Discussion on the Relationship Among the Relative Speed, the Absolute Speed and the Rapidity

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Abstract: Through discussion on the physical meaning of the rapidity in the special relativity, this paper arrived at the conclusion that the rapidity is the ratio between absolute speed and the speed of light in vacuum. At the same time, the mathematical relations amongst rapidity, relative speed, and absolute speed were derived. In addition, through the use of the new definition of rapidity, the Lorentz transformation in special relativity can be expressed using the absolute speed of Newtonian mechanics, and be applied to high energy physics.

Keywords: rapidity; relative speed; absolute speed; special theory of relativity

In the special theory of relativity, Lorentz transformations between an inertial frame of reference (x, t) at rest and a moving inertial frame of reference (x', t') can be expressed by

$$x = \gamma(x' + v t') \tag{1}$$

$$t = \gamma(t' + \frac{v}{{c_0}^2} x') \tag{2}$$

Where $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c_0})^2}}$ (3)

v is the relative speed between the two inertial frames of reference ($v < c_0$). c_0 is the speed of light in vacuum.

Lorentz transformations can be expressed by hyperbolic functions [1] [2]

$$\begin{pmatrix} x \\ c_0 t \end{pmatrix} = \begin{pmatrix} \cosh(\varphi) & \sinh(\varphi) \\ \sinh(\varphi) & \cosh(\varphi) \end{pmatrix} \begin{pmatrix} x' \\ c_0 t' \end{pmatrix} \tag{4}$$

Where $\cosh(\varphi) = \frac{1}{\sqrt{1-(\frac{v}{c_0})^2}}$ (5)

Hyperbolic angle φ is called rapidity. Based on the properties of hyperbolic function

$$\tanh(\varphi) = \frac{v}{c_0} \tag{6}$$

From the above equation, the definition of rapidity can be obtained [3]

$$\varphi = \operatorname{artanh}\left(\frac{v}{c_0}\right) \tag{7}$$

However, the definition of rapidity was derived directly by a mathematical equation (6). The definition is not unique and it can also be defined by other inverse hyperbolic functions. This shows that currently the real physical meaning of rapidity is not entirely accurate and complete, yet it is an often-used physical concept in modern physics. Thus, it must be studied further.

According to the properties of the hyperbolic function, the superposition of hyperbolic angle is linear superposition. Hence let $\varphi = \varphi_1 + \varphi_2$, and substituting into equation (6)

$$\frac{v}{c_0} = \tanh(\varphi_1 + \varphi_2) = \frac{\tanh(\varphi_1) + \tanh(\varphi_2)}{1 + \tanh(\varphi_1)\tanh(\varphi_2)} \tag{8}$$

From (6), $\frac{v_1}{c_0} = \tanh(\varphi_1)$, $\frac{v_2}{c_0} = \tanh(\varphi_2)$, and substituting them into the above equation to obtain the equation of the superposition speed of special relativity

$$v = \frac{v_2 + v_1}{1 + \frac{v_2 v_1}{{c_0}^2}} \tag{9}$$

If $\varphi_1 = \varphi_2 = \varphi_i$, then $v_1 = v_2 = v_i$, $v_i = c_0 \tanh(\varphi_i)$, from equation (9)

$$v = c_0 \tanh(2\varphi_i) = \frac{2v_i}{1 + \frac{{v_i}^2}{{c_0}^2}} \tag{10}$$

It is evident that the superposition of the relative speed is non-linear. However, if v_i is much smaller than the speed of light c_0, then the space-time at this moment is approximately the absolute time and space in Newtonian mechanics. As such, the absolute speed is defined as the magnitude of a velocity vector within the absolute space-time, it is a scalar and satisfies the Galileo transformation. Its speed superposition is linear and its upper limit is infinite. Let the sum of the absolute speed superposition be V, then equation (10) becomes the equation of the absolute speed superposition in Newtonian mechanics.

$$V \approx 2v_i = 2\,c_0 \tanh(\varphi_i) \qquad (v_i \ll c_0) \tag{11}$$

Now extend the above derivation to a broader situation. Suppose there are n+1 inertial frames of reference arranged in numeric sequence, and the frames are superposed in the direction of x-axis, and the speed of the first frame of reference (the observing frame of reference) is defined as zero. Every frame of reference has the same velocity relative to the preceding frame of reference, and they are all moving along in the direction of the positive x-axis. The relative speed of the frame of reference $i+1$ is v_i relative to the frame of reference i. Therefore, the total absolute speed V and total relative speed v of the n moving frames of reference can be obtained by linear superposition and non-linear superposition of the speed v_i, separately. The total absolute speed is

$$\mathrm{V} \approx \mathrm{n}v_i = \mathrm{nc_0} \tanh(\varphi_i) \qquad (v_i \ll \mathrm{c_0}) \tag{12}$$

Let $\varphi = \mathrm{n}\varphi_i$ in equitation (6), the total relative speed is

$$v = \mathrm{c_0}\tanh(\mathrm{n}\varphi_i) \tag{13}$$

The n of both equation (12) and (13) are identical, therefore below equation is established

$$v \approx \mathrm{c_0}\tanh\left(\frac{\mathrm{V}}{\mathrm{c_0}\tanh(\varphi_i)}\varphi_i\right) \tag{14}$$

When n approaches infinity, both v_i and φ_i approach zero. Therefore, the exact equation (15) can be obtained using approximate equation (14) to calculate the limit when φ_i approaches zero is

$$\frac{v}{\mathrm{c_0}} = \lim_{\varphi_i \to 0} \tanh\left(\frac{\mathrm{V}}{\mathrm{c_0}\tanh(\varphi_i)}\varphi_i\right) = \tanh\left(\frac{\mathrm{V}}{\mathrm{c_0}}\right) \tag{15}$$

Comparing equations (15) and (6)

$$\varphi = \frac{\mathrm{V}}{\mathrm{c_0}} \tag{16}$$

From equations (15) and (16)

$$\frac{v}{\mathrm{V}} = \frac{\tanh(\varphi)}{\varphi} \tag{17}$$

Equation (15) reveals the interrelationships among relative speed, absolute speed and the speed of light. That is, with any known relative speed, its corresponding absolute speed can be calculated. Equation (16) shows that the new physical definition of rapidity is the ratio between absolute speed and the speed of light. It illustrates the real physical meaning of rapidity and is more direct and easier to understand than equation (7). Most importantly, this new definition is unique and does not contradict the definition that is currently being used. Equation (17) shows the interrelationships among the relative speed, the absolute speed, and the rapidity. The relative speed of a photon is the speed of light in vacuum, both of its absolute speed and rapidity are infinite.

Substituting equation (16) into (4), the Lorentz transformation can be expressed using absolute speed

$$\begin{pmatrix} x \\ c_0 t \end{pmatrix} = \begin{pmatrix} \cosh\left(\frac{V}{c_0}\right) & \sinh\left(\frac{V}{c_0}\right) \\ \sinh\left(\frac{V}{c_0}\right) & \cosh\left(\frac{V}{c_0}\right) \end{pmatrix} \begin{pmatrix} x' \\ c_0 t' \end{pmatrix} \tag{18}$$

When V approach zero, $\lim_{V\to 0} \sinh\left(\frac{V}{c_0}\right) = \frac{V}{c_0} = 0$, $\lim_{V\to 0} \cosh\left(\frac{V}{c_0}\right) = 1$, from above equation

$$\begin{pmatrix} x \\ c_0 t \end{pmatrix} = \begin{pmatrix} 1 & \frac{V}{c_0} \\ \frac{V}{c_0} & 1 \end{pmatrix} \begin{pmatrix} x' \\ c_0 t' \end{pmatrix} \tag{19}$$

The Galilean transformation can be obtained

$$x = x' + tV \tag{20}$$

$$t = t' \tag{21}$$

The result is exactly the same as the conclusion of the special relativity. Therefore, the new definition of rapidity has been verified.

The concept of rapidity is often utilized in high energy physics, and it can be express as below,

$$y = \frac{1}{2} \ln \frac{E + p_z c_0}{E - p_z c_0} \tag{22}$$

Where y is rapidity, E is the energy of the particle with resting mass of m, p_z is the component of momentum along the direction of the particle beam[4] [5]. The rapidity of the particle calculated using the above equation is precisely the ratio between the absolute speed of the particle and the speed of light. Similarly, the concept of “pseudo-rapidity” in experimental particle physics and other physical quantities related to rapidity can be more intuitive and accurately understood.

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