Title: Solution for Ringel–Kotzig conjecture

Author: Wilber Valgusbitkevyt

Abstract: In this paper, I am creating three new theorems called Victoria Hayanisel Theorem dedicated to Princess Eugenie of York to describe the state of numbers, circles, and lines. Followed by the theorem, I am using the set theory and Fermat's Infinite Descent Method (if my method is different, let me know, and I will name it) to show how the conjecture is true.

The following solution is dedicated to Princess Eugenie of York of United Kingdom while waiting for her response to my marriage proposal letter:

Ringel–Kotzig conjecture states that all trees are graceful. In other words, if there are bunch of circles and lines so that there isn't a polygon by the lines, you can label each line and circle with all different numbers from 1 to e and 0 to e.

I am going to start this conjecture by creating three simple algebraic theorems first. I am going to refer to these theorems as "Victoria Hayanisel Theorem" by taking my middle name and the Princess's middle name.

Victoria Hayanisel Theorem 1: For positive integers excluding zero, if (equation 1) i1^2 + i2^2 = i3^2 + i4^2, and (equation 2) i1 x i2 = i3 x i4, then i1 = i3 and i2 = i4 (or i1 = i4 and i2 = i3, it doesn't make any difference).

Proof: Let i1 = ab and i2 = cd such that a,b,c,d are prime numbers. We have that i1i2 = i3i4. Each number is uniquely factorized by prime numbers. Hence, i3i4 must be divisible by a,b,c,d as well. Obviously, if i1 = i3 and i2 = i4, then equation 1 works. Now, we are going to see how it has to be that way.

There are 2 cases:

Case 1: i3 and i4 has a factor from each i1 and i2. For instance, suppose i3 = ac and i4 = bd.

Then we have a^2 x b^2 + c^2 x d^2 = a^2 x c^2 + b^2 x d^2

Then we have a^2 x (b^2 - c^2) + d^2 x (c^2 - b^2) = 0

Then b has to equal c. Then the following is the i1, i2, i3, i4:

i1 = ab, i2 = cd, i3 = ac = ab, and i4 = bd = cd. In other words, i1 has to be equal to i3.

Case 2: i2's factors are all in i3. Suppose i3 = acd and i4 = b.

Then we have a^2 x b^2 + c^2 x d^2 = a^2 x c^2 x d^2 + b^2

Then we have a^2 x (b^2 - c^2 x d^2) + c^2 x d^2 - b^2 = 0

Then we have a^2 x (b^2 - c^2 x d^2) - (b^2 - c^2 x d^2) = 0

Then we have $(a^2 - 1) \times (b^2 - c^2 \times d^2) = 0$

Then a has to equal 1 or b has to equal cd. Then the following is the i_1, i_2, i_3, i_4:

$i_1 = ab$, $i_2 = cd$, $i_3 = acd = cd$ or ab, and $i_4 = b = b$ or cd. In other words, i_1 ($ab = b$) = i_4 (b) and i_2 (cd) = i_3 (cd) or i_1 (ab) = i_3 (ab) and i_2 (cd) = i_4 (cd).

A factor of either i_1 or i_2 has to be at one of i_3 and i_4, so there are only two possible cases.

Victoria Hayanisel Theorem 2: In Pythagorean Triple $a^2 + b^2 = c^2$ for the positive integers, a and b are unique for a given c.

Proof: Suppose there is d and e such that $d^2 + e^2 = c^2$. Then $a^2 + b^2 = c^2 + d^2$. By Victoria Hayanisel Theorem 1, $a = c$ and $b = d$ (or the other way around, it doesn't make any difference). Hence, a and b are unique.

Victoria Hayanisel Theorem 3: For positive integers excluding zero, if (equation 1) $i_1^2 + i_2^2 + \ldots i_n^2 = j_1^2 + j_2^2 + \ldots + j_n^2$, and (equation 2) $i_1 \times i_2 \times \ldots \times i_n = j_1 \times j_2 \times \ldots \times j_n$, then the set $i = \{i_1, i_2, \ldots, i_n\}$ is equal to the set $j = \{j_1, j_2, \ldots, j_n\}$.

Proof: Consider $i_1^2 + i_2^2 + i_3^2 = i_4^2 + i_5^2 + i_6^2$.

First, let $i_4^2 + i_5^2 = i_7^2$.

Now, we are going to consider 3 equations.

Let $i_1^2 + i_2^2 = i_8^2$.

Then, $i_8^2 + i_3^2 = i_7^2 + i_6^2$.

If $i_8^2 = i_7^2$, then $i_3 = i_6$, $i_1 = i_4$, $i_2 = i_5$ (the order doesn't matter).

If $i_8^2 \neq i_7^2$, $i_6^2 = i_8^2 = i_1^2 + i_2^2$.

Let $i_2^2 + i_3^2 = i_9^2$.

Then, $i_1^2 + i_9^2 = i_7^2 + i_6^2$.

If $i_9^2 = i_7^2$, then $i_1 = i_6$, $i_2 = i_4$, $i_3 = i_5$ (the order doesn't matter).

If $i_9^2 \neq i_7^2$, $i_6^2 = i_9^2 = i_2^2 + i_3^2$.

Let $i_1^2 + i_3^2 = i_{10}^2$.

Then, $i_{10}^2 + i_2^2 = i_7^2 + i_6^2$.

If $i_{10}^2 = i_7^2$, then $i_2 = i_6$, $i_1 = i_4$, $i_3 = i_5$ (the order doesn't matter).

If $i_{10}^2 \neq i_7^2$, $i_6^2 = i_{10}^2 = i_1^2 + i_3^2$.

Hence, if $i_{10}^2 \neq i_7^2$, $i_9^2 \neq i_7^2$, $i_8^2 \neq i_7^2$, then $i_6^2 = i_1^2 + i_3^2 = i_2^2 + i_3^2 = i_1^2 + i_2^2$.

Then $i_1 = i_2 = i_3$, and $i_6^2 = 2 \times i_1^2$. This is a contradiction because the numbers have to be positive integers, and i_6 cannot be a positive integer, square root of 2 is not an integer, if this were possible. Hence, $\{i_1, i_2, i_3\}$ is equivalent to $\{i_4, i_5, i_6\}$. By the recursion (since any number of numbers can be reduced to the form $i_a^2 + i_b^2$ by using Pythagorean Triple multiple times), for positive integers excluding zero, if (equation 1) $i_1^2 + i_2^2 + \dots i_n^2 = j_1^2 + j_2^2 + \dots + j_n^2$, and (equation 2) $i_1 \times i_2 \times \dots \times i_n = j_1 \times j_2 \times \dots \times j_n$, then the set $i = \{i_1, i_2, \dots, i_n\}$ is equal to the set $j = \{j_1, j_2, \dots, j_n\}$.

Now, let's get back to the Ringel–Kotzig conjecture. Proof for Ringel–Kotzig conjecture:

Label all the lines $d_1, \dots, d_e$. Label all the circles $c_1, \dots, c_{e+1}$.

Now, $d_1 = c_1 - c_2, \dots, d_e = c_e - c_{e+1}$. In other words:

$$d_1^2 + \dots + d_e^2 = \deg(c_1)c_1^2 + \deg(c_2)c_2^2 + \dots + \deg(c_{e+1})c_{e+1}^2 - 2\sum c_i c_j$$

$c_i c_j$ refers to all the lines with the combinations of two circles, and $\deg(c_i)$ refers to the degree of ith circle, i.e. the number of lines to the circle.

If it is possible to label each line and circle with all different numbers from 1 to e and 0 to e, then there should be a solution to:

$$d_1^2 + d_2^2 + \dots + d_e^2 = c_1^2 + c_2^2 + \dots + c_{e+1}^2$$

i.e.

$$\deg(c_1)c_1^2 + \deg(c_2)c_2^2 + \dots + \deg(c_{e+1})c_{e+1}^2 - 2\sum c_i c_j = c_1^2 + c_2^2 + \dots + c_{e+1}^2$$

i.e.

$$(\deg(c_1) - 1)c_1^2 + (\deg(c_2) - 1)c_2^2 + \dots + (\deg(c_{e+1}) - 1)c_{e+1}^2 - 2\sum c_i c_j = 0$$

i.e.

$(\deg(c_i) - 1)c_i^2 = 2 c_i \times \sum c_j$ for all c_j connected to c_i such that $j \geq i$, for all $\deg(c_i) > 1$.

i.e.

$(\deg(c_i) - 1)c_i = 2 \sum c_j$ for all c_j connected to c_i such that $j \geq i$, for all $\deg(c_i) > 1$.

Now, we have less than $e + 1$ equations like this (minus 1 further because we can set a circle as zero and division by zero is not possible, and take the equation out of the set of equations). We have $e + 1$ variables. The question is whether there is an element in the solution set such that it satisfies the conjecture. Because the number of equations is less than the number of variables in this homogeneous system, there has to be at least one solution such that:

$d_1^2 + d_2^2 + \dots + d_e^2 = c_1^2 + c_2^2 + \dots + c_{e+1}^2$.

Also, by Victoria Hayanisel Theorem (ignoring 0, since 0 plus something is something), we can tell that the set $d = \{0, d_1, d_2, \dots, d_e\}$ is the same set as the set $c = \{c_1, c_2, \dots, c_{e+1}\}$.

Hence, there is a solution for any set of numbers $\{0, I_1, I_2, \dots, I_e\}$ such that the set of the lines is $\{1, \dots, e\}$ and the set of circles is $\{0, 1, \dots, e\}$ as long as

$I_1 = I_a - I_b$

$I_2 = I_c - I_d$

…………..

Set i_1 to i_e as 1 to e, then we have $\{0, 1, \dots, e\}$.

Now, the last piece of the puzzle is to prove, that one of these number sets is the set of distinct numbers from 0 to e:

1. Consider the circle sets. In order for a graph with e + 1 circles to be a single tree, there has to be e – 1 common circles such that when you separate the lines and draw the corresponding circles at each ends, there are e – 1 common circles for e sets of lines.
2. In e sets of lines, there are 2e circles. Given that the graph is a tree and e – 1 circles are duplicates, e + 1 circles can be distinct circles from 0 to e such that one of them is in each set, which is the sets we are interested in.
3. Consider e + 1 distinct numbers $I_0, \dots, I_e$. Consider number sets such that there are e sets of any numbers from I_0 to I_e, and there are e + 1 distinct numbers in the e sets. Let I_0 be the smallest and let I_e be the largest. The first set has I_0 and I_e in it, and $I_1, \dots, I_{e-1}$ are each in the rest of the sets. There are e – 1 spots remaining each in e – 1 sets, and these can be anything from I_0 to I_e. For these e – 1 spots, the maximum a number can be duplicated is e – 1. If a number is duplicated i times, then the rest of the numbers are duplicated e – i – 1 times altogether.
4. Consider the number sets from 0 to e:
   -There are n ways I_i and I_j can create 1.
   -There are n – 1 ways I_i and I_j can create 2.
   -……………..
   -……………..
   -……………..
   -There are 1 way I_i and I_j can create e.
5. Consider the circle sets again. If there is a circle that has c duplicates, there are c + 1 lines to the circle. Just assign a number for the line such that there are at least c + 1 ways I_i and I_j can create

the number. Assign different numbers to the different lines and circles following the same method until all circles and lines are labeled.

Hence, there is always a combination to write the numbers from 0, …, e for I0, …, Ie such that the 0, …, e and I0, …, Ie correspondences create distinct numbers from 1 to e for the lines.

Hence, all trees are graceful.