Generalized Pythagorean Triples:

$$a_1^2+b_1^2+b_2^2+\cdots+b_n^2=c_n^2$$

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Abstract

In $a^2+b^2=c^2$ there are infinitely many primes a and c solutions. The generalized Pythagorean triples: $a_1^2+b_1^2+b_2^2+\cdots+b_n^2=c_n^2$ has infinitely many integer solutions. There are infinitely many primes $a_1=P$ such that $c_1,\cdots,c_n$ are all prime.

Suppose Pythagorean triples

$$a^2+b^2=c^2, \tag{1}$$

in comprime integers must be of the form

$$a=x^2-y^2,\ b=2xy,\ c=x^2+y^2, \tag{2}$$

where x and y are coprime integers.

Theorem 1. From (2) we have

$$a=(x+y)(x-y) \tag{3}$$

Let $x-y=1$ and $a=x+y=P$, we have,

$$P^2=(x+y)^2=x^2+y^2+2xy=c+b, \tag{4}$$

$$1=(x-y)^2=x^2+y^2-2xy=c-b \tag{5}$$

From (4) and (5) we have

$$a=P,\quad b=\frac{P^2-1}{2},\quad c=\frac{P^2+1}{2}=P_1 \tag{6}$$

There are infinitely many primes P such that P_1 is a prime.

Proof. We have Jiang function [1-3]

$$J_2(\omega)=\prod_{P>2}(P-1-\chi(P)), \tag{7}$$

where $\omega=\prod_{P\ge 2}P$, $\chi(P)$ is the number of solutions of congruence

$$q^2+1\equiv 0(\bmod P), q=1,\cdots,P-1. \tag{8}$$

Substituting (8) into (7) we have

$$J_2(\omega)=\prod_{P>2}(P-2-(-1)^{\frac{P-1}{2}})\neq 0 \qquad (9)$$

Since $J_2(\omega)\neq 0$, we prove that there are infinitely many prime P such that P_1 is a prime.

We have the best asymptotic formula [1-3]

$$\pi_2(N,2)=\left|\{P\leq N: P_1=prime\}\right|\sim\frac{J_2(\omega)\omega}{2\phi^2(\omega)\log^2 N}=\left(1-\frac{1+P(-1)^{\frac{P-1}{2}}}{(P-1)^2}\right)\frac{N}{\log^2 N}, \qquad (11)$$

where $\phi(\omega)=\prod_{P\geq 2}(P-1)$.

Theorem 2. Suppose Pythagorean triples (1)

$$a_1^2+b_1^2=c_1^2, \qquad (12)$$

where

$$a_1=P,\quad b_1=\frac{P^2-1}{2},\quad c_1=\frac{P^2+1}{2}. \qquad (13)$$

Suppose Pythagorean triples (2)

$$a_2^2+b_2^2=c_2^2 \qquad (14)$$

where

$$a_2=c_1,\quad b_2=\frac{c_1^2-1}{2},\quad c_2=\frac{c_1^2+1}{2}, \qquad (15)$$

There are infinitely many primes P such that c_1 and c_2 are primes.

Proof. We have Jiang function [1-3]

$$J_2(\omega)=\prod_{P>2}(P-1-\chi(P)), \qquad (16)$$

where $\chi(P)$ is the number of solutions of congrunce

$$c_1(q)c_2(q)\equiv 0 \pmod{P},\ q=1,2,\cdots,P-1. \qquad (17)$$

From (17) we have $\chi(P)<P-1$ and $J_2(\omega)\neq 0$. We prove that there are infinitely many primes P such that c_1 and c_2 are primes.

We have the best asymptotic formula [1-3]

$$\pi_3(N,2)=\left|\{P\leq N: c_1,c_2=prime\}\right|\sim\frac{J_2(\omega)\omega^2}{8\phi^3(\omega)}\frac{N}{\log^3 N}, \qquad (18)$$

From above we have the generalized Pythagorean triples [4]:

（1） $a_1(P), c_1(P)$ and $c_2(P)$ are primes.

（2） $a_1^2(P)+b_1^2(P)+b_2^2(P)=c_2^2(P)$ has infinitely many integer solutions.

（3） $c_1(P^2)$ and $c_2(P^2)$ are primes.

（4） $a_1^2(P^2)+b_1^2(P^2)+b_2^2(P^2)=c_2^2(P^2)$ has infinitely many integer solutions.

（5） $c_1(P_1P_2)$ and $c_2(P_1P_2)$ are primes.

（6） $a_1^2(P_1P_2)+b_1^2(P_1P_2)+b_2^2(P_1P_2)=c_2^2(P_1P_2)$ has infinitely many integer solutions.

Theorem 3. We rewrite (12)-(15). Suppose Pythagorean triples (1)

$$a_1^2+b_1^2=c_1^2, \tag{12}$$

where

$$a_1=P, \quad b_1=\frac{P^2-1}{2}, \quad c_1=\frac{P^2+1}{2} \tag{13}$$

Suppose Pythagorean triples (2)

$$a_2^2+b_2^2=c_2^2, \tag{14}$$

where

$$a_2=c_1, \quad b_2=\frac{c_1^2-1}{2}, \quad c_2=\frac{c_1^2+1}{2} \tag{15}$$

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Suppose Pythagorean triples (n)

$$a_n^2+b_n^2=c_n^2, \tag{19}$$

where

$$a_n=c_{n-1}, \quad b_n=\frac{c_{n-1}^2-1}{2}, \quad c_n=\frac{c_{n-1}^2+1}{2} \tag{20}$$

There are infinitely many primes P such that $c_1, c_2, \cdots, c_n$ are all prime.

Proof. We have Jiang function [1-3]

$$J_2(\omega)=\prod_{P>2}(P-1-\chi(P)) \tag{21}$$

where $\chi(P)$ is the number of solutions of congruence

$$\prod_{i=1}^{n} c_i(q)\equiv 0(\operatorname{mod} P), q=1,2,\cdots,P-1. \tag{22}$$

From (19) we have $\chi(P)<P-1$ and $J_2(\omega)\neq 0$. We prove that there are infinitely many

primes P such that $c_1, c_2, \cdots, c_n$ are all prime.
We have the best asymptotic formula [2-3]

$$\pi_{n+1}(N,2) = \left|\left\{P \le N : c_1, \cdots, c_n = prime\right\}\right| \sim \frac{J_2(\omega)\omega^n}{(2)^{\frac{n(n+1)}{2}} \phi^{n+1}(\omega)} \frac{N}{\log^{n+1} N} \quad . \tag{23}$$

From (12)-(20) we have the generalized Pythagorean triples [4]:

（1） $a_1(P), c_1(P), c_2(P), \cdots, c_n(P)$ are all prims.

（2） $a_1^2(P) + b_1^2(P) + b_2^2(P) + \cdots b_n^2(P) = c_n^2(P)$ has infinitely many integer solutions.

（3） $c_1(P^2)$, $c_2(P^2), c_3(P^2), \cdots, c_n(p^2)$ are all primes.

（4） $a_1^2(P^2) + b_1^2(P^2) + b_2^2(P^2) + \cdots b_n^2(P^2) = c_n^2(P^2)$ has infinitely many integer solutions.

（5） $c_1(P_1P_2), c_2(P_1P_2), c_3(P_1P_2), \cdots, c_n(P_1P_2)$ are all prime.

（6） $a_1^2(P_1P_2) + b_1^2(P_1P_2) + b_2^2(P_1P_2) + \cdots b_n^2(P_1P_2) = c_n^2(P_1P_2)$ has infinitely many integer solutions.

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