quad + \langle \boldsymbol{\nabla}, \boldsymbol{E}\rangle \boldsymbol{E} \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) \\ &\quad - \rho_0(q)\, \boldsymbol{E}(q) \\ &= \boldsymbol{H}(\boldsymbol{E}) + \boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{f}(q) = \boldsymbol{\mathcal{T}}(q) - \boldsymbol{f}(q)\end{aligned} \tag{6}$$

$\mathcal{T}(\mathrm{q})$ is the linear momentum flux tensor.

The linear momentum of the field contained in volume $V$ surrounded by surface $S$ is:

$$\begin{aligned}\boldsymbol{P}_{field} &= \int_V \boldsymbol{g}_{field}\, dV \\ &= \int_V \rho_0 \boldsymbol{\phi}\, dV \\ &\quad + \int_V \langle \nabla \boldsymbol{\phi}, \boldsymbol{E}\rangle\, dV + \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{EA}\rangle dS\end{aligned} \tag{7}$$

$$\boldsymbol{f}(q) = \boldsymbol{\rho}(q) \times \boldsymbol{B}(q) + \rho_0(q)\, \boldsymbol{E}(q) \tag{8}$$

Physically, $\boldsymbol{f}(q)$ is the Lorentz force density. It equals the time rate change of the mechanical linear momentum density $\boldsymbol{g}_{mechanical}$.

$$\boldsymbol{g}_{mechanical}(q) = \rho_{0m}(q)\boldsymbol{v}(q) \tag{9}$$

The force acted upon a single particle that is contained in a volume $V$ is:

$$\boldsymbol{F} = \int_V \boldsymbol{f}\, dV = \int_V (\boldsymbol{\rho} \times \boldsymbol{B} + \rho_0 \boldsymbol{E})\, dV \tag{10}$$

Brought together this gives:

$$\nabla_0 \left( \boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{T}}(q) \rangle \tag{11}$$

This is the continuity equation for linear momentum.

The component $\mathcal{T}_{ij}$ is the linear momentum in the i-th direction that passes a surface element in the j-th direction per unit time, per unit area.

Total change within $V$ = flow into $V$ + production inside $V$ (12)

$$\boldsymbol{g}(q) = \boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \tag{13}$$

$$\boldsymbol{P} = \boldsymbol{P}_{field} + \boldsymbol{P}_{mechanical} = \int_V \boldsymbol{g}\, dV \tag{14}$$

$$\frac{d}{dt}\int_V \boldsymbol{g}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{T}} \rangle dS + \int_V \boldsymbol{s}_{\boldsymbol{g}}\, dV \tag{15}$$

Here the source $\boldsymbol{s}_{\boldsymbol{g}} = 0$.

## 13.5 Conservation of angular momentum

### 13.5.1 *Field angular momentum*

The angular momentum relates to the linear momentum.

$$\boldsymbol{h}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}(q) \tag{1}$$

$$\boldsymbol{h}_{field}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{field}(q) \tag{2}$$

$$\boldsymbol{h}_{mechanical}(q) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{mechanical}(q) \tag{3}$$

$$\boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{\mathcal{T}}(q) \tag{4}$$

This enables the balance equation for angular momentum:

$$\nabla_0 \left( \boldsymbol{h}_{field}(\boldsymbol{q}_c) + \boldsymbol{h}_{mechanical}(\boldsymbol{q}_c) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) \rangle \tag{5}$$

Total change within $V$ = flow into $V$ + production inside $V$

$$\boldsymbol{J} = \boldsymbol{J}_{field} + \boldsymbol{J}_{mechanical} = \int_V \boldsymbol{h}\, dV \tag{6}$$

$$\frac{d}{dt} \int_V \boldsymbol{h}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{K}} \rangle dS + \int_V \boldsymbol{s}_{\boldsymbol{h}}\, dV \tag{7}$$

Here the source $\boldsymbol{s}_{\boldsymbol{h}} = 0$.

For a localized charge density contained within a volume $V$ holds for the mechanical torsion:

$$\tau(\boldsymbol{q}_c) = \int_V (\boldsymbol{q}' - \boldsymbol{q}_c) \times \boldsymbol{f}(q')dV \tag{8}$$

$$= \int_V (\boldsymbol{q}' - \boldsymbol{q}_c) \times \left(\rho_0(q')\boldsymbol{E}(q') + \boldsymbol{j}(q') \times \boldsymbol{B}(q')\right)dV$$

$$= Q(\boldsymbol{q} - \boldsymbol{q}_c) \times (\boldsymbol{E}(q) + \boldsymbol{v}(q) \times \boldsymbol{B}(q))$$

$$\boldsymbol{J}_{field}(\boldsymbol{q}_c) = \boldsymbol{J}_{field}(\boldsymbol{0}) + \boldsymbol{q}_c \times \boldsymbol{P}(q) \tag{9}$$

Using

$$\langle \nabla \boldsymbol{a}, \boldsymbol{b} \rangle = \boldsymbol{n}_\nu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{10}$$

$$\langle \boldsymbol{b}, \nabla \boldsymbol{a} \rangle = \boldsymbol{n}_\mu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{11}$$

holds

$$\boldsymbol{J}_{field}(\boldsymbol{0}) = \int_V \boldsymbol{q}' \times \mathfrak{S}(q')dV \tag{12}$$

$$= \int_V \boldsymbol{q}' \times \boldsymbol{E}(q') \times \nabla \times \boldsymbol{\phi}(q')\, dV$$

$$= \int_V (\boldsymbol{q}' \times \langle(\boldsymbol{\nabla\phi}), \boldsymbol{E}\rangle - \langle \boldsymbol{q}' \times \boldsymbol{E}, (\boldsymbol{\nabla\phi})\rangle)\, dV$$

$$= \int_V \boldsymbol{q}' \times \langle(\boldsymbol{\nabla\phi}), \boldsymbol{E}\rangle dV$$

$$+ \int_V \boldsymbol{E} \times \boldsymbol{\phi}\, dV - \int_V \langle \boldsymbol{\nabla}, \boldsymbol{E}\boldsymbol{q}' \times \boldsymbol{\phi}\rangle dV$$

$$+ \int_V (\boldsymbol{q}' \times \boldsymbol{\phi})\langle \boldsymbol{\nabla}, \boldsymbol{E}\rangle dV$$

### 13.5.2 Spin

Define the non-local spin term, *which does not depend on* $\boldsymbol{q}'$ as:

$$\boldsymbol{\Sigma}_{field} = \int_V \boldsymbol{E}(q) \times \boldsymbol{\phi}(q) dV \tag{13}$$

Notice

$$\boldsymbol{\phi}(q) \times \boldsymbol{\nabla}\phi_0(q) = \phi_0 \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) + \boldsymbol{\nabla} \times \left(\phi_0(q)\boldsymbol{\phi}(q)\right)$$

And

$$\boldsymbol{L}_{field}(\boldsymbol{0}) = \int_V \boldsymbol{q}' \times \langle(\boldsymbol{\nabla\phi}), \boldsymbol{E}\rangle dV + \int_V \boldsymbol{q}' \times \rho_0 \boldsymbol{\phi} dV \tag{14}$$

Using Gauss:

(15)

$$\int_V \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{a} \rangle dS$$

And

$$\rho_0 = \langle \boldsymbol{\nabla}, \boldsymbol{E} \rangle \tag{16}$$

Leads to:

$$\boldsymbol{J}_{field}(\boldsymbol{0}) = \boldsymbol{\Sigma}_{field} + \boldsymbol{L}_{field}(\boldsymbol{0}) + \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{E}\boldsymbol{q}' \times \boldsymbol{\phi} \rangle dS \tag{17}$$

### 13.5.3 Spin discussion

The spin term is defined by:

$$\boldsymbol{\Sigma}_{field} = \int_V \boldsymbol{E}(q) \times \boldsymbol{\phi}(q) dV \tag{1}$$

In free space the charge density $\rho_0$ vanishes and the scalar potential $\phi_0$ shows no variance. Only the vector potential $\boldsymbol{\phi}$ may vary with $q_0$. Thus:

$$\boldsymbol{E} = \boldsymbol{\nabla}\phi_0 - \nabla_0 \boldsymbol{\phi} \approx -\nabla_0 \boldsymbol{\phi} \tag{2}$$

$$\boldsymbol{\Sigma}_{field} \approx \int_V (\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q) dV \tag{3}$$

Depending on the selected field $\boldsymbol{\Sigma}_{field}$ has two versions that differ in their sign. These versions can be combined in a single operator:

(4)

$$\boldsymbol{\Sigma}_{field} = \begin{bmatrix} \boldsymbol{\Sigma}^{+}{}_{field} \\ \boldsymbol{\Sigma}^{-}{}_{field} \end{bmatrix}$$

If $\frac{\boldsymbol{\phi}(q)}{|\boldsymbol{\phi}(q)|}$ can be interpreted as tantrix $(q_0)$ ) and $\frac{\nabla_0 \boldsymbol{\phi}(q)}{|\nabla_0 \boldsymbol{\phi}(q)|}$ can be interpreted as the principle normal $\boldsymbol{N}(q_0)$, then $\frac{(\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q)}{|(\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q)|}$ can be interpreted as the binormal $\boldsymbol{B}(q_0)$.

From these quantities the curvature and the torsion[149] can be derived.

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{5}$$

[149]Path characteristics

# 14 Lorentz transformation

Differences between positions in subsequent members of the sequence of HBM pages can be interpreted as displacements. The displacement is a coordinate transformation. For the properties of this transformation it does not matter where the displacement starts or in which direction it is taken.

The same holds for displacements that concern sequence members that are located further apart. The corresponding displacements form a group. The displacement is a function of both the position and the sequence number. The displacement $z, t \rightarrow z', t'$ can be interpreted as a coordinate transformation and can be described by a matrix.

$$\begin{bmatrix} t' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & \delta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} \tag{1}$$

The matrix elements are interrelated. When the displacement concerns a uniform movement, the interrelations of the matrix elements become a function of the speed $v$. The group properties together with the isomorphism of space fix the interrelations.

$$\begin{bmatrix} t' \\ z' \end{bmatrix} = 1/\sqrt{1 + kv^2} \begin{bmatrix} 1 & kv \\ -v & 1 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} \tag{2}$$

If $k$ is positive, then there may be transformations with $kv^2 \gg 1$ which transform time into a spatial coordinate and vice versa. This is considered to be unphysical. The Hilbert book model also supports that vision.

The condition k = 0 corresponds to a Galilean transformation

$$\begin{bmatrix} t' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} \tag{3}$$

The condition $k < 0$ corresponds to a Lorentz transformation. We can set $kc^2 = -1$, where $c$ is an invariant speed that corresponds to the maximum of $v$.

$$\begin{bmatrix} t' \\ z' \end{bmatrix} = 1/\sqrt{1 - v^2/c^2} \begin{bmatrix} 1 & -v/c^2 \\ -v & 1 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} \tag{4}$$

The Lorentz transformation corresponds with the situation in which a maximum speed occurs.

Since in each progression step photons step with a non-zero space step and both step sizes are fixed, the speed of the photon at microscopic scale is fixed. No other particle goes faster, so in the model a maximum speed occurs. With other words when sequence members at different sequence number are compared, then the corresponding displacements can be described by Lorentz transformations.

Lorentz transformations introduce the phenomena that go together with relativity, such as length contraction, time dilatation and relativity of simultaneity that occur when two inertial reference frames are considered.

$$\Delta t_c = \left(\Delta t_p - \Delta z_p \, v/c^2\right)/\sqrt{1 - v^2/c^2} \tag{5}$$

The term $\Delta z_p \, v/c^2$ introduces time dilatation. If $\Delta t_p = 0$ then depending on $v$ and $\Delta z_p$ the time difference $\Delta t_c$ is non-zero.

These phenomena occur in the Hilbert Book Model when different members of the sequence of Hilbert spaces are compared. Usually the inertial frames are spread over a range of Hilbert book pages.

Since the members of the sequence represent static status quos, the relativity of simultaneity restricts the selection of the inertial frames. Only one of the inertial frames can be situated completely in a single member of the sequence. In that case the other must be taken from a range of sequence elements.

# 15 Quaternionic metric

Wiki: In the mathematical field of differential geometry, a **metric tensor** is a type of function defined on a manifold (such as a surface in space) which takes as input a pair of tangent vectors $v$ and $w$ and produces a real number (scalar) $g(v,w)$ in a way that generalizes many of the familiar properties of the inner product of vectors in Euclidean space. In the same way as an inner product, a metric tensor is used to define the length of and angle between tangent vectors, but is not required to be positive-definite.

The Palestra is a continuous and compact space. It is characterized by a continuous quaternionic distribution $\wp$ that has a single sign flavor. This means that in the Palestra no reflections occur. Every location of the Palestra is characterized by a metric tensor, which defines for every location and every direction an infinitesimal distance. This distance is an invariant with respect to transformations that represent continuous symmetries. These transformations form a Lie group. Apart from a small deviation, we follow

the ideas of Mendel Sachs[150]. The infinitesimal distance in the curved Palestra is defined as[151]

$$ds = \sum_{\mu} \frac{\partial \wp}{\partial x_\mu} dx_\mu = q^\mu(x)\, dx_\mu \tag{1}$$

The quaternionic distribution $\wp(x)$ specifies the metric of the Palestra. $x_\mu$ are coordinates in the **flat** parameter space of $\wp(x)$.

$ds$ is quaternion valued. $dx_\mu$ is real valued.

We define the factors $q^\mu(x)$ as derivatives of the quaternionic distribution $\wp(x)$

$$q^\mu(x) = \frac{\partial \wp}{\partial x_\mu} \tag{2}$$

They belong to the flat parameter space of the quaternionic distribution $\wp(x)$.

$dx_0$ is the infinitesimal spacetime interval $d\varsigma$. (In this discussion we suppose it to be time-like). Apart from a constant factor it equals the progression interval $d\tau$. It is directly related to the proper time interval $d\tau$.

$$dx_0 = c\, d\tau \tag{3}$$

$\tau$ is the proper time. It is measured at the location of the observed item. Thus it is independent of who is observing.

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[150] See: Symmetry in electrodynamics; M. Sachs

[151] This approach differs from the approach of Mendel Sachs.

$$dx_0^2 = c^2 d\tau^2 = c^2\, dt^2 - q_\mu q_\nu^* dx^\mu dx^\nu;\ \mu,\nu = 1,2,3 \tag{4}$$

$ds$ appears to relate to the coordinate time interval $c\ dt$.
The coordinate time interval $dt$ follows from

$$ds\ ds^* = q_\mu q_\nu^* dx^\mu dx^\nu;\ \mu,\nu = 0,1,2,3 \tag{5}$$

$$= c^2 dt^2 = c^2\, d\tau^2 + q_\mu q_\nu^* dx^\mu dx^\nu;\ \mu,\nu = 1,2,3$$

$t$ is the coordinate time. It is measured at the location of the observer.

The infinitesimal progression interval $d\tau$ is a model invariant of the HBM. The infinitesimal spacetime interval $d\text{ş}$ is a physical invariant. For that reason the infinitesimal spacetime interval $d\text{ş}$ is used for the definition of the local metric tensor.

By using the spacetime interval for defining the metric, the Palestra becomes a pseudo-Riemannian manifold with a Minkowski signature.

When instead the coordinate time interval is used as the controlling interval for the metric, the Palestra is a Riemannian manifold with an Euclidean signature. The coordinate time interval is not a physical invariant.

So $ds$ of formula (1) is related wit $c\ dt$. $t$ is the coordinate time. It is measured at the location of the observer.

In his version of formula (1), Mendel Sachs uses the proper time interval instead of the coordinate time interval.

The spacetime interval is a measure which is independent from the observer. The corresponding Lie group is the Einstein group[152]. The Einstein group relates to a 1+3D pseudo Riemannian space with Minkowski signature.

The Einstein group does not contain any discrete transformations, such as the Poincare group does. With other words, the spacetime interval corresponds with the Einstein group, rather than with the Poincare group and the metric tensor has 16 components, instead of 10.

~~The electromagnetic field components of the metric tensor are given by:~~

$$~~F^{\mu\nu} = \frac{1}{8}QR(q^{\mu}q^{\nu} - q^{\nu}q^{\mu}) = \frac{1}{4}QR\ im(q^{\mu}q^{\nu})~~ \qquad ~~(6)~~$$

~~R is the scalar curvature. Q is the magnetic flux.~~

~~This formula defines 6 of the 16 components of the metric tensor.~~

[152] http://en.wikipedia.org/wiki/Einstein_group#The_Einstein_group
See: Symmetry in electrodynamics; M. Sachs

# 16 The universe of items

All particles have properties. Some of these properties expose as sources of corresponding fields. Via superposition these fields interact. Long range fields such as the gravitation field have universe wide effects.

## 16.1 Inertia

The influence of items in universe may decrease with distance according to some function $f(r)$ of the distance $r$[153]. However the number of contributing items increases with the distance. Depending on function $f(r)$ the most probable result is that the strongest influence comes from the cooperative activity of the most distant items. Due to the enormous number of items in the universe, any variation of the influences of the distant items averages away. This also holds for the density distribution of the items. So there exists a fairly uniform background influence caused by the universe of items. What will happen, can be deduced from an equivalent of Dennis Sciama's analysis[154][155][156]. We will take his analysis as a guide. Sciama's analysis uses a different setting: the (observed) 3D space and coordinate time and Sciama applies Maxwell field theory. We use the coordinate space defined by an appropriate coordinate operator that resides in the Gelfand triple of the separable Hilbert space and the progression parameter $t$ that relates to the progression step counter as our setting. A location in this coordinate

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[153] http://en.wikipedia.org/wiki/Bertrand's_theorem and Role of the particle locator operator

[154] http://arxiv.org/abs/physics/0609026v4.pdf

[155] http://www.adsabs.harvard.edu/abs/1953MNRAS.113...34S

[156] http://rmp.aps.org/abstract/RMP/v36/i1/p463_1

space represents a location on the unit sphere of Gelfand triple. This last location is taken by the eigenvector that corresponds to the first location.

As stated before, the unit sphere of Gelfand triple is an affine space. This means that we must treat position as relative data. With other words, the eigenspace of the coordinate operator has no absolute origin. Instead of Sciama's usage of Maxwell fields we will use quaternionic field theory that is applied to quaternionic probability amplitude distributions (QPAD's).

We may specify a QPAD for usage in a continuity equation. In that case we specify in fact the combination of a charge density distribution and a current density distribution. As long as the charges and the currents stay static, the QPAD is a static object.

In the continuity equation we consider the influence of the QPAD on the whole universe. Here we consider the influence of the universe on a local charge or current. For this purpose we can use similar QPAD's and volume integrals!

At large distances, the density $\rho$ of the contributing items can be considered to be uniformly distributed. Also any variance in strength other than the dependence on $r$ becomes negligible because the differences between functions $\{f(r)\}$ average away. We take the average of the strength of $\{f(r)\}$ as the significant parameter. We combine it with $\rho$. Therefore the average of $\rho$ can be taken out of the integral.

The total potential $\Phi$ at the location of the influenced subject is[157]

[157] http://en.wikipedia.org/wiki/Newtonian_potential

$$\Phi = -\int_V \frac{\rho}{r} dV = -\rho \int_V \frac{dV}{r} = 2\pi R^2 \rho \quad (1)$$

What we have here is the reverse of the definition of the potential that goes together with a charge distribution.

The integral is taken over the coordinate space volume $V$. Indirectly, the integral is taken over the unit sphere of the Gelfand triple. This is an affine space. The parameter $r$ is the length of the vector from the actor to the location of the subject. The considered subject is located somewhere in the affine coordinate space. All other subjects have positions relative to that considered subject. Thus, apart from its dependence on the average value of $\rho$, $\Phi$ is a huge constant. Sciama relates $\Phi$ to the gravitational constant $G$.

$$G = {-c^2}/{\Phi}$$

As a consequence we can consider the universe as a very large rigid body. If nothing else happens then all influences compensate each other.

In contrast to Sciama, we use imaginary quaternions rather than 3D vectors. This also avoids the distracting factor *i*.

If the considered subject moves relative to the universe with a uniform speed **v**, then a vector potential **A** is generated.

$$\mathbf{A} = -\int_V \frac{\mathbf{v}\,\rho}{c\,r} dV \quad (2$$

Both $\rho$ and $\mathbf{v}$ are independent of $r$. The product $\mathbf{v}\,\rho$ represents a current. Together with the constant $c$ they can be taken out of the integral. Thus

$$\mathbf{A} = \Phi \cdot \mathbf{v}/c \qquad (3$$

The notions of charge and current correspond to equivalent notions in Noether's theorem[158]. Here we talk about inertia. Thus charge may symbolize mass. Or even better; it symbolizes the coupling factor that plays the role of mass.

The progression parameter t plays the role of "time". Be aware, in our setting it is the progression parameter, which is not the usual notion of time.

According to the Helmholtz theorem the field that is derived from the above potential can be split into a divergence free part and a rotation free part. The Helmholtz decomposition theorem only concerns the static versions of the derived field. It is related to the fact that the Fourier transform of a vector field can be split in a longitudinal and a transversal version. A corresponding split of the multi-dimensional Dirac delta function in a longitudinal and a transversal version exists as well.

According to Maxwell field theory as well as according to quaternionic field theory, a variation of $\mathbf{v}$ goes together with a variation of $\mathbf{A}$. On its turn this goes together with a non-zero field $\dot{\boldsymbol{A}}(\boldsymbol{r}, t)$ which is a **dynamical** part of the QPAD. Thus, with varying $\mathbf{v}$ the QPAD is no longer static.[159]

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[158] http://en.wikipedia.org/wiki/Noether%27s_theorem

[159] See Differentiation

Sciama uses a Maxwell equation to explain the relation between $\partial \mathbf{v}/\partial \mathrm{t}$ and $\dot{\boldsymbol{A}}(\boldsymbol{r}, t)$. Our setting differs, but the quaternionic field theory delivers the same results.

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\boldsymbol{\nabla}\Phi(\boldsymbol{r}, t) - \frac{1}{c} \cdot \dot{\boldsymbol{A}}(\boldsymbol{r}, t) \tag{4}$$

$$\widetilde{\boldsymbol{E}}(\boldsymbol{k}, \omega) = -\boldsymbol{k} \cdot \widetilde{\Phi}(\boldsymbol{k}, \omega) - \frac{1}{c} \cdot \omega \widetilde{\boldsymbol{A}}(\boldsymbol{k}, \omega) \tag{5}$$

If we exclude the first term because it is negligible small, we get:

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\Phi}{c^2} \cdot \frac{\partial \mathbf{v}}{\partial \mathrm{t}} = G \cdot \frac{\partial \mathbf{v}}{\partial \mathrm{t}} \tag{6}$$

Remark: As soon as we turn to the dynamic version (4) an extra component $\dot{\boldsymbol{A}}$ of field $\boldsymbol{E}$ appears that corresponds to acceleration $\partial \mathbf{v}/\partial \mathrm{t}$.[160]

As already claimed, in our setting the component $\boldsymbol{\nabla}\Phi$ of the field $\boldsymbol{E}$ is negligible. With respect to this component the items compensate each other's influence. This means that if the influenced subject moves with uniform speed **v**, then $\boldsymbol{E} \approx 0$. However, a vector potential $\boldsymbol{A}$ is present due to the movement of the considered item. Like $\rho$ and $\rho\mathbf{v}$, $\Phi$ and $\boldsymbol{A}$ together form a QPAD. Any acceleration of the considered item goes together with an extra non-zero $\boldsymbol{E}$ field. In this way the universe of items causes inertia in the form of a

[160] See http://www.plasma.uu.se/CED/Book; formula 3.25 or Appendix; Maxwell equations

force that acts upon the accelerating item's charge. The item is the carrier of the charge $\rho$.

The situation in curved space differs. When the path of the item coincides with a **geodesic**, then it can be travelled free of extra generated fields. Thus, a uniform movement along the geodesic does not on itself generate a reaction of the universe of items. Any alteration of that uniform movement will go together with the existence of an extra field. The physical name for this reaction is **action**. It usually gets the symbol **S**.

On the other hand, as we see from inertia, any field change goes together with a corresponding acceleration. Uniform movements do cause displacement of charges. In a curved environment it changes the configuration of the QPAD. Thus, in that case, even an originally static QPAD may be affected.

We may reverse the conclusion of the analysis:

**An extra field component goes together with an acceleration of the local item.**

The acceleration can be seen as the consequence of a local curvature and vice versa. Thus, the extra field goes together with a local curvature.

It must be noticed that the original analysis of Sciama uses observable position space rather coordinate space and it uses a different notion of time. However, the general conclusion stays the same. Sciama's analysis is criticized because it uses infinite speed of information transfer. Since we do not work in observable position space, we do not encounter coordinate time. So for the setting of our analysis, this criticism is misplaced. Most part of the story plays in a stationary QPAD condition. As long as the movement

stays uniform, the QPAD is static. Any acceleration deviates from this stationary condition. This deviation goes together with an extra field component and it goes together with a local curvature.

Coordinate time[161] relates to observations of position. It is a local player in the game, where the progression parameter is a global player.

The situation with electromagnetic fields is different, because with this field positive and negative charges compensate each other's long range influence. For that reason there exists no electromagnetic background influence. The masses of the gravitational and inertial fields only compensate each other's long range influences through geometrical circumstances. Still in combination, they create gigantic potentials.

The particles outside the information horizons also contribute to the inertia.

Thus when through uniform movement the local field configuration changes, then that change goes together with an acceleration of the local item.

## 16.2 Nearby items

Items that are located nearby have a different effect. In general their influence will not have its strength equal to the average strength. Further these items are not uniformly distributed. Still at macroscopic distances their influence depends on inter-distance as $f = -k/r$. As a consequence their influences form a landscape of

[161] Dynamics; Relativity

which the effects will become sensible in the action of the fields that surround the considered item. For observers, this landscape will form a curved action space. The considered item will try to follow a geodesic through that curved space.

## 16.3 Rotational inertia

Besides linear inertia there exists rotational inertia. In a non-rotating universe hold near the origin $\mathbf{A} = 0$ and $\Phi = -c^2/G$. We choose units such that c=G=1. In a universe rotating slowly with angular speed ω hold

$$A_x = \omega \cdot y \tag{1}$$

$$A_y = -\omega \cdot x \tag{2}$$

$$A_z = 0 \tag{3}$$

$$\Phi = -\sqrt{1 + (\omega \cdot r)^2} \tag{4}$$

A constant angular movement meets the fields that correspond to a centripetal force.

The field $\boldsymbol{E}$ has the form

$$\boldsymbol{E} = \frac{\omega^2 \boldsymbol{r}}{\sqrt{1 + \omega^2 r^2}} \tag{5}$$

An added uniform speed $\boldsymbol{v}$ meets the fields corresponding to a Coriolis force.

$$\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{A} = 2 \cdot \boldsymbol{\omega} \tag{6}$$

$$\boldsymbol{v} \times \boldsymbol{H} = 2 \cdot \boldsymbol{v} \times \boldsymbol{\omega} \tag{7}$$

The forces are usually considered as *fictitious* but they are actually caused by inertia. Sciama treats them in section 5 of his paper. Like fields of linear inertia these rotation related fields correspond to actions of the manipulator.

## 16.4 Computation of the background QPAD

The same line of thinking that lead to the formula for the local potential in section 12.1-(1) can be applied to the computation of the QPAD that represents the local background field.

The ensemble $\{ \psi_i(r, q_i)\}$ is distributed randomly over the center points $\{q_i\}$ in an affine parameter space. At a given point P in this space the superposition of all $\{ \psi_i(r, P)\}$ will be constructed.

This superposition will be renormalized and then indicated by $\Phi(r, P)$.

Thus,

$$\int_V |\Phi(r, P)|^2 \, dV = 1 \tag{1}$$

In this superposition the largest contribution comes from the $\psi_i(r, q_i)$ for which the $q_i$ is farthest from P. Further the directions of the imaginary part of $\Phi(r, P)$.are reversed with respect to the directions in the $\psi_i(r, q_i)$.

Especially at long distances, all differences are smoothed away via an averaging process.

The result is that for the average QPAD $\Psi(r, P)$:

$$\Phi(r, P) = \Psi^*(r, P) \tag{2}$$

We will interpret $\Phi(r, 0)$ as the background QPAD.

Since we are talking about quaternionic distributions it is possible that every sign flavor has its own background QPAD.

# 17 Path characteristics

The Frenet-Serret frame is devised for describing curved paths of particles

Let $\{\boldsymbol{\alpha}_{qt}\}_t = \boldsymbol{\alpha}(q,t)$ describe a curved path consisting of infinitesimal steps through a landscape $\{\boldsymbol{\alpha}_q\}_q = \boldsymbol{\alpha}(q)$ of imaginary quaternions $\boldsymbol{\alpha}_{qt}$, such that $||\dot{\boldsymbol{\alpha}}(q(t))|| = 1$ for all t.

The 3D Frenet-Serret frame for the above path is given by:

$$\boldsymbol{T}(q(t)) \coloneqq \frac{\partial \boldsymbol{\alpha}(q(t))}{\partial t} = \boldsymbol{T}(t) = \dot{\boldsymbol{\alpha}}(t) \tag{1}$$

$$\kappa(t) \coloneqq ||\dot{\boldsymbol{T}}(t)|| \tag{2}$$

$$\kappa(t) \cdot \boldsymbol{N}(t) \coloneqq \dot{\boldsymbol{T}}(t) \tag{3}$$

$$\boldsymbol{B}(t) \coloneqq \boldsymbol{T}(t) \times \boldsymbol{N}(t) \tag{4}$$

$$||\boldsymbol{T}(t)|| = ||\boldsymbol{N}(t)|| = ||\boldsymbol{B}(t)|| = 1 \tag{5}$$

$\boldsymbol{T}(t)$ is the **tantrix** of curve $\boldsymbol{\alpha}(q(t))$ at instance t.
$\boldsymbol{N}(t)$ is the **principal normal** of curve $\boldsymbol{\alpha}(q(t))$ at instance t. It is only defined when $\kappa(t) \neq 0$.
$\boldsymbol{B}(t)$ is the **binormal** of curve $\boldsymbol{\alpha}(q(t))$ at instance t.
$\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$ and $\boldsymbol{B}(t)$ are imaginary quaternions.
$\kappa(t)$ is the curvature of curve at $\boldsymbol{\alpha}(q(t))$ at instance t.
$r(t) = 1/\kappa(t)$ is the radius of curvature at instance t.
$\tau(t)$ is the torsion of curve $\boldsymbol{\alpha}(q(t))$ at instance t.

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{6}$$

The Frenet-Serret curves have particular characteristics. The path may be curved and curled. The path is completely determined by its tantrix, curvature and torsion given by functions of t. Each coordinate of the quaternionic function $\boldsymbol{\alpha}(q(t))$ has its own set of characteristics. This means that for a given quaternionic function these characteristics are quaternions rather than real numbers and they are all functions of parameter t.

## 17.1 Path equations

The path equations are given by

$$\dot{\boldsymbol{T}}(t) = \kappa(t) \cdot \boldsymbol{N}(t) \tag{1}$$

$$\begin{aligned} \dot{\boldsymbol{N}}(t) &= -\kappa(t) \cdot \boldsymbol{T}(t) + \tau(t) \cdot \boldsymbol{B}(t) \\ &= -\kappa(t) \cdot \boldsymbol{T}(t) + \tau(t) \cdot \boldsymbol{T}(t) \times \boldsymbol{N}(t) \end{aligned} \tag{2}$$

$$\begin{aligned} \dot{\boldsymbol{B}}(t) &= -\tau(t) \cdot \boldsymbol{N}(t) = \boldsymbol{T}(t) \times \dot{\boldsymbol{N}}(t) + \dot{\boldsymbol{T}}(t) \times \boldsymbol{N}(t) \\ &= \tau(t) \cdot \boldsymbol{T}(t) \times \boldsymbol{B}(t) \end{aligned} \tag{3}$$

## 17.2 Curve length

The curve length $l(a, b)$ is defined by:

$$l(a, b) = \int_{x=a}^{x=b} |\dot{\boldsymbol{\alpha}}(q(x))| \, dx \tag{1}$$

The integration over the square of the modulus delivers the **action** S of the curve.

$$S(a,b) = \int_{x=a}^{x=b} |\dot{\boldsymbol{\alpha}}(q(\mathrm{x}))|^2 \, dx \tag{2}$$

## 17.3 Reparameterization

The path characteristics κ(t) and τ(t) together with the curve length and the curve action are independent of any reparameterization $s(t)$ of the progression parameter t.

A natural reparameterization is given by $s(t) = l(t_0, t)$.

This turns the curve $\boldsymbol{\alpha}(q(\mathrm{t}))$ into a **natural** curve $\boldsymbol{\gamma}\big(q(\mathrm{s})\big)$:

$$\boldsymbol{\gamma}\big(q(\mathrm{s})\big) = \boldsymbol{\alpha}(q(\mathrm{t})) \tag{1}$$

Curves on a surface which minimize length between the endpoints are called geodesics.

The natural curve corresponds to a geodesic[162].

The consequence is that in three-dimensional space the corresponding movement obeys the geodesic equation[163]. The Lagrangian is an equivalent of this equation.

## 17.4 Non unity path

Instead of the unity sped path $\alpha(q,t)$ we use a vector function $\boldsymbol{\beta}(q,t)$. The function $\boldsymbol{\beta}(q,t)$ is supposed to be regular

.

$$\boldsymbol{T}(t) = \frac{\dot{\boldsymbol{\beta}}(q,t)}{\|\dot{\boldsymbol{\beta}}(q,t)\|} \tag{1}$$

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[162] http://en.wikipedia.org/wiki/Geodesic

[163] Euler Lagrange equations

$$\boldsymbol{B}(t) = \frac{\dot{\boldsymbol{\beta}}(q,t)\times\ddot{\boldsymbol{\beta}}(q,t)}{\left\|\dot{\boldsymbol{\beta}}(q,t)\times\ddot{\boldsymbol{\beta}}(q,t)\right\|} \tag{2}$$

$$\boldsymbol{N}(t) = \boldsymbol{B}(t) \times \boldsymbol{T}(t) \tag{3}$$

$$\kappa(\mathrm{t}) = \frac{\left\|\dot{\boldsymbol{\beta}}(q,t) \times \ddot{\boldsymbol{\beta}}(q,t)\right\|}{\left\|\dot{\boldsymbol{\beta}}(q,t)\right\|^3} \tag{4}$$

$$\tau(\mathrm{t}) = \frac{\left[\dot{\boldsymbol{\beta}}(q,t), \ddot{\boldsymbol{\beta}}(q,t), \dddot{\boldsymbol{\beta}}(q,t)\right]}{\left\|\dot{\boldsymbol{\beta}}(q,t) \times \ddot{\boldsymbol{\beta}}(q,t)\right\|^2} \tag{5}$$

$$= \frac{\langle \dot{\boldsymbol{\beta}}(q,t) \times \ddot{\boldsymbol{\beta}}(q,t), \dddot{\boldsymbol{\beta}}(q,t)\rangle}{\left\|\dot{\boldsymbol{\beta}}(q,t) \times \ddot{\boldsymbol{\beta}}(q,t)\right\|^2}$$

where

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c}\rangle \tag{6}$$

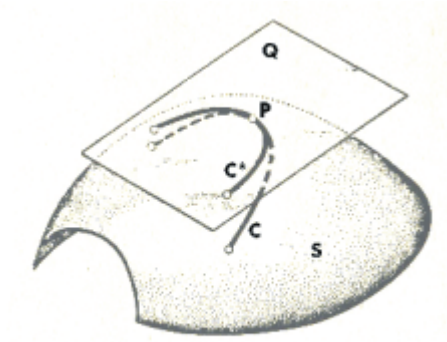


Fig. 3

### 17.5 Geodesic curvature

Let C be a curve on a surface S. The geodesic curvature $\kappa_g$ of C at a given point P is defined as the curvature, at P, of the orthogonal projection of C onto the plane Q tangent to S at point P. See Fig. 3, where C* is the projection of C onto the tangent plane Q. The geodesic curvature of C at P is defined then as the curvature of C* at P[164].

The total curvature $\boldsymbol{\kappa}$ is composed of the geodesic curvature $\kappa_g$ and the normal curvature $\boldsymbol{\kappa_n}$.

$$\boldsymbol{\kappa} = \boldsymbol{\kappa_g} + \boldsymbol{\kappa_n};\ \kappa_g = \|\boldsymbol{\kappa_g}\| \tag{1}$$

[164] See: http://www.solitaryroad.com/c335.html

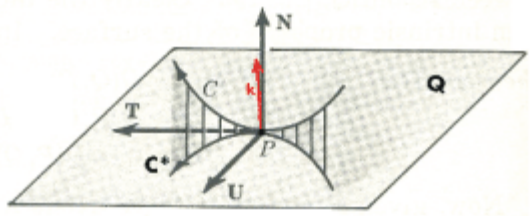


Fig. 4

$$\boldsymbol{U}(t) = \boldsymbol{N}(t) \times \boldsymbol{T}(t) \tag{2}$$

$\boldsymbol{\kappa_g}$ is directed along $\boldsymbol{U}(t)$.

$\boldsymbol{\kappa_g}$ and $\boldsymbol{\kappa_n}$ can be expressed in Christoffel symbols

## 17.6 Geodesic equations

$$\boldsymbol{\gamma}(s) = \boldsymbol{x}(u(s(t), v(s(t)) = \boldsymbol{\beta}(q, t) \tag{1}$$

$s(t)$ is the arc length. From here we interpret $\dot{\boldsymbol{\gamma}}(s)$ as

$$\dot{\boldsymbol{\gamma}}(s) = \frac{\partial \boldsymbol{\gamma}(s)}{\partial s} = \boldsymbol{T} \tag{2}$$

$$\boldsymbol{S} = \boldsymbol{n} \times \boldsymbol{T};\ \boldsymbol{S} \perp \boldsymbol{n};\ \boldsymbol{S} \perp \boldsymbol{T} \tag{3}$$

$$\dot{\boldsymbol{\gamma}}(s) = \boldsymbol{T} \tag{4}$$

$$\ddot{\boldsymbol{\gamma}}(s) = \dot{\boldsymbol{T}} = \kappa \boldsymbol{N} = \kappa_n \boldsymbol{n} + \kappa_g \boldsymbol{S} \tag{5}$$

$$\kappa_n = \langle \ddot{\boldsymbol{\gamma}}, \boldsymbol{n} \rangle \tag{6}$$

$$\kappa_g = \langle \ddot{\boldsymbol{\gamma}}, \boldsymbol{S} \rangle \tag{7}$$

$$\boldsymbol{\gamma}(s) = \boldsymbol{x}(u(s), v(s)) \tag{8}$$

$$\boldsymbol{T} = \dot{\boldsymbol{\gamma}}(s) = \frac{\partial \boldsymbol{x}}{\partial u}\big(u(s), v(s)\big)\dot{u}(s) + \frac{\partial \boldsymbol{x}}{\partial v}(u(s), v(s))\dot{v}(s) \tag{9}$$

$$\begin{aligned} \ddot{\boldsymbol{\gamma}} &= \left(\frac{\partial^2 \boldsymbol{x}}{\partial u^2}\dot{u} + \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v}\dot{v}\right)\dot{u} + \frac{\partial \boldsymbol{x}}{\partial u}\ddot{u} \\ &\quad + \left(\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v}\dot{u} + \frac{\dot{\partial}^2 \boldsymbol{x}}{\partial v^2}\dot{v}\right)\dot{v} + \frac{\partial \boldsymbol{x}}{\partial v}\ddot{v} \\ &= \frac{\partial^2 \boldsymbol{x}}{\partial u^2}\dot{u}^2 + 2\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v}\dot{u}\dot{v} + \frac{\partial^2 \boldsymbol{x}}{\partial v^2}\dot{v}^2 + \frac{\partial \boldsymbol{x}}{\partial u}\ddot{u} + \frac{\partial \boldsymbol{x}}{\partial v}\ddot{v} \\ &= \dot{u}^2 \left(\Gamma^{\mathrm{u}}_{\mathrm{uu}}\frac{\partial \boldsymbol{x}}{\partial u} + \Gamma^{\mathrm{v}}_{\mathrm{uu}}\frac{\partial \boldsymbol{x}}{\partial v} + L_{uu}\boldsymbol{n}\right) \\ &\quad + 2\dot{u}\dot{v} \left(\Gamma^{\mathrm{u}}_{\mathrm{uv}}\frac{\partial \boldsymbol{x}}{\partial u} + \Gamma^{\mathrm{v}}_{\mathrm{uv}}\frac{\partial \boldsymbol{x}}{\partial v} + L_{uv}\boldsymbol{n}\right) \\ &\quad + \dot{v}^2 \left(\Gamma^{\mathrm{u}}_{\mathrm{vv}}\frac{\partial \boldsymbol{x}}{\partial u} + \Gamma^{\mathrm{v}}_{\mathrm{vv}}\frac{\partial \boldsymbol{x}}{\partial v} + L_{vv}\boldsymbol{n}\right) + \ddot{u}\frac{\partial \boldsymbol{x}}{\partial u} + \ddot{v}\frac{\partial \boldsymbol{x}}{\partial v} \end{aligned} \tag{10}$$

$$= (\dot{u}^2\,\Gamma^{u}_{uu} + \,2\dot{u}\dot{v}\,\Gamma^{u}_{uv} + \dot{v}^2\,\Gamma^{u}_{vv} + \ddot{u})\frac{\partial \boldsymbol{x}}{\partial u}$$

$$+(\dot{u}^2\Gamma^{v}_{uu} + \,2\dot{u}\dot{v}\,\Gamma^{v}_{uv} + \dot{v}^2\,\Gamma^{v}_{vv} + \ddot{v})\frac{\partial \boldsymbol{x}}{\partial v}$$

$$+(\dot{u}^2\,L_{uu} + \,2\dot{u}\dot{v}\,L_{uv} + \dot{v}^2\,L_{vv})\boldsymbol{n}$$

$$= \left\{[\dot{u}\dot{v}]\begin{bmatrix}\Gamma^{u}_{uu} & \Gamma^{u}_{uv}\\ \Gamma^{u}_{vu} & \Gamma^{u}_{vv}\end{bmatrix}\begin{bmatrix}\dot{u}\\ \dot{v}\end{bmatrix} + \ddot{u}\right\}\frac{\partial \boldsymbol{x}}{\partial u}$$

$$+\left\{[\dot{u}\dot{v}]\begin{bmatrix}\Gamma^{v}_{uu} & \Gamma^{v}_{vu}\\ \Gamma^{v}_{uv} & \Gamma^{v}_{vv}\end{bmatrix}\begin{bmatrix}\dot{u}\\ \dot{v}\end{bmatrix} + \ddot{v}\right\}\frac{\partial \boldsymbol{x}}{\partial v}$$

$$+[\dot{u}\dot{v}]\begin{bmatrix}L_{uu} & L_{uv}\\ L_{vu} & L_{vv}\end{bmatrix}\begin{bmatrix}\dot{u}\\ \dot{v}\end{bmatrix}\boldsymbol{n}$$

The first two terms form the tangential component and the third term is the normal component.

$$\kappa_n = [\dot{u}\dot{v}]\begin{bmatrix}L_{uu} & L_{uv}\\ L_{vu} & L_{vv}\end{bmatrix}\begin{bmatrix}\dot{u}\\ \dot{v}\end{bmatrix}$$

In geodesic curves the tangential components are zero ($\kappa_g = \mathbf{0}$)
From this condition follow the geodesic equations:

$$\dot{u}^2\,\Gamma^{u}_{uu} + \,2\dot{u}\dot{v}\,\Gamma^{u}_{uv} + \dot{v}^2\,\Gamma^{u}_{vv} + \ddot{u} = 0 \tag{11}$$

$$\dot{u}^2\Gamma^{v}_{uu} + \,2\dot{u}\dot{v}\,\Gamma^{v}_{uv} + \dot{v}^2\,\Gamma^{v}_{vv} + \ddot{v} = 0 \tag{12}$$

# 18 Metric tensor field

The metric tensor is an example of a tensor field. This means that relative to a locally non-affected coordinate system[165] on the manifold, a metric tensor takes on the form of a symmetric matrix whose entries transform covariantly under changes to the coordinate system. Thus a metric tensor is a covariant symmetric tensor[166]. From the coordinate-independent point of view, a metric tensor is defined to be a non-degenerate symmetric bilinear form[167] on each tangent space that varies smoothly from point to point.

## 18.1 Curved path

In a Riemannian manifold[168] $M$ with metric tensor[169] $g$, the length of a continuously differentiable curve $\gamma: [a, b] \to M$ is defined by

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt \tag{1}$$

The distance $d(p, q)$ between two points $p$ and $q$ of $M$ is defined as the infimum[170] of the length taken over all continuous, piecewise continuously differentiable curves $\gamma: [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. With this definition of distance, geodesics in a

[165] http://en.wikipedia.org/wiki/Local_coordinate_system
[166] http://en.wikipedia.org/wiki/Symmetric_tensor
[167] http://en.wikipedia.org/wiki/Symmetric_bilinear_form
[168] http://en.wikipedia.org/wiki/Riemannian_manifold
[169] http://en.wikipedia.org/wiki/Metric_tensor
[170] http://en.wikipedia.org/wiki/Infimum

Riemannian manifold are then the locally distance-minimizing paths, in the above sense.

The minimizing curves of $L$ in a small enough open set[171] of $M$ can be obtained by techniques of calculus of variations[172]. Typically, one introduces the following action[173] or energy functional[174]

$$E(\gamma) = \tfrac{1}{2} \int_a^b g_{\gamma(t)(\dot{\gamma}(t),\dot{\gamma}(t))}\, dt \tag{2}$$

It is then enough to minimize the functional $E$, owing to the Cauchy–Schwarz inequality[175]

$$L(\gamma)^2 \leq 2(b-a)\, E(\gamma) \tag{3}$$

with equality if and only if $|d\gamma/dt|$ is constant.

The Euler–Lagrange[176] equations of motion for the functional $E$ are then given in local coordinates by

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \cdot \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} = 0 \tag{4}$$

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[171] http://en.wikipedia.org/wiki/Open_set

[172] http://en.wikipedia.org/wiki/Calculus_of_variations

[173] http://en.wikipedia.org/wiki/Action_(physics)

[174] http://en.wikipedia.org/wiki/Energy_functional

[175] http://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality

[176] Appendix; Euler Langrange equations

where $\Gamma^{\lambda}_{\mu\nu}$are the Christoffel symbols[177] of the metric. This is the **geodesic equation**.

## 18.2 Calculus of variations

Techniques of the classical calculus of variations[178] can be applied to examine the energy functional $E$. The first variation[179] of energy is defined in local coordinates by

$$\delta E(\gamma)(\varphi) = \frac{\partial}{\partial t}\Big|_{t=0} E(\gamma + t\,\varphi) \tag{1}$$

The critical points[180] of the first variation are precisely the geodesics. The second variation is defined by

$$\delta^2 E(\gamma)(\varphi, \psi) = \frac{\partial^2}{\partial t^2}\Big|_{t=0} E(\gamma + t\,\varphi + s\psi) \tag{2}$$

In an appropriate sense, zeros of the second variation along a geodesic γ arise along Jacobi fields[181]. Jacobi fields are thus regarded as variations through geodesics.

By applying variational techniques from classical mechanics[182], one can also regard geodesics as Hamiltonian flows[183]. They are so-

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[177] Metric tensor field; Christoffel symbols

[178] http://en.wikipedia.org/wiki/Calculus_of_variations

[179] http://en.wikipedia.org/wiki/First_variation

[180] http://en.wikipedia.org/wiki/Critical_point_(mathematics)

[181] http://en.wikipedia.org/wiki/Jacobi_field

[182] http://en.wikipedia.org/wiki/Classical_mechanics

[183] http://en.wikipedia.org/wiki/Geodesics_as_Hamiltonian_flows

lutions of the associated Hamilton–Jacobi equations[184], with (pseudo-)Riemannian metric taken as Hamiltonian[185].

## 18.3 Affine geometry

A **geodesic** on a smooth manifold $M$ with an affine connection[186] $\nabla$ is defined as a curve $\boldsymbol{\gamma}(t)$ such that parallel transport[187] along the curve preserves the tangent vector to the curve, so

$$\nabla_{\dot{\boldsymbol{\gamma}}}\dot{\boldsymbol{\gamma}}(t) = 0 \tag{1}$$

at each point along the curve, where $\dot{\boldsymbol{\gamma}}$ is the derivative with respect to $t$. More precisely, in order to define the covariant derivative of $\dot{\boldsymbol{\gamma}}$ it is necessary first to extend $\dot{\boldsymbol{\gamma}}$ to a continuously differentiable imaginary Quaternionic distribution in an open set[188]. However, the resulting value of the equation is independent of the choice of extension.

Using local coordinates[189] on $M$, we can write the **geodesic equation** (using the summation convention[190]) as

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \cdot \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} = 0 \tag{2}$$

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184 http://en.wikipedia.org/wiki/Hamilton%E2%80%93Jacobi_equation

185 http://en.wikipedia.org/wiki/Hamiltonian_mechanics

186 http://en.wikipedia.org/wiki/Affine_connection

187 http://en.wikipedia.org/wiki/Parallel_transport

188 http://en.wikipedia.org/wiki/Open_set

189 http://en.wikipedia.org/wiki/Local_coordinates

190 http://en.wikipedia.org/wiki/Summation_convention

where $x^{\mu}(t)$ are the coordinates of the curve $\boldsymbol{\gamma}(t)$ and $\Gamma^{\lambda}_{\mu\nu}$ are the Christoffel symbols[191] of the connection $\nabla$. This is just an ordinary differential equation for the coordinates. It has a unique solution, given an initial position and an initial velocity.

From the point of view of classical mechanics, geodesics can be thought of as trajectories of free particles in a manifold. Indeed, the equation $\nabla_{\dot{\boldsymbol{\gamma}}}\dot{\boldsymbol{\gamma}}(t) = 0$ means that the acceleration of the curve has no components in the direction of the surface (and therefore it is perpendicular to the tangent plane of the surface at each point of the curve). So, the motion is completely determined by the bending of the surface. This is also the idea of the general relativity where particles move on geodesics and the bending is caused by the gravity.

## 18.4 Christoffel symbols

If $x^i$, $i = 1,2,...,n$, is a local coordinate system on a manifold $M$, then the tangent vectors

$$e_{\mu} = \frac{\partial}{\partial x_{\mu}}, \qquad \mu = 1, 2, \dots, n \tag{1}$$

define a basis of the tangent space of $M$ at each point. The Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ are defined as the unique coefficients such that the equation

$$\nabla_{\mu} e_{\nu} = \Gamma^{\lambda}_{\mu\nu} \cdot e_{\lambda} \tag{2}$$

[191] http://en.wikipedia.org/wiki/Christoffel_symbol

holds, where $\nabla_\mu$ is the Levi-Civita connection[192] on *M* taken in the coordinate direction $e_\mu$.

The Christoffel symbols can be derived from the vanishing of the covariant derivative of the metric tensor $g_{ik}$:

$$0 = \nabla_\lambda g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x_\lambda} - g_{\eta\mu} \cdot \Gamma^\eta_{\mu\lambda} - g_{\mu\eta} \cdot \Gamma^\eta_{\nu\lambda} \tag{3}$$

By permuting the indices, and re-summing, one can solve explicitly for the Christoffel symbols as a function of the metric tensor:

$$\Gamma^\mu_{\nu\lambda} = ½ \cdot g^{\mu\nu} \cdot \left(\frac{\partial g_{\eta\nu}}{\partial x^\lambda} + \frac{\partial g_{\eta\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\eta}\right) \tag{4}$$

where the matrix $(g^{\mu\nu})$ is an inverse of the matrix $(g_{\mu\nu})$, defined as (using the Kronecker delta, and Einstein notation for summation)

$$g^{\lambda\mu} \cdot g_{\mu\nu} = \delta^\lambda_\nu \tag{5}$$

Although the Christoffel symbols are written in the same notation as tensors with index notation, they are **not** tensors, since they do not transform like tensors under a change of coordinates.

Under a change of variable from $(x^1, …., x^n)$ to $(y^1, …., y^n)$, vectors transform as

$$\frac{\partial}{\partial y^i} = \frac{\partial x^k}{\partial y^i} \cdot \frac{\partial}{\partial x^k} \tag{6}$$

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[192] http://en.wikipedia.org/wiki/Levi-Civita_connection

and so

$$\underline{\Gamma}_{ij}^{k} = \frac{\partial x^{p}}{\partial y^{i}} \cdot \frac{\partial x^{q}}{\partial y^{j}} \cdot \underline{\Gamma}_{pq}^{r} \cdot \frac{\partial y^{k}}{\partial x^{r}} + \frac{\partial y^{k}}{\partial x^{m}} \cdot \frac{\partial^{2} x^{m}}{\partial y^{i} \partial y^{j}} \tag{7}$$

where the underline denotes the Christoffel symbols in the *y* coordinate frame. Note that the Christoffel symbol does **not** transform as a tensor, but rather as an object in the jet bundle.

At each point, there exist coordinate systems in which the Christoffel symbols vanish at the point. These are called (geodesic) normal coordinates, and are often used in Riemannian geometry.

The Christoffel symbols are most typically defined in a coordinate basis, which is the convention followed here. However, the Christoffel symbols can also be defined in an arbitrary basis of tangent vectors $e_{\mu}$ by

$$\nabla_{e_{\mu}} e_{\nu} = \Gamma_{\mu\nu}^{\lambda} \cdot e_{\lambda} \tag{8}$$

## 18.5 Local metric equation

The local metric equation relates the local value of the metric tensor field to the influence of the properties of the local particles on the local curvature.

In order to do this it requires a non-affected coordinate system and a way to qualify the influence that the local value of the particle properties have on the resulting curved coordinate system.

For example the Kerr Newman metric equation uses the per category summed property values of the local coupling factors, the

electric charges of the local particles and the angular momenta of the local particles in order to relate these to the local curvature[193].

### 18.5.1 Kerr-Newman metric equation

The Kerr–Newman metric equation describes the geometry of spacetime in the vicinity of a rotating mass M with charge Q. The formula for this metric depends upon what coordinates or coordinate conditions are selected.

It uses three local properties. These properties are:

- The coupling factor $m$
- The electric charge $Q$
- The angular momentum $J$

The angular momentum $J$ includes the spin $s$.

In most cases, the simplest interpretation of the Kerr-Newman metric can be taken on the surface of a sphere that has a selected radius $r$. This metric uses the sum of a category of properties that are collected within the observed sphere. However, the summation produces different centers of activity for different property categories. Thus, these centers need not be at the same location. However, for large enough selected radius $r$ and applied to black holes or single particles, these centers coincide.

The formula uses three characteristic radii, whose prominence usually differs with the content of the investigated sphere.

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[193] See next part.

The metric uses a non-curved coordinate system to start with. Several coordinate systems can be used. The most common coordinate systems for a non-curved three dimensional space are:

- Cartesian coordinates
- Spherical coordinates

Alternatives for spherical coordinates are:

- Schwarzschild coordinates[194]
- Kruskal-Szekeres coordinates[195]
- Lemaitre coordinates[196]
- Eddington–Finkelstein coordinates[197]

The advantage of the alternative coordinates is that they avoid unnecessary singularities.

#### 18.5.1.1 Spherical coordinates

The line element dτ in spherical coordinates is given by:

$$c^2\, d\tau^2 = -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + (c\, dt - \alpha\ sin^2(\theta)\ d\phi)^2 \frac{\Delta}{\rho^2} \tag{1}$$

[194] http://en.wikipedia.org/wiki/Schwarzschild_coordinates
[195] http://en.wikipedia.org/wiki/Kruskal-Szekeres_coordinates
[196] http://en.wikipedia.org/wiki/Lemaitre_coordinates
[197] http://en.wikipedia.org/wiki/Eddington%E2%80%93Finkelstein_coordinates

$$-\left((r^2+\alpha^2)\,d\phi-\alpha\,c\,dt\right)^2\,\frac{sin^2(\theta)}{\rho^2}$$

where the coordinates r, θ and ϕ are the parameters of the standard spherical coordinate system. The length-scales α, ρ and Δ have been introduced for brevity.

$$\alpha=\frac{J}{M\,c} \tag{2}$$

$$\rho^2=r^2+\alpha^2\,cos^2(\theta) \tag{3}$$

$$\Delta=r^2-r_s\,r+\alpha^2+r_Q^2 \tag{4}$$

$r_s$ is the Schwarzschild radius[198] (in meters) of the massive body, which is related to its mass $M$ by

$$r_s=\frac{2GM}{c^2} \tag{5}$$

where $G$ is the gravitational constant[199]. In case of a single encapsulated elementary particle, $M$ stands for the coupling constant m.

Compare this with the Planck length, $l_{Pl}=\sqrt{\hbar G/c^3}$

The Schwarzschild radius is radius of a spherical geo-cavity with mass $M$. The escape speed from the surface of this geo-cavity equals the speed of light. Once a stellar remnant collapses within

[198] http://en.wikipedia.org/wiki/Schwarzschild_radius

[199] http://en.wikipedia.org/wiki/Gravitational_constant

this radius, light cannot escape and the object is no longer visible. It is a characteristic radius associated with every quantity of mass.

$r_Q$ is a length-scale corresponding to the electric charge Q of the mass

$$r_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4} \tag{6}$$

where $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's force constant[200].

Next for simplicity we use the dimension adapted parameter m.

The radius where the ergo region[201] of a black hole starts can be specified by:

$$r = m + \sqrt{m^2 - r_Q^2 - \alpha^2 \cos^2(\theta)} \tag{7}$$

And the radius of the horizon by

$$r = m + \sqrt{m^2 - r_Q^2 - \alpha^2} \tag{8}$$

where

$$\alpha = S/mc \tag{9}$$

[200] http://en.wikipedia.org/wiki/Coulomb%27s_law

[201] http://en.wikipedia.org/wiki/Black_hole#Ergosphere

is the dimensionless spin parameter, q is the electric charge and m is the mass of the particle that is set by the coupling factors[202].

Between these radii lays the ergo-region. That is the place where for any item it is impossible to stand still. This is the result of a process known as frame-dragging; general relativity predicts that any rotating mass will tend to slightly "drag" along the spacetime immediately surrounding it. Any object near the rotating mass will tend to start moving in the direction of rotation.

The region where the considered item can be considered as a black hole is defined by:

$$m^2 > r_Q^2 + \alpha^2$$

#### 18.5.1.2 Kerr-Newman limit

The lowest mass $\mathfrak{M}$ where a horizon exists is set by

$$\mathfrak{M} \equiv m = \sqrt{r_Q^2 + (S/cm)^2}$$

Where $S$ is the elementary spin s.

#### 18.5.1.3 Cartesian coordinates

The Kerr–Newman metric can be expressed in "Kerr–Schild" form, using a particular set of Cartesian coordinates

$$g_{\mu\nu} = \eta_{\mu\nu} + f\, k_\mu\, k_\nu \qquad (1)$$

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[202] .Misner, C. W., Thorne, K. S. and Wheeler, J. A., Gravitation, W. H. Freeman and Co., 1973. (Box 33.2)

$$f = \frac{G\, r^2}{r^4 + a^2\, z^2}\left[2\, M\, r - Q^2\right] \tag{2}$$

$$k_x = \frac{r\, x + a\, y}{r^2 + a^2} \tag{3}$$

$$k_y = \frac{r\, y - a\, x}{r^2 + a^2} \tag{4}$$

$$k_0 = 1 \tag{5}$$

Notice that $\boldsymbol{k}$ is a unit vector. Here $M$ is the constant mass of the spinning object, $Q$ is the constant charge of the spinning object, $\eta$ is the Minkowski tensor, and $a$ is a constant rotational parameter of the spinning object. It is understood that the vector $\boldsymbol{a}$ is directed along the positive z-axis. The quantity $r$ is not the radius, but rather is implicitly defined like this:

$$1 = \frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} \tag{6}$$

Notice that the quantity r becomes the usual radius $R = \sqrt{x^2 + y^2 + z^2}$ when the rotational parameter $a$ approaches zero. In this form of solution, units are selected so that the speed of light is unity ($c = 1$).

In order to provide a complete solution of the Einstein–Maxwell Equations, the Kerr–Newman solution not only includes a formula for the metric tensor, but also a formula for the electromagnetic potential:

(7)

$$A_\mu = \frac{Q\, r^3}{r^4 + a^2\, z^2} k_\mu$$

At large distances from the source (R>>a), these equations reduce to the Reissner-Nordstrom metric[203] with:

$$A_\mu = \left(-\phi, A_x, A_y, A_z\right) \tag{8}$$

The static electric and magnetic fields are derived from the vector potential and the scalar potential like this:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi \tag{9}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{10}$$

### 18.5.2 Schwarzschild metric

#### 18.5.2.1 Schwarzschild coordinates

Specifying a metric tensor[204] is part of the definition of any Lorentzian manifold[205]. The simplest way to define this tensor is to define it in compatible local coordinate charts and verify that the same tensor is defined on the overlaps of the domains of the charts. In this article, we will only attempt to define the metric tensor in the domain of a single chart.

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[203] http://en.wikipedia.org/wiki/Reissner%E2%80%93Nordstr%C3%B6m_metric

[204] http://en.wikipedia.org/wiki/Metric_tensor

[205] http://en.wikipedia.org/wiki/Lorentzian_manifold

In a Schwarzschild chart[206] (on a static spherically symmetric spacetime), the line element ds takes the form

$$ds^2 = -\big(f(r)\big)^2 dt + \big(g(r)\big)^2 dr + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2) \tag{1}$$

$$-\infty < t < \infty,\ r_0 < r < r_1, 0 < \theta < \pi, -\pi < \phi < \pi$$

In the Schwarzschild chart, the surfaces $t = t_0, r = r_0$ appear as round spheres (when we plot loci in polar spherical fashion), and from the form of the line element, we see that the metric restricted to any of these surfaces is

$$d\sigma = r_0^2(d\theta^2 + \sin^2(\theta)\, d\phi^2), \qquad 0 < \theta < \pi, -\pi < \phi < \pi \tag{2}$$

That is, these nested coordinate spheres do in fact represent geometric spheres with
surface area

$$A = 4\pi r_0^2 \tag{3}$$

And Gaussian curvature
$K = 1/r_0^2$

That is, they are geometric round spheres. Moreover, the angular coordinates $\theta, \phi$ are exactly the usual polar spherical angular coordinates: $\theta$ is sometimes called the colatitude and $\phi$ is usually called the longitude. This is essentially the defining geometric feature of the Schwarzschild chart.

[206] http://casa.colorado.edu/~ajsh/schwp.html

With respect to the Schwarzschild chart, the Lie algebra of Killing vector fields is generated by the time-like irrotational Killing vector field $\boldsymbol{\partial_t}$ and three space-like Killing vector fields $\boldsymbol{\partial_\phi}, \sin(\phi)\,\boldsymbol{\partial_\theta} + \cot(\theta)\cos(\phi)\;\boldsymbol{\partial_\phi}, \cos(\phi)\;\boldsymbol{\partial_\theta} - \cot(\theta)\sin(\phi)\;\boldsymbol{\partial_\phi}$

Here, saying that $\boldsymbol{\partial_t}$ is irrotational means that the vorticity tensor of the corresponding time-like congruence vanishes; thus, this Killing vector field is hyper-surface orthogonal. The fact that our spacetime admits an irrotational time-like Killing vector field is in fact the defining characteristic of a static spacetime. One immediate consequence is that the constant time coordinate surfaces $t = t_0$ form a family of (isometric) spatial hyper-slices. (This is not true for example in the Boyer-Lindquist chart for the exterior region of the Kerr vacuum, where the time-like coordinate vector is not hyper-surface orthogonal.)

It may help to add that the four Killing fields given above, considered as abstract vector fields on our Lorentzian manifold, give the truest expression of both the symmetries of a static spherically symmetric spacetime, while the particular trigonometric form which they take in our chart is the truest expression of the meaning of the term Schwarzschild chart. In particular, the three spatial Killing vector fields have exactly the same form as the three non-translational Killing vector fields in a spherically symmetric chart on E3; that is, they exhibit the notion of arbitrary Euclidean rotation about the origin or spherical symmetry.

However, note well: in general, the Schwarzschild radial coordinate does not accurately represent radial distances, i.e. distances taken along the space-like geodesic congruence which arise as the integral curves of $\partial r$. Rather, to find a suitable notion of 'spatial distance' between two of our nested spheres, we

should integrate $g(r)dr$ along some coordinate ray from the origin:

$$\Delta\rho = \int_{r_1}^{r_2} g(r)dr \tag{4}$$

Similarly, we can regard each sphere as the locus of a spherical cloud of idealized observers, who must (in general) use rocket engines to accelerate radially outward in order to maintain their position. These are static observers, and they have world lines of form $r = r_0, \theta = \theta_0, \phi = \phi_0$, which of course have the form of vertical coordinate lines in the Schwarzschild chart.

In order to compute the proper time interval between two events on the world line of one of these observers, we must integrate $f(r)dt$ along the appropriate coordinate line:

$$\Delta\tau = \int_{t_1}^{t_2} f(r)dt \tag{5}$$

#### 18.5.2.2 Schwarzschild metric

In Schwarzschild coordinates[207], the Schwarzschild metric has the form:

$$c^2\,d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2\,dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2(\theta)\,d\phi^2) \tag{6}$$

where:

[207] http://en.wikipedia.org/wiki/Schwarzschild_coordinates

- $\tau$ is the proper time (time measured by a clock moving with the particle) in seconds,
- $c$ is the speed of light in meters per second,
- $t$ is the time coordinate (measured by a stationary clock at infinity) in seconds,
- $r$ is the radial coordinate (circumference of a circle centered on the star divided by 2π) in meters,
- $\theta$ is the colatitude (angle from North) in radians,
- $\varphi$ is the longitude in radians, and
- $r_s$ is the Schwarzschild radius (in meters) of the massive body.

#### 18.5.2.3 Lemaître coordinates

In Schwarzschild coordinates the Schwarzschild metric has a singularity. Georges Lemaître was the first to show that this is not a real physical singularity but simply a manifestation of the fact that the static Schwarzschild coordinates cannot be realized with material bodies inside the gravitational radius[208]. Indeed inside the gravitational radius everything falls towards the center and it is impossible for a physical body to keep a constant radius.

A transformation of the Schwarzschild coordinate system from $\{t, r\}$ to the new coordinates $\{\tau, \rho\}$,

$$d\tau = dt + \frac{\sqrt{r_s/r}}{\left(1 - \frac{r_s}{r}\right)} dr \tag{1}$$

(2)

[208] http://en.wikipedia.org/wiki/Lemaitre_coordinates

$$d\rho = dt + \frac{\sqrt{r/r_s}}{\left(1 - \frac{r_s}{r}\right)}\, dr$$

leads to the Lemaître coordinate expression of the metric,

$$ds^2 = d\tau^2 - \frac{r_s}{r} d\rho^2 - r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2) \tag{3}$$

Where

$$r = r_s^{1/3}\left[\frac{3\,(\rho - \tau)}{2}\right]^{2/3} \tag{4}$$

In Lemaître coordinates there is no singularity at the gravitational radius, which instead corresponds to the point $\frac{3\,(\rho-\tau)}{2} = r_s$. However, there remains a genuine gravitational singularity at the centrum, where $\rho - \tau = 0$, which cannot be removed by a coordinate change.

The Lemaître coordinate system is synchronous, that is, the global time coordinate of the metric defines the proper time of co-moving observers. The radially falling bodies reach the gravitational radius and the center within finite proper time.

Along the trajectory of a radial light ray,

$$dr = \left(\pm 1 - \sqrt{r_s/r}\right) d\tau \tag{5}$$

therefore no signal can escape from inside the Schwarzschild radius, where always $dr < 0$ and the light rays emitted radially inwards and outwards both end up at the origin.

# 19 The action along the live path

Where the Hamiltonian is derived from the balance equation, is the Lagrangian derived from the path equation.

The integrated action $S_{ab}$ is performed over a distance along the action trail or equivalently over a period of coordination time

$$S_{ab} = -\int_a^b m \cdot c^2 \cdot ds + matter\ terms \tag{1}$$

$$= -\int_{\tau_a}^{\tau_b} m \cdot c^2 \cdot \sqrt{1 - \left(\frac{v}{c}\right)^2} \cdot d\tau + matter\ terms$$

$$= \int_{\tau_a}^{\tau_b} \mathcal{L} \cdot d\tau$$

$m$ is the mass of the considered item.
$v$ is the speed in Q space.
$\mathcal{L}$ is the Lagrangian.

The first line of this formula can be considered as an integral along the trail in coordinate space or equivalently over the trail in Hilbert space. The next lines concern integrals over the corresponding path in observed space combined with coordinate time. It must be noticed that these spaces have different signature.

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + \text{matter terms} \tag{2}$$

In general relativity, the first term generalizes (includes) both the classical kinetic energy and interaction with the Newtonian gravitational potential. It becomes:

$$m \cdot c^2 \cdot \frac{ds}{d\tau} = -m \cdot c \cdot \sqrt{g_{\alpha\beta} \cdot \dot{q}_\alpha \cdot \dot{q}_\beta} \tag{3}$$

$g_{\alpha\beta}$ is the rank 2 symmetric metric tensor which is also the gravitational potential. Notice that a factor of $c$ has been absorbed into the square root.

The matter terms in the Lagrangian $\mathcal{L}$ differ from those in the integrated action $S_{ab}$.

$$S_{ab\_matter} = -\int_a^b e \cdot A_\gamma \cdot dq^\gamma + \text{other matter terms} \tag{4}$$

The matter term in the Lagrangian due to the presence of an electromagnetic field is given by:

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + e \cdot \dot{q^\gamma} \cdot A_\gamma + \text{other matter terms} \tag{5}$$

$A_\gamma$ is the electromagnetic 4-vector potential.

## 19.1 Noether's theorem

When the Lagrangian does not vary with one or more of its parameters, then this corresponds with a corresponding symmetry of the system. By Noether's theorem[209], such symmetries of the system

[209] http://en.wikipedia.org/wiki/Noether%27s_theorem

correspond to conservation laws[210]. In particular, the invariance of the Lagrangian with respect to time $\tau$ implies the conservation of energy.

By partial differentiation of the above Lagrangian, we find:

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial q_i} = \frac{\partial U}{\partial q_i} = F_i \tag{5}$$

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial \dot{q}_i} = m \cdot \dot{q}_i = p_i \tag{6}$$

where the force is $F = -\nabla U$ (the negative gradient of the potential, by definition of conservative force), and p is the momentum. By substituting these into the Euler–Lagrange equation, we obtain a system of second-order differential equations for the coordinates on the particle's trajectory,

$$F_i = \frac{d(m\dot{q}_i)}{dt} = m \cdot \ddot{q}_ı = \dot{p}_ı \tag{7}$$

which is Newton's second law.

[210] http://en.wikipedia.org/wiki/Conservation_law

# 20 Euler Lagrange equations of field

## 20.1 First order equations

The Dirac Lagrangian density is

$$\mathcal{L} = \psi^* \left[ \gamma^\mu \left( i \frac{\partial}{\partial x_\mu} - eA_\mu \right) - m \right] \psi \tag{1}$$

The corresponding Euler-Lagrange equation

$$\left[ \gamma^\mu \left( i \frac{\partial}{\partial x_\mu} - eA_\mu \right) - m \right] \psi = 0 \tag{2}$$

The Dirac 4-current is

$$j^\mu = \psi^* \gamma^\mu \psi \tag{3}$$

The density is the 0-component

$$\rho_{Dirac} = \psi^* \psi \tag{4}$$

The Dirac Hamiltonian density is

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \dot{\psi} - \mathcal{L} = \psi^* \left[ \alpha^\mu \left( i \frac{\partial}{\partial x_\mu} - eA_\mu \right) + \beta m + eV \right] \psi \tag{5}$$

$$\mathcal{H} = \psi^* [ -\langle \boldsymbol{\alpha}, \boldsymbol{\nabla} + e\boldsymbol{A} \rangle + \beta m + eV ] \psi \tag{6}$$

The Dirac equation runs

$$\mathcal{H}\psi = [-\langle \boldsymbol{\alpha}, \boldsymbol{\nabla} + e\boldsymbol{A}\rangle + \beta m + eV]\psi \tag{7}$$

$$\begin{aligned}\mathcal{H}\begin{bmatrix}\psi_L\\ \psi_R\end{bmatrix} &= [-\langle \boldsymbol{\alpha}, \boldsymbol{\nabla} + e\boldsymbol{A}\rangle + eV]\begin{bmatrix}\psi_L\\ \psi_R\end{bmatrix}\\ &\quad + m\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_L\\ \psi_R\end{bmatrix}\\ &= [-\langle \boldsymbol{\alpha}, \boldsymbol{\nabla} + e\boldsymbol{A}\rangle + eV]\begin{bmatrix}\psi_L\\ \psi_R\end{bmatrix} + m\begin{bmatrix}\psi_R\\ \psi_L\end{bmatrix}\end{aligned} \tag{8}$$

$$\begin{aligned}\mathcal{H}\psi_L &= (-\boldsymbol{\nabla} - e\boldsymbol{A} + eV)\psi_L + m\psi_R\\ &= (-\boldsymbol{D} + eV)\psi_L + m\psi_R\end{aligned} \tag{9}$$

$$\begin{aligned}\mathcal{H}\psi_R &= (\boldsymbol{\nabla} + e\boldsymbol{A} + eV)\psi_R + m\psi_L\\ &= (\boldsymbol{D} + eV)\psi_R + m\psi_L\end{aligned} \tag{10}$$

The mass m couples $\psi_L$ and $\psi_R$. The fact $m = 0$ decouples $\psi_L$ and $\psi_R$.

The Dirac Hamiltonian density (6) as well as the Dirac Hamiltonian (7) *do not contain a derivative of $\psi$ with respect to time.*

Now, the form of an energy eigenfunction is

$$\psi(\boldsymbol{x}, t) = \chi(\boldsymbol{x})\exp(-\boldsymbol{E}t) \tag{11}$$

$$\mathcal{H}\psi = [-\langle \boldsymbol{\alpha}, \boldsymbol{\nabla} + e\boldsymbol{A}\rangle + eV]\psi + m\psi^* \tag{12}$$

# 21 Lagrangians in quantum field theory

## 21.1 Dirac Lagrangian

The Lagrangian density for a Dirac field[211] is:

$$\mathcal{L} = \frac{i\hbar c}{2}\left(\bar{\psi}\,\gamma^{\mu}\frac{\partial\psi}{\partial x_{\mu}} - \psi\,\gamma^{\mu}\frac{\partial\bar{\psi}}{\partial x_{\mu}}\right) - m\,c^{\,2}\bar{\psi}\,\psi \tag{1}$$

where $\psi$ is a Dirac spinor[212] (annihilation operator), $\bar{\psi}$ is its Dirac adjoint[213] (creation operator)

$$\bar{\psi} = \psi^{\dagger}\,\gamma^{0} \tag{2}$$

## 21.2 Quantum electrodynamic Lagrangian

The Lagrangian density for quantum electro dynamics[214] is:

$$\mathcal{L}_{QED} = \frac{i\hbar c}{2}\left(\bar{\psi}\,\gamma^{\mu}D_{\mu}\psi - \psi\,\gamma^{\mu}D_{\mu}\bar{\psi}\right) - m\,c^{\,2}\bar{\psi}\,\psi - \frac{1}{4\mu_{0}}F_{\mu\nu}F^{\mu\nu} \tag{3}$$

where $F_{\mu\nu}$ is the electromagnetic tensor[215], $D$ is the gauge covariant derivative.

(4)

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[211] http://en.wikipedia.org/wiki/Fermionic_field#Dirac_fields
[212] http://en.wikipedia.org/wiki/Dirac_spinor
[213] http://en.wikipedia.org/wiki/Dirac_adjoint
[214] http://en.wikipedia.org/wiki/Quantum_electrodynamics
[215] http://en.wikipedia.org/wiki/Electromagnetic_tensor

$$F^{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\mu\nu}F^{\mu\nu} = B^2 - E^2/c^2 \tag{5}$$

This is a Lorentz scalar.

The equation of motion is

$$i\gamma^\mu \partial_\mu \psi - m\psi = e\gamma^\mu (A^\mu + B^\mu)\psi \tag{6}$$

The left-hand side is like the original Dirac equation and the right-hand side is the interaction with the electromagnetic field.

$$\partial_\nu F^{\mu\nu} = e\bar{\psi}\,\gamma^\mu\,\psi \tag{7}$$

Now, if we impose the Lorenz-Gauge condition, i.e., that the divergence of the four potential vanishes then we get

$$\Box A^\mu = e\bar{\psi}\,\gamma^\mu\,\psi \tag{8}$$

The d'Alembert operator $\Box$ is defined as:

$$\Box = \partial_\mu \partial^\mu = g_{\mu\nu}\partial^\mu \partial^\nu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \tag{9}$$

$$= \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \Delta$$

### 21.3 Quantum chromodynamic Lagrangian

The Lagrangian density for quantum chromodynamics[216]:

$$\mathcal{L}_{QCD} = \frac{i\hbar c}{2}\left(\bar{\psi}_n \gamma^\mu D_\mu \psi_n - \psi_n \gamma^\mu D_\mu \bar{\psi}_n\right) - m_n c^2 \bar{\psi}_n \psi_n - \frac{1}{4} G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha \quad (1)$$

where $D$ is the QCD gauge covariant derivative, $n = 1...6$ counts the quark types, and $G^\alpha_{\mu\nu}$ is the gluon field strength tensor.

# 22 Path of the quantum state function

The path equation treats the path of a particle that moves with uniform speed along a curved path. Now let us investigate the path of a static QPAD. Any uniform movement with speed **v** will be represented by the currents in the static QPAD.

Any deviation of that uniform speed will go together with the existence of an extra field. This deviation can be split in three parts.

1. A deviation along the original direction.
2. A deviation in a perpendicular direction that follows the curvature of the path.
3. A deviation perpendicular to both the original path and the curvature deviation.

When the QPAD follows a geodesic then the first deviation is zero.

[216] http://en.wikipedia.org/wiki/Quantum_chromodynamics

The second deviation $\dot{\mathbf{v}}$ goes together with a curvature field and the third deviation $\ddot{\mathbf{v}}$ goes together with a torque field.
These extra fields depend on the existence of inertia fields[217], a scalar potential field $\phi$ and a vector potential field $\boldsymbol{A}$

$$A = \Phi \cdot v/c \tag{1}$$

$$E_{curve} = -\nabla\Phi - \frac{1}{c} \cdot \Phi \cdot \dot{v} \tag{2}$$

The field $\Phi$ is considered to be rather static and $\boldsymbol{\nabla\Phi}$ is considered to be small.

$$F_{torque} \approx -\frac{1}{c} \cdot \Phi \cdot \ddot{v} = \dot{E}_{curve} \tag{3}$$

Via the equation

$$\nabla \times B = \mu_0 \left( j + \varepsilon_0 \frac{\partial E}{\partial t} \right) \tag{4}$$

Follows for the torque field

$$F_{torque} = \frac{\nabla \times B - \mu_0 j}{\varepsilon_0} \tag{5}$$

Thus, even when the QPAD travels a geodesic, the curvature and the torque of the path go together with the existence of extra fields.

[217] See The universe of items; Inertia

# Index

The Hilbert Book Model
Author: Hans van Leunen

This book starts from the axioms of traditional quantum logic and extends this model such that it incorporates physical fields as well as dynamics.

It uses the isomorphism between the set of propositions of traditional quantum logic and the set of closed subspaces of an infinite dimensional separable Hilbert space that uses quaternions in order to specify its inner products.

In order to implement dynamics, the developed model applies a sequence of extended quantum logics or equivalently a sequence of extended separable Hilbert spaces. Each of the members of the sequence represents a static status quo of the universe. This leads to a new model of physics:

# The Hilbert Book Model

The book explains the existence of all known elementary particles of the standard model and shows how these particles get their mass and inertia. Due to its usage of quaternionic quantum state functions the model offers a completely new view into the undercrofts of fundamental physics. This complements the view that is obtained in conventional quantum physics by using complex quantum state functions.

http://vixra.org/author/j_a_j_van_leunen

http://www.crypts-of-physics.eu

The crypt is under a friend’s house in Nesle, France