Non-Associative Linear Algebras

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PREFACE

In this book authors for the first time introduce the notion of non associative vector spaces and non associative linear algebras over a field. We construct non associative space using loops and groupoids over fields.

In general in all situations, which we come across to find solutions may not be associative; in such cases we can without any difficulty adopt these non associative vector spaces/linear algebras. Thus this research is a significant one.

This book has six chapters. First chapter is introductory in nature. The new concept of non associative semilinear algebras is introduced in chapter two. This structure is built using groupoids over semifields. Third chapter introduces the notion of non associative linear algebras. These algebraic structures are built using the new class of loops. All these non associative linear algebras are defined over the prime characteristic field Z_p, p a prime. However if we take polynomial non associative, linear algebras over Z_p, p a prime; they are of infinite dimension over Z_p.

We in chapter four introduce the notion of groupoid vector spaces of finite and infinite order and their generalizations.

Only when this study becomes popular and familiar among researchers several applications will be found. The final chapter suggests around 215 problems some of which are at research level. We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

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Chapter One

BASIC CONCEPTS

In this chapter we just recall the notion of two basic structures viz; groupoids and loops. We have used $Z_n = \{0, 1, 2, \ldots, n-1\} = \{g_0, g_1, \ldots, g_{n-1}\}$ to be the ring of modulo integers. We define an operation * on Z_n as follows; for $g_i, g_j \in Z_n$,

$$g_i * g_j = tg_i + ug_j = g_{ti + uj \text{ (mod n)}}$$

where $t, u \in Z_n$.

$\{Z_n, *, (t, u)\}$ is a groupoid of order n.

Depending on $t, u \in Z_n$ we can get different types of groupoids.

For more about these groupoids please refer [35].

Example 1.1: Let $G = \{Z_{10}, *, (7, 3)\}$ be a groupoid of order 10. If $g_3, g_1 \in Z_{10}$ then $g_3 * g_1 = g_{3.7+1.3(\text{mod}10)} = g_4 \in Z_{10}$.

It is easily verified * on Z_n is non associative and non commutative.

Example 1.2: Let $G = \{Z_{12}, *, (4, 6)\}$ be a groupoid of order 12.

Example 1.3: Let G = $\{Z_{13}, *, (1, 9)\}$ be a groupoid of order 13.

Example 1.4: Let G = $\{Z_{40}, *, (19, 0)\}$ be a groupoid of order 40.

Now using these type of groupoids built using Z_n we construct matrix groupoids, polynomials with groupoid coefficients and polynomials with matrix groupoids under the groupoid operation '*' on Z_n.

We will illustrate all these situations only by examples.

Example 1.5: Let

$$G = \{(g_1, g_2, g_3, g_4) \mid g_i \in Z_{15}, *, (2, 7); 1 \leq i \leq 4\}$$

be a row matrix groupoid. If x = (g_7, g_9, g_1, g_0) and y = (g_3, g_2, g_4, g_6) are in g then

$$\begin{aligned} x * y &= (g_7, g_9, g_1, g_0) * (g_3, g_2, g_4, g_6) \\ &= (g_7 * g_3, g_9 * g_2, g_1 * g_4, g_0 * g_6) \\ &= (g_{14+21 \text{ (mod 15)}}, g_{18+14 \text{ (mod 15)}}, g_{2+28 \text{ (mod 15)}}, g_{0+42 \text{ (mod 15)}}) \\ &= (g_5, g_2, g_0, g_{12}) \in G. \end{aligned}$$

Thus G is a row matrix groupoid.

Example 1.6: Let

$$G = \left\{ \begin{bmatrix} g_i \\ g_j \\ g_k \\ g_t \\ g_s \end{bmatrix} \middle| \; g_i, g_j, g_k, g_t, g_s \in Z_{20}, (10, 4), * \right\}$$

be a column matrix groupoid.

For any x = $\begin{bmatrix} g_1 \\ g_4 \\ g_6 \\ g_7 \\ g_8 \end{bmatrix}$ and y = $\begin{bmatrix} g_{18} \\ g_7 \\ g_9 \\ g_1 \\ g_{10} \end{bmatrix}$ $\in$ G;

$$x * y = \begin{bmatrix} g_1 * g_{18} \\ g_4 * g_7 \\ g_6 * g_9 \\ g_7 * g_1 \\ g_8 * g_{10} \end{bmatrix} = \begin{bmatrix} g_{10+72 (\text{mod } 20)} \\ g_{40+28 (\text{mod } 20)} \\ g_{60+36 (\text{mod } 20)} \\ g_{70+4 (\text{mod } 20)} \\ g_{80+40 (\text{mod } 20)} \end{bmatrix} = \begin{bmatrix} g_2 \\ g_8 \\ g_{16} \\ g_{14} \\ g_0 \end{bmatrix} \in G.$$

Thus G is a groupoid.

Example 1.7: Let

$$G = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| a_i \in Z_7; (2, 5), 1 \le i \le 18 \right\}$$

be a matrix groupoid.

Example 1.8: Let

$$H = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in Z_{21}; (3, 12), *, 1 \le i \le 9 \right\}$$

be a matrix groupoid.

Now we can proceed onto define polynomial with coefficients from the groupoid.

We illustrate this situation by some examples.

Example 1.9: Let

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{26}; *, (3, 8)\right\}$$

be a polynomial groupoid.

Example 1.10: Let

$$T = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{12}; *, (9, 0)\right\}$$

be a polynomial groupoid.

Example 1.11: Let

$$P = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{13}; *, (3, 3)\right\}$$

be a polynomial groupoid.

Now we can have polynomial groupoids with matrix coefficients from the groupoid.

Example 1.12: Let

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (x_1, x_2, \ldots, x_{15}) \right.$$
$$\left. \text{where } x_j \in Z_{12}; *, (3, 4), 1 \le j \le 15\right\}$$

be a polynomial groupoid with row matrix coefficients.

Example 1.13: Let

$$P = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_9 \end{bmatrix} \text{ where } y_j \in Z_{10}; *, (7, 0), 1 \leq j \leq 9\right\}$$

be a polynomial groupoid with column matrix coefficients.

Example 1.14: Let

$$P = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \\ t_5 & t_6 & t_7 & t_8 \\ t_9 & t_{10} & t_{11} & t_{12} \\ t_{13} & t_{14} & t_{15} & t_{16} \end{bmatrix}\right.$$

$$\left. \text{where } t_j \in Z_{29}; *, (17, 13), 1 \leq j \leq 16\right\}$$

be a polynomial groupoid with square matrix coefficients.

Example 1.15: Let

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 & m_2 & \ldots & m_{10} \\ m_{11} & m_{12} & \ldots & m_{20} \\ m_{21} & m_{22} & \ldots & m_{30} \end{bmatrix}\right.$$

$$\left. \text{where } m_j \in Z_{40}; *, (10, 1), 1 \leq j \leq 30\right\}$$

be a polynomial groupoid with matrix coefficients.

Example 1.16: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \text{ where } t_j \in Z_{12}; *, (3, 0), 1 \leq j \leq 4 \right\}$$

be a polynomial groupoid with square matrix coefficients.

We just recall the definition of the new class of loops

$L_n(m)$ = {e, 1, 2, …, n with *, m, n > 3 and n odd}
= {e, g_1, g_2, …, g_n with *, m

(where $g_i \in Z_n$; $g_1 = 1$, $g_2 = 2$, …, $g_n = 0$) where $1 < m < n$ such that [m, n] = 1, [m – 1, n] = 1, n > 3 and n odd}.

For $g_i, g_j \in L_n(m)$ define $g_i * g_j = g_{(m-1)i} + m_{j \pmod n} \in L_n(m)$.

$L_n(m)$ is a loop of order n + 1 that is $L_n(m)$ is always of even order greater than or equal to 6. For more about these loops please refer [37].

All properties discussed in case of groupoids can be done for the case of loops. However the resultant product may not be a loop in general. For the concept of semifield refer [41]. We will be using these loops and groupoids to build linear algebra and semilinear algebras which are non associative.

Chapter Two

NON ASSOCIATIVE SEMILINEAR ALGEBRAS

In this chapter we for the first time introduce the notion of non associative semilinear algebra and illustrate them will examples.

DEFINITION 2.1: *Let $(V, *)$ be a groupoid. F a semifield. We say V is a nonassociative semilinear algebra over F if the following conditions are true.*

(i) For all $v \in V$ and $a \in F$, $av \in V$.
(ii) For all $s \in F$ and $u, v \in V$ $s(u * v) = su * sv$
(iii) $1.u = u$ for all $u \in V$; $1 \in F$

or equivalently we define a non associative semilinear algebra V over a semifield F as follows:

(i) $(V, *)$ is groupoid
($*$ a non associative binary operation on V).
(ii) For all $v \in V$ and $a \in F$ $av \in V$.
(iii) $s(u * v) = su * sv$ for all $s \in F$ and $u, v \in V$.
(iv) $1.u = u$ for all $u \in V$ and $1 \in F$.

We will illustrate this situation by some examples.

Example 2.1: Let $(Q^+ \cup \{0\}, *, (8, 7)) = G$ be the groupoid. G is a non associative semilinear algebra over the semifield $Q^+ \cup \{0\}$.

It is interesting to note that G is not a non associative semilinear algebra over $R^+ \cup \{0\}$; however G is a non associative semilinear algebra over $Z^+ \cup \{0\}$.

Example 2.2: Let $V = \{3Z^+ \cup \{0\}, *, (3, 11)\}$ be a groupoid and $S = Z^+ \cup \{0\}$ be a semifield. V is not a non associative semilinear algebra over the semifield $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$.

Example 2.3: Let $M = \{(a, b, c) \mid a, b, c \in Z^+ \cup \{0\}, *, (7, 1)\}$ be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

Example 2.4: Let

$$P = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \middle| a, b, c, d, e \in Q^+ \cup \{0\}, *, (4, 9) \right\}$$

be a non associative semilinear algebra over the semifield $S = Q^+ \cup \{0\}$.

Example 2.5: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| a_i \in Q^+ \cup \{0\}, *, (11, 3) \right\}$$

be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

For if x = $\begin{bmatrix} 7 & 2 & 4 & 7/5 \\ 0 & 2/5 & 0 & 4/3 \end{bmatrix}$ and

y = $\begin{bmatrix} 0 & 1 & 2 & 8/3 \\ 1/2 & 0 & 2/7 & 0 \end{bmatrix} \in$ R then

$$x * y \quad = \begin{bmatrix} 7 & 2 & 4 & 7/5 \\ 0 & 2/5 & 0 & 4/3 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 2 & 8/3 \\ 1/2 & 0 & 2/7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7*0 & 2*1 & 4*2 & 7/5*8/5 \\ 0*1/2 & 2/5*0 & 0*2/7 & 4/3*0 \end{bmatrix}$$

$$= \begin{bmatrix} 77 & 25 & 50 & 117/5 \\ 3/2 & 22/5 & 6/7 & 44/3 \end{bmatrix} \in \text{R}.$$

Also if 20 $\in$ S then 20.x = 20 $\times \begin{bmatrix} 7 & 2 & 4 & 7/5 \\ 0 & 2/5 & 0 & 4/3 \end{bmatrix}$

$$= \begin{bmatrix} 140 & 40 & 80 & 28 \\ 0 & 8 & 0 & 80/3 \end{bmatrix} \in \text{R}.$$

Example 2.6: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in R^+ \cup \{0\}, 1 \leq i \leq 9, *, (3, 2)\right\}$$

be a non associative semilinear algebra over the semifield $S = Q^+ \cup \{0\}$.

Consider $x = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ and $y = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \in V$

then

$$x * y = \begin{bmatrix} 3a_1 + 2b_1 & 3a_2 + 2b_2 & 3a_3 + 2b_3 \\ 3a_4 + 2b_4 & 3a_5 + 2b_5 & 3a_6 + 2b_6 \\ 3a_7 + 2b_7 & 3a_8 + 2b_8 & 3a_9 + 2b_9 \end{bmatrix} \in V.$$

Example 2.7: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \, a_i \in R^+ \cup \{0\},\ 1 \leq i \leq 15 \right\}$$

be a non associative semilinear algebra over the semifield $S = Q^+ \cup \{0\}$.

Now having seen examples of non associative semilinear algebras, we proceed onto define substructures.

DEFINITION 2.2: *Let V be a non associative semilinear algebra defined over the semifield F. Suppose $W \subseteq V$ be a proper subset of V; if W itself is a non associative semilinear algebra then we define W to be a non associative semilinear subalgebra of V over the semifield F.*

We will first give examples of them.

Example 2.8: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in Z^+ \cup \{0\}, *, (3, 13),\ 1 \leq i \leq 4 \right\}$$

be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

Consider

$$W_1 = \left\{ \begin{bmatrix} 0 & a_1 \\ a_2 & a_3 \end{bmatrix} \middle| a_i \in Z^+ \cup \{0\}, *, (3, 13),\ 1 \leq i \leq 3 \right\} \subseteq V,$$

W_1 is a non associative semilinear subalgebra over the semifield $S = Z^+ \cup \{0\}$.

$$W_2 = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \middle| a_i \in Z^+ \cup \{0\}, *, (3, 13),\ 1 \leq i \leq 2 \right\} \subseteq V$$

is a non associative semilinear subalgebra of V over the semifield S.

Now

$$W_1 \cap W_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \middle| a \in Z^+ \cup \{0\}, *, (3, 13) \right\} \subseteq V$$

is again a non associative semilinear subalgebra of V over the semifield S.

Suppose

$$W_3 = \left\{ \begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix} \middle| a_i \in Z^+ \cup \{0\}, *, (3, 13),\ 1 \leq i \leq 2 \right\} \subseteq V$$

be a non associative semilinear subalgebra of V over S.

We see $W_2 \cap W_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. However $W_2 + W_3 \neq V$ we cannot call this as a direct sum of non associative semilinear subalgebras of V over S.

THEOREM 2.1: *Let V be a non associative semilinear algebra over the semifield S. Suppose $W_1, W_2, \ldots, W_t$ be t non associative semilinear subalgebras of V over S.*

(i) $\bigcap_{i=1}^{t} W_i$ *= W is a non associative semilinear subalgebra of V or (0).*

(ii) $\bigcup_{i=1}^{t} W_i$ *in general is not a non associative semilinear algebra over V.*

The proof of this theorem is direct and hence is left as an exercise to the reader.

Example 2.9: Let $V = \{(a_1, a_2, a_3, a_4) \mid a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 4, (8, 9), *\}$ be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

Consider
$W_1 = \{(a_1, a_2, a_3, a_4) \mid a_i \in 3Z^+ \cup \{0\}; 1 \leq i \leq 4, (8, 9), *\} \subseteq V$ and $W_2 = \{(a_1, a_2, a_3, a_4) \mid a_i \in 5Z^+ \cup \{0\}; 1 \leq i \leq 4, (8, 9), *\} \subseteq V$ be two non associative semilinear subalgebras of V over $S = Z^+ \cup \{0\}$. We see $W_1 \cap W_2 = \{(a_1, a_2, a_3, a_4) \mid a_i \in 15Z^+ \cup \{0\}; 1 \leq i \leq 4, (8, 9), *\} \subseteq V$ is a non associative semilinear subalgebra of V over $S = Z^+ \cup \{0\}$.

But $W_1 \cup W_2 = \{(a_1, a_2, a_3, a_4) \mid a_i \in 5Z^+ \cup \{0\}; 1 \leq i \leq 4, (8, 9), *\} \subseteq V$ is only a subset for take x = (3, 3, 0, 0) and y = (5, 5, 5, 0) $\in W_1 \cup W_2$

$$\begin{aligned} x * y &= (3, 3, 0, 0) * (5, 5, 5, 0) \\ &= (3*5, 3*5, 0*5, 0*0) \end{aligned}$$

$= (69, 69, 45, 0) \notin W_1 \cup W_2$ so $W_1 \cup W_2$ is not a non associative semilinear subalgebra of V over $S = Z^+ \cup \{0\}$.

We cannot say for any non associative semilinear algebra V over a semifield S; V is a direct sum of non associative semilinear subalgebras or pseudo direct sum of non associative semilinear algebras as it is a difficult task.

Example 2.10: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z^+ \cup \{0\}, *, (3, 7)\right\}$$

be a non associative semilinear algebra over the semifield $Z^+ \cup \{0\} = S$. Take $x = 9 + 20x^2 + 3x^3$ and $y = 3x + 2x^2 + 4x^3 + 7$ in V.

$$\begin{aligned} x * y &= (9 + 20x^2 + 3x^3) * (3x + 2x^2 + 4x^3 + 7) \\ &= 9 * 3x + 20x^2 * 3x + 3x^3 * 3x + 9 * 2x^2 + 20x^2 * 2x^2 + 3x^2 * 2x^2 + 3x^3 * 4x^3 + 9 * 4x^3 + 20x^2 * 4x^3 + 9 * 7 + 20x^2 * 7 + 3x^3 * 7 \\ &= 48x + 81x^3 + 30x^4 + 41x^2 + 74x^4 + 23x^4 + 37x^6 + 57x^3 + 88x^5 + 76 + 109x^2 + 58x^3 \\ &= 76 + 48x + 150x^2 + 196x^3 + 127x^4 + 88x^5 + 37x^6. \end{aligned}$$

Thus it is easily verified $x * y \neq y * x$.

Example 2.11: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R^+ \cup \{0\}, *, (3/2, \sqrt{2}\,)\right\}$$

be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

Example 2.12: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Q^+ \cup \{0\}, *, (7, 3) \right\}$$

be a non associative semilinear algebra over the semifield S = $Q^+ \cup \{0\}$.

Example 2.13: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z^+ \cup \{0\}, *, (9, 0) \right\}$$

be a non associative semilinear algebra over the semifield S = $Z^+ \cup \{0\}$.

Example 2.14: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Q^+ \cup \{0\}, *, (9/7, 0) \right\}$$

be a non associative semilinear algebra over S = $Z^+ \cup \{0\}$, the semifield.

Take $p(x) = 35 + 8x + 9x^2$ and $q(x) = 40 + 7x^2 + 14x^3 + 45x^5$ in T.

$$\begin{aligned} p(x) * q(x) &= (35 + 8x + 9x^2) * (40 + 7x^2 + 14x^3 + 45x^5) \\ &= (35 \times 9) / 7 + (8/7) \times 9x + 9^2 x^2 / 7 \\ &= 45 + (72/7)x + (81/7)x^2 \in T. \end{aligned}$$

Example 2.15: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z^+ \cup \{0\}, *, (2, 2) \right\}$$

be a non associative semilinear algebra over the semifield S = $Z^+ \cup \{0\}$.

Consider $p(x) = 3 + 2x + x^3$, $q(x) = 1 + 4x^2$ and $r(x) = 2+7x$ in M.

$$
\begin{aligned}
&(p(x)) * (q(x) * r(x)) \\
&\quad = p(x) * [(2+4) + (2+14)x + (8+4)x^2 + (8+14)x^3] \\
&\quad = p(x) * (6 + 16x + 12x^2 + 22x^3) \\
&\quad = (3 + 2x + x^3)(6 + 16x + 12x^2 + 22x^3) \\
&\quad = (6 + 12) + (6 + 32)x + (6 + 24)x^2 + (6 + 44)x^2 + \\
&\qquad (4+12)x + (4+32)x^2 + (4+24)x^3 + (4 + 44)x^4 + \\
&\qquad (2+12)x^3 + (2+32)x^4 + (2 + 24)x^5 + (2+44)x^6 \\
&\quad = 18 + 38x + 30x^2 + 50x^2 + 16x + 36x^2 + 28x^3 + 48x^4 + \\
&\qquad 14x^3 + 34x^4 + 26x^5 + 46x^6 \\
&\quad = 18 + 54x + 116x^2 + 42x^3 + 82x^4 + 26x^5 + 46x^6.
\end{aligned}
$$

$$
\begin{aligned}
&(p(x) * q(x)) * r(x) = \\
&\quad = ((3 + 2x + x^3) * (1+4x^2)) * (2 + 7x) \\
&\quad = ((6 + 2) + (4 + 2)x + (2 + 2)x^3 + (6 + 8)x^2 + (4 + 8)x^3 \\
&\qquad + (2 + 8)x^5) * (2 + 7x) \\
&\quad = (8 + 6x + 14x^2 + 16x^3 + 10x^5) * (2 + 7x) \\
&\quad = (16 + 4) + (12 + 14)x^2 + (16 + 14)x + (4 + 12)x + \\
&\qquad (28 + 4)x^2 + (28 + 14)x^3 + (32 + 4)x^3 + (20 + 4)x^5 + \\
&\qquad (32 + 14)x^4 + (20 + 14)x^6. \\
&\quad = 20 + 46x + 58x^2 + 78x^3 + 46x^4 + 24x^5 + 34x^6.
\end{aligned}
$$

Clearly $p(x) * (q(x) * r(x)) \neq (p(x) * q(x)) * r(x)$.

Thus M is a non associative semilinear algebra of polynomials over the semifield.

Now we proceed onto define strong non associative semilinear algebras over the field.

DEFINITION 2.3: *Let V = {G, *, (t, u)} be a groupoid. F be any fields, V is a strong non associative semilinear algebra over the field F if the following conditions are satisfied.*

(i) For every $x \in V$ and $a \in F$
$ax = xa \in V$.
(ii) $a(x*y) = ax * ay$ for all $a \in F$ and $x, y \in F$.
(iii) For $1 \in F$, $1.v = v$ for all $v \in V$.
(iv) $0. v = 0$ for all $v \in V$ and $0 \in F$.

We give examples of them.

Example 2.16: Let V = {Q, *, (7, 3)} be a groupoid; V is a strong non associative semilinear algebra over the field Q.

Example 2.17: Let V = {R, *, (0, 8)} be a strong non associative semilinear algebra over the field Q.

Example 2.18: Let M = {R, *, (7, 7)} be a strong non associative semilinear algebra over the field Q.
Take 2, 3, 5 $\in$ R. $2 * (3 * 5) = 2 * (21+35) = 2 * 56$

$$= 14 + 56 \times 7$$
$$= 14 + 392 = 408$$

Now $(2 * 3) * 5 = (14 + 21) * 5$

$$= 35 * 5$$
$$= 245 + 35$$
$$= 280.$$

Clearly $(2 * 3) * 5 \neq 2 * (3 * 5)$. Thus M is a strong non associative semilinear algebra over the field Q.

Example 2.19: Let T = {R, *, (2, 5)} be a strong non associative semilinear algebra over the field R.

Example 2.20: Let P = {Q, *, (3/2, 7/9)} be a strong non associative semilinear algebra over the field Q.

Example 2.21: Let S = {(a, b, c, d) | a, b, c, d, $\in$ Q, *, (8, 11)} be a strong row matrix non associative semilinear algebra over the field Q.

Example 2.22: Let

$M = \{(a_1, a_2, a_3) \mid a_i \in Q, 1 \leq i \leq 3, *, (10, 3)\}$ be a strong row matrix non associative semilinear algebra over the field Q.

Example 2.23: Let $M = \{(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9) \mid m_i \in R, 1 \leq i \leq 9, *, (0, 19)\}$ be a strong row matrix non associative semilinear algebra over the field Q.

Example 2.24: Let

$T = \{(t_1, t_2, t_3, t_4) \mid ti \in R, 1 \leq i \leq 4, *, (3/7, 19/5)\}$

be a strong row matrix non associative semilinear algebra over the field Q.

Example 2.25: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in R, 1 \leq i \leq 5, *, (3, \sqrt{7}) \right\}$$

be a strong column matrix non associative semilinear algebra over the field Q.

Example 2.26: Let

$$S = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{20} \end{bmatrix} \middle| b_i \in Q, 1 \leq i \leq 20, *, (10, 7/11) \right\}$$

be a strong non associative semilinear algebra of column matrices over the field Q.

Example 2.27: Let

$$P = \left\{ \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_{10} \end{bmatrix} \middle| t_i \in R, 1 \leq i \leq 10, *, (\sqrt{3}, \sqrt{11}) \right\}$$

be a strong column matrix non associative semilinear algebra over the field Q (or R).

Example 2.28: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in Q, 1 \leq i \leq 12, *, (3, 13) \right\}$$

be a strong square matrix semilinear algebra over the field Q.

$$\text{Consider } x = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \text{ in S.}$$

$$x * y = \begin{bmatrix} 13 & 3 & 19 \\ 9 & 26 & 13 \\ 16 & 29 & 0 \end{bmatrix} \in S.$$

It is easily verified '*' is a non associative operation on S.

Example 2.29: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in Q, 1 \leq i \leq 5, *, (3,0) \right\}$$

be a strong non associative column matrix semilinear algebra over the semifield Q.

$$\text{For } x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \text{ and } y = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \in S.$$

$$x * y = \begin{bmatrix} 3a_1 \\ 3a_2 \\ 3a_3 \\ 3a_4 \\ 3a_5 \end{bmatrix} \in S,$$

'*' is clearly a non associative operation on S.

$$\text{For take } x = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}, y = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 7 \end{bmatrix} \text{ and } z = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \text{ in S.}$$

$$\text{Consider } x * (y*z) = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \\ 5 \end{bmatrix} * \left(\begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 7 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \\ 5 \end{bmatrix} * \begin{bmatrix} 0 \\ 6 \\ 12 \\ 15 \\ 21 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 0 \\ 3 \\ 15 \end{bmatrix}.$$

$$\text{If we take } (x*y)*z = \left(\begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \\ 5 \end{bmatrix} * \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 7 \end{bmatrix} \right) * \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 6 \\ 0 \\ 3 \\ 15 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 27 \\ 18 \\ 0 \\ 9 \\ 15 \end{bmatrix}.$$

Cleary $x*(y*z) \neq (x*y)*z$.

Thus S is a strong non associative column matrix semilinear algebra over the field Q.

Further x * y ≠ y * x, for consider

$$x*y = \begin{bmatrix} 9 \\ 6 \\ 0 \\ 3 \\ 15 \end{bmatrix} \text{ and } y*x = \begin{bmatrix} 0 \\ 6 \\ 12 \\ 15 \\ 21 \end{bmatrix}$$

so that '*' operation on S is also non commutative.

Example 2.30: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| a_i \in Q, 1 \leq i \leq 6, *, (2,2)\right\}$$

be a strong non associative matrix semilinear algebra over the field Q.

$$\text{Consider } x = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 5 & 4 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ in M.}$$

$$x*y = \begin{bmatrix} 8 & 6 \\ 6 & 12 \\ 20 & 20 \end{bmatrix} \text{ and}$$

$$y*x = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} * \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 6 & 12 \\ 20 & 20 \end{bmatrix}.$$

$y*x = x*y$ and is in M.

Take $x * (y * z)$; where $z = \begin{bmatrix} 4 & 2 \\ 6 & 1 \\ 0 & 3 \end{bmatrix}$ is in M.

$$x * (y*z) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 5 & 4 \end{bmatrix} * \begin{bmatrix} 10 & 8 \\ 18 & 10 \\ 10 & 18 \end{bmatrix} = \begin{bmatrix} 26 & 18 \\ 36 & 24 \\ 30 & 44 \end{bmatrix}.$$

$$\text{Consider } (x*y) * z = \begin{bmatrix} 8 & 6 \\ 6 & 12 \\ 20 & 20 \end{bmatrix} * \begin{bmatrix} 4 & 2 \\ 6 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 20 \\ 24 & 14 \\ 40 & 46 \end{bmatrix}.$$

We see $x * (y*z) \neq (x*y)*z$.

Thus M is a strong non associative matrix semilinear algebra over the field Q.

Now having seen examples of strong non associative matrix semilinear algebras defined over the field Q we now proceed onto give substructures. The definition of the strong non associative matrix semilinear subalgebra is a matter of routine.

We give one or two examples of them.

Example 2.31: Let $M = \{(a_1, a_2, a_3) \mid a_i \in Q, 1 \leq i \leq 3, *, (8, 1)\}$ be a strong row matrix non associative semilinear algebra over the field Q. Clearly $P = \{(a, 0, 0) \mid a \in Q, *, (8, 1)\} \subseteq M$ is not a strong row matrix non associative semilinear algebra over the field Q.

Consider $R = \{(a, a, a) \mid a \in Q; *, (8, 1)\} \subseteq M$ is a strong row matrix non associative semilinear sub algebra of M over Q.

For if x = (a, a, a) and y = (b, b, b) are in P then x * y = (a, a, a) * (b, b, b) = (8a + b, 8a+b, 8a+b) $\in$ P further if t $\in$ Q then tx = (ta, ta, ta) $\in$ P hence R is a strong row matrix non associative semilinear subalgebra of M over Q.

Example 2.32: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| a_i \in R, 1 \leq i \leq 10, *, (5, 0) \right\}$$

be a strong column matrix non associative semilinear algebra over the field R.

Consider

$$S = \left\{ \begin{bmatrix} a \\ a \\ \vdots \\ a \end{bmatrix} \middle| a \in R, *, (5, 0) \right\} \subseteq T,$$

S is a strong column matrix non associative semilinear subalgebra of T for if

$$x = \begin{bmatrix} a \\ a \\ \vdots \\ a \end{bmatrix} \text{ and } y = \begin{bmatrix} b \\ b \\ \vdots \\ b \end{bmatrix} \text{ then}$$

$$x*y = \begin{bmatrix} 5a \\ 5a \\ \vdots \\ 5a \end{bmatrix} \in S,$$

$$\text{further if } t \in R \text{ then } tx = \begin{bmatrix} ta \\ ta \\ \vdots \\ ta \end{bmatrix} \in S,$$

thus S is a strong column matrix non associative semilinear subalgebra of T over R.

We see

$$W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix} \middle| \, a, b \in R, *, (5, 0) \right\} \subseteq T$$

is also a strong column matrix non associative semilinear subalgebra of T over the field R.

Now having see strong matrix non associative semilinear subalgebras, we can also have the notion of quasi strong matrix non associative semilinear subalgebra over a subfield.

We give a few examples of them.

Example 2.33: Let

$$V = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in R, 1 \leq i \leq 4, *, (2, 5)\right\}$$

be a strong non associative matrix semilinear algebra over the field R.

Consider

$$W = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in Q, 1 \leq i \leq 4, *, (2, 5)\right\} \subseteq V,$$

W is a quasi strong matrix non associative semilinear subalgebra of V over Q. Q a subfield of R.

Example 2.34: Let

$$M = \left\{\begin{bmatrix} a_1 & a_6 \\ a_2 & a_7 \\ a_3 & a_8 \\ a_4 & a_9 \\ a_5 & a_{10} \end{bmatrix} \middle| a_i \in R, 1 \leq i \leq 10, *, (5, 10)\right\}$$

be a strong non associative quasi semilinear algebra over the field Q. Clearly M has no quasi strong non associative quasi semilinear subalgebras as Q has no subfield so we define M to be quasi simple.

In view of this we have the following theorem.

THEOREM 2.2: *Let V be a strong non associative semilinear algebra over the prime field F. V has no quasi strong non associative semilinear subalgebras so V is quasi simple.*

COROLLARY 2.1: Let V be strong non associative semilinear algebra over the non prime field F (F has subfields). V is not quasi simple.

The proof is direct and hence left as an exercise to the reader.

Now we can have strong non associative semilinear algebras using polynomials and the operation on the polynomials forms only a groupoid.

We will give only examples of them.

Example 2.35: Let

$$S = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R, *, (3, 19)\right\}$$

be a strong non associative polynomial semilinear algebra over the field R. S is not quasi simple.

Example 2.36: Let

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Q, *, (2, 0)\right\}$$

be a strong non associative polynomial semilinear algebra over the field Q. M is quasi simple.

Example 2.37: Let

$$P = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Q, *, (3, 3)\right\}$$

be a strong non associative polynomial quasi simple semilinear algebra over the field Q.

Example 2.38: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in R, *, (3, 9) \right\}$$

be a strong non associative polynomial semilinear algebra over the field Q. M has strong non associative polynomial semilinear subalgebras over the subfield Q of R.

Example 2.39: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (m_1, m_2, \ldots, m_7);\, m_i \in Q, *, (8, 0) \right\}$$

be a strong non associative row matrix coefficient quasi simple semilinear algebra over the field Q.

Example 2.40: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (n_1, n_2, \ldots, n_{10});\, n_j \in R, *, 1 \leq j \leq 10; (3, 3) \right\}$$

be a strong non associative row matrix coefficient polynomial semilinear algebra over R. P is a commutative non associative semilinear algebra over R.

Example 2.41: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (t_1, t_2, t_3);\, t_j \in Q, *, 1 \leq j \leq 3; (29, 7) \right\}$$

be a strong non commutative non associative row matrix coefficient polynomial semilinear algebra over the field R.

Now we can construct column matrix coefficient polynomial semilinear algebra over field F; we give a few examples of them.

Example 2.42: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{12} \end{bmatrix}; m_j \in R, *, 1 \leq j \leq 12; (0, 9) \right\}$$

be a strong column matrix coefficient polynomial non associative semilinear algebra over the field Q. Clearly M is non commutative and non associative.

Example 2.43: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{20} \end{bmatrix}; t_j \in R, *, 1 \leq j \leq 20; (\sqrt{5}, \sqrt{5}) \right\}$$

be a strong commutative non associative column matrix coefficient polynomial semilinear algebra over the field Q.

Example 2.44: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{26} \end{bmatrix}; m_j \in Q, *, 1 \leq j \leq 26; (3/7, 19/2) \right\}$$

be a strong non commutative non associative column matrix coefficient semilinear algebra over the field Q.

Now we can have any matrix coefficient polynomial non associative semilinear algebra over a field.

We give a few examples of them.

Example 2.45: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ \vdots & \vdots & \vdots \\ m_{28} & m_{29} & m_{30} \end{bmatrix}; m_j \in R, *, 1 \leq j \leq 30; \right.$$

$$\left. (3, 0) \right\}$$

be a strong non associative matrix coefficient polynomial semilinear algebra over Q.

Example 2.46: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (x_1, x_2, x_3, x_4); x_j \in Q, *, 1 \leq j \leq 4; (7, 1) \right\}$$

be a strong non associative semilinear algebra with row matrix coefficient polynomials over the field Q.

Take $p(x) = (2, 0, 1, 5) + (0, 3, 0, 1)x + (1, 1, 0, 2)x^2$ and $q(x) = (1, 1, 0, 0) + (3, 0, 1, 0)x^2 + (0, 1, 1, 0)x^3$ in T.

$$\begin{aligned} p(x) * q(x) = {} & (2, 0, 1, 5) * (1, 1, 0, 0) + (2, 0, 1, 5) * \\ & (3, 0, 1, 0)x^2 + (2, 0, 1, 5) * (0, 1, 1, 0)x^3 + \\ & (0, 3, 0, 1) * (1, 1, 0, 0)x + (0, 3, 0, 1) * \\ & (3, 0, 1, 0)x^3 + (0, 3, 0, 1) * (0, 1, 1, 0)x^4 + \\ & (1, 1, 0, 2) * (1, 1, 0, 0)x^2 + (1, 1, 0, 2) * \\ & (3, 0, 1, 0)x^4 + (1, 1, 0, 2)\ (0, 1, 1, 0)x^5 \\ = {} & (15, 1, 8, 35)x^3 + (1, 22, 0, 7)x + (3, 21, 1, 7)x^3 \\ & + (0, 22, 1, 7)x^4 + (8, 8, 0, 14)x^2 + \\ & (10, 7, 1, 14)x^4 + (7, 8, 1, 14)x^5 \\ = {} & (15, 1, 7, 35) + (25, 8, 8, 49)x^2 + (1, 22, 0, 7)x + \\ & (17, 22, 9, 42)x^3 + (10, 29, 2, 21)x^4 + \\ & (7, 8, 1, 14)x^5. \end{aligned}$$

This is the way operation * is performed on the strong non associative row matrix coefficient polynomial semilinear algebra over a field.

Example 2.47: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}; m_j \in Q, *, 1 \leq j \leq 6; (0,3) \right\}$$

be a strong non associative column matrix coefficient polynomial semilinear algebra over the field Q.

$$\text{Take } p(x) = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} x^2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} x^4 \text{ and}$$

$$q(x) = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} x^3 + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x^5 \text{ be in M.}$$

We find p(x) * q(x) as follows.

$$p(x) * q(x) = \begin{bmatrix}0\\1\\2\\0\\3\\0\end{bmatrix} * \begin{bmatrix}2\\0\\2\\0\\2\\0\end{bmatrix} x + \begin{bmatrix}0\\1\\2\\0\\3\\0\end{bmatrix} * \begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix} x^3$$

$$+ \begin{bmatrix}0\\1\\2\\0\\3\\0\end{bmatrix} * \begin{bmatrix}1\\0\\1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\3\\0\\1\\0\end{bmatrix} * \begin{bmatrix}2\\0\\2\\0\\2\\0\end{bmatrix} x^3 + \begin{bmatrix}1\\0\\3\\0\\1\\0\end{bmatrix} * \begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix} x^5 + \begin{bmatrix}1\\0\\3\\0\\1\\0\end{bmatrix} * \begin{bmatrix}1\\0\\1\\1\\1\\1\end{bmatrix} x^7$$

$$+ \begin{bmatrix}0\\0\\0\\1\\1\\1\end{bmatrix} * \begin{bmatrix}2\\0\\2\\0\\2\\0\end{bmatrix} x^5 + \begin{bmatrix}0\\0\\0\\1\\1\\1\end{bmatrix} * \begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix} x^7 + \begin{bmatrix}0\\0\\0\\1\\1\\1\end{bmatrix} * \begin{bmatrix}1\\0\\1\\1\\1\\1\end{bmatrix} x^9$$

$$= \begin{bmatrix}6\\0\\6\\0\\6\\0\end{bmatrix} x + \begin{bmatrix}0\\3\\0\\3\\0\\3\end{bmatrix} x^3 + \begin{bmatrix}3\\0\\3\\3\\3\\3\end{bmatrix} x^5 + \begin{bmatrix}6\\0\\6\\0\\6\\0\end{bmatrix} x^3 + \begin{bmatrix}0\\3\\0\\3\\0\\3\end{bmatrix} x^5 + \begin{bmatrix}3\\0\\3\\3\\3\\3\end{bmatrix} x^7 +$$

$$\begin{bmatrix}6\\0\\6\\0\\6\\0\end{bmatrix}x^5 + \begin{bmatrix}0\\3\\0\\3\\0\\3\end{bmatrix}x^7 + \begin{bmatrix}3\\0\\3\\3\\3\\3\end{bmatrix}x^9$$

$$= \begin{bmatrix}6\\0\\6\\0\\6\\0\end{bmatrix}x + \begin{bmatrix}6\\3\\6\\3\\6\\3\end{bmatrix}x^3 + \begin{bmatrix}9\\3\\9\\6\\9\\6\end{bmatrix}x^5 + \begin{bmatrix}0\\3\\0\\3\\0\\3\end{bmatrix}x^7 + \begin{bmatrix}3\\0\\3\\3\\3\\3\end{bmatrix}x^9 \text{ is in M.}$$

This is the way the '*' operation on M is defined to make the strong column matrix non associative semilinear algebra.

Example 2.48: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix}; m_j \in R, *, 1 \leq j \leq 9; \right.$$

$$\left. (2,3) \right\}$$

be a strong non associative semilinear algebra of square matrix coefficient polynomials over the field Q.

Consider

$$p(x) = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 4 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix} x^3,$$

$$q(x) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 7 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} x^3 + \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x^4$$

in M.

$$p(x) * q(x) = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 4 & 6 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 7 & 0 \end{bmatrix} x +$$

$$\begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 4 & 6 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} x^3 + \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 4 & 6 & 0 \end{bmatrix} * \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x^4 +$$

$$\begin{bmatrix} 6 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 7 & 0 \end{bmatrix} x^2 + \begin{bmatrix} 6 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} x^4 +$$

$$\begin{bmatrix} 6 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x^5 + \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 4 & 6 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 7 & 0 \end{bmatrix} x^4 +$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} x^6 + \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix} * \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x^7$$

$$= \begin{bmatrix} 3 & 6 & 3 \\ 2 & 6 & 3 \\ 8 & 33 & 0 \end{bmatrix} x + \begin{bmatrix} 3 & 6 & 8 \\ 5 & 3 & 4 \\ 14 & 12 & 3 \end{bmatrix} x^3 + \begin{bmatrix} 6 & 9 & 5 \\ 5 & 0 & 7 \\ 8 & 15 & 0 \end{bmatrix} x^4 +$$

$$\begin{bmatrix} 15 & 2 & 7 \\ 0 & 8 & 3 \\ 4 & 21 & 2 \end{bmatrix} x^2 + \begin{bmatrix} 15 & 2 & 10 \\ 3 & 5 & 0 \\ 10 & 0 & 5 \end{bmatrix} x^4 + \begin{bmatrix} 22 & 5 & 7 \\ 3 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix} x^5 +$$

$$\begin{bmatrix} 12 & 5 & 7 \\ 5 & 0 & 5 \\ 0 & 13 & 0 \end{bmatrix} x^7 + \begin{bmatrix} 9 & 2 & 7 \\ 2 & 6 & 5 \\ 0 & 31 & 0 \end{bmatrix} x^4 + \begin{bmatrix} 9 & 2 & 10 \\ 5 & 3 & 2 \\ 6 & 10 & 3 \end{bmatrix} x^6$$

$$= \begin{bmatrix} 3 & 6 & 3 \\ 2 & 6 & 3 \\ 8 & 33 & 0 \end{bmatrix} x + \begin{bmatrix} 3 & 6 & 8 \\ 5 & 3 & 4 \\ 14 & 12 & 3 \end{bmatrix} x^3 + \begin{bmatrix} 15 & 2 & 7 \\ 0 & 8 & 3 \\ 4 & 21 & 2 \end{bmatrix} x^2 +$$

$$\begin{bmatrix} 30 & 13 & 22 \\ 10 & 11 & 12 \\ 18 & 46 & 5 \end{bmatrix} x^4 + \begin{bmatrix} 22 & 5 & 7 \\ 3 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix} x^5 + \begin{bmatrix} 9 & 2 & 10 \\ 5 & 3 & 2 \\ 6 & 10 & 3 \end{bmatrix} x^6 +$$

$$\begin{bmatrix} 12 & 5 & 7 \\ 5 & 0 & 5 \\ 0 & 13 & 0 \end{bmatrix} x^7 \text{ is in M.}$$

This is the way '*' operation on M is performed.

Example 2.49: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5 \\ m_6 & m_7 & m_8 & m_9 & m_{10} \\ m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{16} & m_{17} & m_{18} & m_{19} & m_{20} \end{bmatrix}; m_j \in Q, *, \right.$$

$$\left. 1 \leq j \leq 20; (2,2) \right\}$$

be a strong non associative matrix coefficient polynomial semilinear algebra over the field Q.

Let

$$p(x) = \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 4 & 0 & 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 3 & 3 & 3 & 0 & 0 \\ 1 & 2 & 3 & 5 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} x +$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 1 & 1 & 3 & 2 \\ 1 & 3 & 1 & 1 & 1 \end{bmatrix} x^3 \text{ and}$$

$$q(x) = \begin{bmatrix} 0 & 1 & 0 & 1 & 4 \\ 4 & 0 & 2 & 1 & 2 \\ 1 & 1 & 6 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 5 & 0 & 1 & 0 & 6 \\ 0 & 3 & 0 & 1 & 2 \\ 4 & 5 & 0 & 6 & 0 \end{bmatrix} x^4 \text{ in P.}$$

Now

$$p(x) * q(x) = \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 4 & 0 & 1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 1 & 4 \\ 4 & 0 & 2 & 1 & 2 \\ 1 & 1 & 6 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} x$$

$$+ \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 4 & 0 & 1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 5 & 0 & 1 & 0 & 6 \\ 0 & 3 & 0 & 1 & 2 \\ 4 & 5 & 0 & 6 & 0 \end{bmatrix} x^4$$

$$+ \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 4 & 0 & 1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 1 & 4 \\ 4 & 0 & 2 & 1 & 2 \\ 1 & 1 & 6 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} x^2$$

$$+ \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 3 & 3 & 3 & 0 & 0 \\ 1 & 2 & 3 & 5 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 5 & 0 & 1 & 0 & 6 \\ 0 & 3 & 0 & 1 & 2 \\ 4 & 5 & 0 & 6 & 0 \end{bmatrix} x^5$$

$$+ \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 1 & 1 & 3 & 2 \\ 1 & 3 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 1 & 4 \\ 4 & 0 & 2 & 1 & 2 \\ 1 & 1 & 6 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} x^4$$

$$+ \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 1 & 1 & 3 & 2 \\ 1 & 3 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 5 & 0 & 1 & 0 & 6 \\ 0 & 3 & 0 & 1 & 2 \\ 4 & 5 & 0 & 6 & 0 \end{bmatrix} x^7$$

$$= \begin{bmatrix} 0 & 4 & 4 & 2 & 14 \\ 10 & 0 & 8 & 6 & 4 \\ 2 & 4 & 12 & 0 & 4 \\ 8 & 2 & 2 & 2 & 4 \end{bmatrix} x + \begin{bmatrix} 2 & 2 & 8 & 2 & 6 \\ 12 & 0 & 4 & 4 & 12 \\ 0 & 8 & 0 & 2 & 8 \\ 16 & 10 & 2 & 12 & 2 \end{bmatrix} x^4$$

$$+ \begin{bmatrix} 2 & 4 & 2 & 6 & 12 \\ 14 & 6 & 10 & 2 & 4 \\ 4 & 6 & 18 & 10 & 0 \\ 0 & 4 & 2 & 4 & 6 \end{bmatrix} x^2 + \begin{bmatrix} 4 & 2 & 6 & 6 & 4 \\ 16 & 6 & 8 & 0 & 12 \\ 2 & 10 & 6 & 12 & 4 \\ 8 & 12 & 2 & 14 & 4 \end{bmatrix} x^5$$

$$+ \begin{bmatrix} 2 & 6 & 2 & 6 & 10 \\ 12 & 2 & 8 & 4 & 8 \\ 8 & 4 & 14 & 6 & 4 \\ 2 & 8 & 2 & 4 & 4 \end{bmatrix} x^4 + \begin{bmatrix} 4 & 4 & 6 & 6 & 2 \\ 14 & 2 & 6 & 2 & 14 \\ 6 & 8 & 2 & 8 & 8 \\ 10 & 16 & 2 & 14 & 2 \end{bmatrix} x^7.$$

$$= \begin{bmatrix} 0 & 4 & 4 & 2 & 14 \\ 10 & 0 & 8 & 6 & 4 \\ 2 & 4 & 12 & 0 & 4 \\ 8 & 2 & 2 & 2 & 4 \end{bmatrix} x + \begin{bmatrix} 2 & 4 & 2 & 6 & 12 \\ 14 & 6 & 10 & 2 & 4 \\ 4 & 6 & 18 & 10 & 0 \\ 0 & 4 & 2 & 4 & 6 \end{bmatrix} x^2$$

$$+ \begin{bmatrix} 4 & 8 & 10 & 8 & 16 \\ 24 & 2 & 12 & 8 & 20 \\ 8 & 12 & 14 & 8 & 12 \\ 18 & 18 & 4 & 16 & 6 \end{bmatrix} x^4 + \begin{bmatrix} 4 & 2 & 6 & 6 & 4 \\ 16 & 6 & 8 & 0 & 12 \\ 2 & 10 & 6 & 12 & 4 \\ 8 & 12 & 2 & 14 & 4 \end{bmatrix} x^5$$

$$+ \begin{bmatrix} 4 & 4 & 6 & 6 & 2 \\ 14 & 2 & 6 & 2 & 14 \\ 6 & 8 & 2 & 8 & 8 \\ 10 & 16 & 2 & 14 & 2 \end{bmatrix} x^7$$

is in P. Thus we have shown how * operates on strong non associative with coefficient matrix polynomial semilinear algebras defined over a field.

All these strong structures where built using fields of characteristic zero are of infinite cardinality.

Now we define strong non associative semilinear algebras of different prime characteristic and of finite cardinality.

DEFINITION 2.4: *Let V = {Z_p, *, (t, u), t, u $\in Z_p$} be a groupoid. V is a strong non associative semilinear algebra defined over the field Z_p.*

We will examples of them.

Example 2.50: Let P = {Z_5, *, (2, 3)} be a strong non associative semilinear algebra defined over the field Z_5.

Example 2.51: Let V = {Z_{11}, *, (2, 7)} be a strong non associative semilinear algebra defined over the field Z_{11}. For x = 10 and y = 6 in V and 3 $\in Z_{11}$. We see

$$3\,(x*y) = 3\,(10*6) = 3\,(20+42) = 3.7 = 9.$$

$$3 \times 10 * 3 \times 6 = 10 * 18$$
$$= 8 * 7 = 16 + 49 = 65 = 9$$

Thus V is a strong semilinear non associative algebra over the field Z_{11}.

Example 2.52: Let V = {Z_{23}, *, (11, 0)} be a strong non associative semilinear algebra defined over the field Z_{23}.

Example 2.53: Let V = {(x_1, x_2, x_3, x_4) where $x_i \in Z_{43}$, $1 \leq i \leq 4$, 0, (0, 7)} be a strong non associative semilinear algebra defined over the field Z_{43}.

Consider x = (x_1, x_2, x_3, x_4) = (7, 1, 0, 2) and y = (1, 3, 4, 0) in V.

$x * y = (7, 1, 0, 2) * (1, 3, 4, 0) = (7, 21, 28, 0) \in V.$

Example 2.54: Let $V = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in Z_{19}, 0, (3, 3)\}$ be a strong non associative semilinear algebra defined over the field Z_{19}.

Take $x = (8, 1, 10, 17, 3, 1)$ and $y = (2, 3, 0, 1, 5, 1)$ in V.

$$x * y = (8, 1, 10, 17, 3, 1) * (2, 3, 0, 1, 5, 1)$$

$$= (11, 12, 11, 16, 5, 6) \in V.$$

Clearly number of elements in V is finite. But V is commutative but non associative.

For take $z = (0, 1, 2, 0, 0, 0) \in V$

$$(x * y) * z = (11, 12, 11, 16, 5, 6) * (0, 1, 2, 0, 0, 0)$$
$$= (14, 1, 8, 10, 15, 18).$$

Consider $x * (y * z)$

$$= (8, 1, 10, 17, 3, 1) * [(2, 3, 0, 1, 5, 1) * (0, 1, 2, 0, 0, 0)]$$
$$= (8, 1, 10, 17, 3, 1) * (6, 12, 6, 3, 15, 3)$$
$$= (17, 1, 10, 3, 16, 12).$$

Clearly $x * (y*z) \neq (x*y) * z$ for $x, y, z \in V$ so V is a non associative structure under ‘*’.

Now we proceed onto give examples of strong non associative column matrix semilinear algebra.

Example 2.55: Let

$$V = \left\{ \left[\begin{matrix} m_1 \\ m_2 \\ \vdots \\ m_8 \end{matrix}\right] \middle| m_i \in Z_{17}, 1 \leq i \leq 10, *, (3,7)\right\}$$

be a strong non associative semilinear algebra defined over the field Z_{17}.

$$\text{We see for } x = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ in } V$$

$$x * y = \begin{bmatrix} 7 \\ 0 \\ 6 \\ 13 \\ 2 \\ 0 \\ 7 \\ 3 \end{bmatrix} \in V.$$

Example 2.56: Let

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| x_i \in Z_3, 1 \leq i \leq 4, *, (2,1) \right\}$$

be a strong non associative semilinear algebra defined over the field Z_3.

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \in V,$$

$$x*y = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \in V.$$

Clearly V is of finite order.

Example 2.57: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in Z_2, 1 \le i \le 6, *, (1,0) \right\}$$

be a strong non associative semilinear algebra over the field S = Z_2.

Example 2.58: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in Z_7, 1 \le i \le 9, *, (3,5) \right\}$$

be a strong non associative semilinear algebra of finite order.

$$\text{If } x = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ are in V, then}$$

$$x * y = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 5 \end{bmatrix} \in V.$$

Example 2.59: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \, a_i \in Z_{13}, 1 \leq i \leq 15, *, (8,0) \right\}$$

be a strong non associative semilinear algebra of finite order.

$$\text{For } p = \begin{bmatrix} 8 & 2 & 1 \\ 6 & 0 & 7 \\ 0 & 7 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } q = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 6 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \\ 8 & 7 & 6 \end{bmatrix} \text{ in P,}$$

$$\text{we see } p * q = \begin{bmatrix} 12 & 3 & 8 \\ 9 & 0 & 4 \\ 0 & 4 & 0 \\ 8 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} \in P.$$

Example 2.60: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_{10} & a_{11} & a_{12} & a_{13} \\ a_4 & a_5 & a_6 & a_{14} & a_{15} & a_{16} & a_{17} \\ a_7 & a_8 & a_9 & a_{18} & a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| a_i \in Z_7, \right.$$

$$\left. 1 \le i \le 21, *, (3,4) \right\}$$

be a strong non associative semilinear algebra of finite order over the field Z_7.

Example 2.61: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ \vdots & \vdots & & \vdots \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix} \middle| a_i \in Z_{47}, 1 \le i \le 25, *, (3,10) \right\}$$

be a strong non associative semilinear algebra over the field Z_{47}.

We have the following theorem.

THEOREM 2.3: *Let M be a strong matrix non associative semilinear algebra over the field Z_p (p a prime). M is quasi simple.*

Proof: Obvious from the fact Z_p is a prime field.

Now interested reader can find substructures and study them.

Now we give polynomial matrix coefficient strong non associative semilinear algebra over the finite field Z_p.

We give mainly examples of them and they are of infinite order.

Example 2.62: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_7, *, (3, 0) \right\}$$

be a strong non associative semilinear algebra of polynomials with coefficients from Z_7. Clearly M is of infinite order.

Example 2.63: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{11}, *, (10, 3) \right\}$$

be a strong non associative semilinear algebra of polynomials with coefficients from Z_{11} of infinite order over Z_{11}.

Example 2.64: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_3, *, (2, 1) \right\}$$

be a strong non associative semilinear algebra over the field Z_3.

Example 2.65: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{29}, *, (13, 13) \right\}$$

be a strong non associative semilinear algebra over the field Z_{29}.

Example 2.66: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{19}, *, (14, 7) \right\}$$

be a strong non associative semilinear algebra over the field Z_{19} of infinite order.

Example 2.67: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (t_1, t_2, \ldots, t_{10});\ t_j \in Z_{11},\ 1 \le j \le 10, *, (3, 7) \right\}$$

be a strong non associative semilinear algebra over the field Z_{11} of infinite order.

Example 2.68: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (m_1, m_2, \ldots, m_{20});\ m_j \in Z_{23},\ 1 \le j \le 20, *, (3, 11) \right\}$$

be a strong non associative semilinear algebra over the field Z_{23}.

Example 2.69: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (n_1, n_2, \ldots, n_9);\ n_j \in Z_{13},\ 1 \le j \le 9, *, (9, 4) \right\}$$

be a strong non associative semilinear algebra over the field Z_{13}.

Example 2.70: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (m_1, m_2, \ldots, m_8);\ m_j \in Z_{17},\ 1 \le j \le 8, *, (3, 0) \right\}$$

be a strong non associative semilinear algebra of row matrix coefficient polynomial of infinite order over the finite field Z_{17}.

Example 2.71: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_6 \end{bmatrix}; p_j \in Z_5, 1 \leq j \leq 6, *, (2, 2) \right\}$$

be a strong non associative semilinear algebra of column matrix coefficient polynomial semilinear algebra over the field Z_5. M is commutative.

$$\text{Consider } x = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \\ 3 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \text{ in M.}$$

$$x * y = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \\ 2 \\ 3 \end{bmatrix} \text{ and } y * x = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

Example 2.72: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}; x_j \in Z_{13}, 1 \le j \le 4, *, (3, 10)\right\}$$

be a strong non associative column matrix coefficient semilinear algebra over the field Z_{13}.

$$\text{If } p(x) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 7 \\ 0 \end{bmatrix} x^3$$

$$\text{and } q(x) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} x + \begin{bmatrix} 1 \\ 7 \\ 8 \\ 2 \end{bmatrix} x^4 \text{ are in M then}$$

$$p(x) * q(x) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 5 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 2 \\ 5 \end{bmatrix} * \begin{bmatrix} 1 \\ 7 \\ 8 \\ 2 \end{bmatrix} x^4 + \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} x^2 +$$

$$\begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix} * \begin{bmatrix} 1 \\ 7 \\ 8 \\ 2 \end{bmatrix} x^5 + \begin{bmatrix} 1 \\ 0 \\ 7 \\ 0 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 7 \\ 0 \end{bmatrix} * \begin{bmatrix} 1 \\ 7 \\ 8 \\ 2 \end{bmatrix} x^7$$

$$= \begin{bmatrix} 10 \\ 10 \\ 6 \\ 10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \\ 8 \\ 9 \end{bmatrix} x^4 + \begin{bmatrix} 7 \\ 6 \\ 3 \\ 11 \end{bmatrix} x^2 + \begin{bmatrix} 10 \\ 1 \\ 5 \\ 0 \end{bmatrix} x^5 + \begin{bmatrix} 10 \\ 10 \\ 8 \\ 5 \end{bmatrix} x$$

$$+ \begin{bmatrix} 0 \\ 5 \\ 10 \\ 7 \end{bmatrix} x^7$$

$$= \begin{bmatrix} 7 \\ 7 \\ 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \\ 8 \\ 9 \end{bmatrix} x^4 + \begin{bmatrix} 7 \\ 6 \\ 3 \\ 11 \end{bmatrix} x^2 + \begin{bmatrix} 10 \\ 1 \\ 5 \\ 0 \end{bmatrix} x^5 + \begin{bmatrix} 0 \\ 5 \\ 10 \\ 7 \end{bmatrix} x^7 \in M.$$

Example 2.73: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}; d_i \in Z_3, 1 \leq i \leq 4, *, (1,2) \right\}$$

be a strong non associative semilinear algebra of matrix coefficient polynomials over the field Z_3.

Example 2.74: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \end{bmatrix}; p_j \in Z_7, \right.$$

$$\left. 1 \leq j \leq 8, *, (0,4) \right\}$$

be a strong non associative semilinear algebra of matrix coefficient polynomials over the field Z_7.

Consider p(x) = $\begin{bmatrix} 0 & 2 & 1 & 5 \\ 6 & 0 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}x^3$

and q(x) = $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}x^4 + \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 \end{bmatrix}x^6$ in P.

Now p(x) * q(x) = $\begin{bmatrix} 0 & 2 & 1 & 5 \\ 6 & 0 & 2 & 0 \end{bmatrix} * \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}x^4$

$$+ \begin{bmatrix} 0 & 2 & 1 & 5 \\ 6 & 0 & 2 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 \end{bmatrix}x^6 + \begin{bmatrix} 6 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} *$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}x^7 + \begin{bmatrix} 6 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 \end{bmatrix}x^9$$

$$= \begin{bmatrix} 1 & 4 & 4 & 4 \\ 4 & 4 & 4 & 1 \end{bmatrix}x^4 + \begin{bmatrix} 4 & 1 & 5 & 2 \\ 6 & 3 & 0 & 4 \end{bmatrix}x^6$$

$$+ \begin{bmatrix} 1 & 4 & 4 & 4 \\ 4 & 4 & 4 & 1 \end{bmatrix}x^7 + \begin{bmatrix} 4 & 1 & 5 & 2 \\ 6 & 3 & 0 & 4 \end{bmatrix}x^9 \in 9.$$

This is the way the * operation is performed on P. Clearly p(x) * q(x) ≠ q(x) * p(x).

Example 2.75: Let

$$S = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \\ t_5 & t_6 & t_7 & t_8 \\ t_9 & t_{10} & t_{11} & t_{12} \\ t_{13} & t_{14} & t_{15} & t_{16} \end{bmatrix}; t_j \in Z_{23},\right.$$

$$\left. 1 \le j \le 16, *, (2,1)\right\}$$

be a strong non associative semilinear algebra of matrix coefficient polynomial over the field Z_{23}.

Consider

$$p(x) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 4 \end{bmatrix} x^2 + \begin{bmatrix} 3 & 4 & 0 & 5 \\ 1 & 0 & 3 & 2 \\ 6 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} x^5$$

and $q(x) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 4 & 3 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & 0 & 2 \end{bmatrix} x^4$ be elements of S.

$$p(x) * q(x) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 4 & 3 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & 0 & 2 \end{bmatrix} x^5$$

$$+ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 4 & 3 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & 0 & 2 \end{bmatrix} x^6 +$$

$$\begin{bmatrix} 3 & 4 & 0 & 5 \\ 1 & 0 & 3 & 2 \\ 6 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 4 & 3 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & 0 & 2 \end{bmatrix} x^9.$$

$$= \begin{bmatrix} 1 & 4 & 4 & 7 \\ 2 & 5 & 4 & 9 \\ 5 & 0 & 7 & 2 \\ 11 & 3 & 4 & 2 \end{bmatrix} x^5 + \begin{bmatrix} 3 & 4 & 2 & 3 \\ 0 & 5 & 4 & 7 \\ 7 & 0 & 7 & 0 \\ 7 & 9 & 2 & 10 \end{bmatrix} x^6$$

$$+ \begin{bmatrix} 7 & 10 & 0 & 11 \\ 2 & 1 & 10 & 7 \\ 13 & 0 & 1 & 2 \\ 5 & 3 & 4 & 2 \end{bmatrix} x^9 \in S.$$

Study of special identities are properties about strong non associative semilinear algebras defined over a field of characteristic p are given as theorems. For proof refer [ ].

THEOREM 2.4: *Let*

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in z_p,\ (t,\ t),\ * \right\}$$

be a strong polynomial non associative semilinear algebra over the field Z_p. V is a P-semilinear algebra (A semilinear algebra is said to be a P-semilinear algebra if $p(x) * (q(x) * p(x)) = (p(x) * q(x)) * p(x)$ for all $p(x), q(x) \in V$).

Proof is direct hence left as an exercise to the reader.

Example 2.76: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ \vdots & \vdots & & \vdots \\ a_{82} & a_{83} & \dots & a_{90} \end{bmatrix} \middle|\; a_i \in Z_7,\ 1 \le i \le 90,\ *,\ (3, 3) \right\}$$

be a strong non associative P-semilinear algebra over the field Z_7.

Example 2.77: Let $T = \{(a_1, a_2, \ldots, a_{40}) \mid a_i \in Z_{41}, 1 \leq i \leq 40, *, (19, 19)\}$ be a strong non associative P-semilinear algebra over the field Z_{41}.

Example 2.78: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{20} \end{bmatrix} \middle| a_i \in Z_{23}, 1 \leq i \leq 23, *, (10, 10) \right\}$$

be a strong non associative P-semilinear algebra over the field Z_{23}.

Example 2.79: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| a_i \in Z_{73}, 1 \leq i \leq 16, *, (24, 24) \right\}$$

be a strong non associative P-semilinear algebra over the field Z_{73}.

A semilinear algebra V is an alternative semilinear algebra if for all $x, y \in V$;

$$(x * y) * y = x * (y * y).$$

In view of this definition we give a few examples.

Example 2.80: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in Z_{29}, 1 \leq i \leq 9, *, (9, 9) \right\}$$

be a strong non associative semilinear algebra over the field Z_{29} which is not alternative.

THEOREM 2.5: *Let $V = \{a_i \in Z_p, *, (t, t)\}$ be a strong non associative semilinear algebra defined over a field Z_p. V is never an alternative semilinear algebra.*

For we see for any x, y $\in$ V.

$$(x * y) * y = (tx + ty) * y \quad = t^2x + t^2y + ty \qquad \text{I}$$

$$\text{and } x * (y * y) = x * (ty + ty) \quad = tx + t^2y + t^2y \qquad \text{II}$$

I and II are not equal for equality forces $t^2 = t \pmod p$ which is impossible in Z_p.

Next we proceed onto define the notion of Smarandache non associative semilinear algebras which are defined over a Smarandache ring.

DEFINITION 2.5: *Let V be a groupoid. S be a Smarandache ring. If V is a non associative semilinear algebra over the S-ring S we call V a Smarandache non associative semilinear algebra over the S-ring.*

We will illustrate this situation by some examples.

Example 2.81: Let $V = \{(a_1, a_2) \mid a_i \in Z_6, *, (3, 5)\}$ be a S-non associative semilinear algebra over the S-ring Z_6 (For in the ring Z_6, $T = \{0, 3\} \subseteq Z_6$ is a field.

Example 2.82: Let

$V = \{(a_1, a_2, a_3, a_4) \mid a_i \in Z_{10}, *, (3, 7), 1 \leq i \leq 4\}$ be a S-non associative semilinear algebra over the S-ring Z_{10}. (Z_{10} is a S-ring as $\{0, 5\} \subseteq Z_{10}$ is a field).

Example 2.83: Let $V = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots, a_{12}) \mid a_i \in Z_{38}, *, (0, 20), 1 \leq i \leq 12\}$ be a Smarandache strong non associative semilinear algebra over the S-ring Z_{38}.

In view of all these examples we have the following theorem.

THEOREM 2.56: *Let Z_{2n} be the ring of modulo integers, Z_{2n} is a S-ring (Here n is a prime).*

Proof: Consider $\{0, n\} = A \subseteq Z_{2n}$ we see $n + n \equiv 2n = 0 \pmod{2n}$.

Thus $\{0, n\} = A$ is a field isomorphic to $Z_2 = \{0, 1\}$. Hence the claim.

THEOREM 2.7: *Let*

$V = \{(a_1, \ldots, a_m) \mid a_i \in Z_{2n}$ (n a prime), $*, (t, u); t, u \in Z_{2n}\}$ be a groupoid. V is a S-non associative semilinear algebra over the S-ring Z_{2n}.

This proof is also direct and hence is left as an exercise to the reader.

We can also have the Smarandache non associative column matrix semilinear algebras defined over S-rings.

We give examples of this situation.

Example 2.84: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \, a_i \in Z_{14}, 1 \le i \le 10, *, (3, 7) \right\}$$

be a Smarandache non associative column matrix semilinear algebra over the S-ring Z_{14}.

Example 2.85: Let

$$M = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \middle| \, x_i \in Z_{22}, *, (10, 9), 1 \le i \le 6 \right\}$$

be a Smarandache non associative column matrix semilinear algebra over the S-ring Z_{22}.

Example 2.86: Let

$$P = \left\{ \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{20} \end{bmatrix} \middle| \, t_i \in Z_{86}, 1 \le i \le 20, *, (2, 70) \right\}$$

be a Smarandache non associative column matrix semilinear algebra over the S-ring Z_{86}.

Example 2.87: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| a_i \in Z_{94}, *, (0, 8), 1 \leq i \leq 7 \right\}$$

be a Smarandache non associative column matrix semilinear algebra over the S-ring Z_{94}.

Example 2.88: Let

$$M = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| a, b, c \in Z_6, *, (2, 2) \right\}$$

be a Smarandache non associative column matrix semilinear algebra over the S-ring Z_6. M is a commutative structure.

Now we proceed onto give examples Smarandache non associative matrix semilinear algebras over a S-ring.

Example 2.89: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{25} & a_{26} & a_{27} \end{bmatrix} \middle| a_i \in Z_{15}, 1 \leq i \leq 27, *, (8, 1) \right\}$$

is a Smarandache non associative matrix semilinear algebra over the S-ring Z_{15}.

Example 2.90: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in Z_{21}, 1 \leq i \leq 12, *, (3, 3) \right\}$$

be a Smarandache non associative matrix semilinear algebra over the S-ring Z_{15}. V is commutative.

Example 2.91: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| a_i \in Z_{33}, 1 \leq i \leq 16, *, (0, 10) \right\}$$

be a Smarandache non associative matrix semilinear algebra over the S-ring Z_{33}.

Example 2.92: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in Z_{57}, 1 \leq i \leq 9, *, (2, 2) \right\}$$

be a Smarandache non associative matrix semilinear algebra over the S-ring Z_{57}.

Now we can define on similar line Smarandache non associative polynomial semilinear algebra and Smarandache non associative matrix polynomial coefficient semilinear algebra defined over the S-ring.

These are illustrated by some examples.

Example 2.93: Let

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{15}, *, (3, 2)\right\}$$

be a Smarandache non associative polynomial semilinear algebra over the S-ring Z_{15}.

If $p(x) = 7 + 8x^3 + 9x^2 + 10x^4$ and
$q(x) = 6 + 2x + 4x^2 + 10x^5$ are in M then
$p(x) * q(x) = (7 + 8x^3 + 9x^2 + 10x^4) * (7 + 8x^3 + 9x^2 + 10x^4)$

$= 7 * 6 + 8 * 6x^3 + 9 * 6x^2 + 10 * 6x^4 + 7 * 2x + 8 * 2x^4 + 9 * 2x^3 + 10 * 2x^5 + 7 * 4x^2 + 8 * 4x^5 + 8 * 4x^5 + 9 * 4x^4 + 10 * 4x^6 + 7 * 10x^5 + 8 * 10x^8 + 9 * 10x^7 + 10x^9.$

$= 3 + 6x^3 + 9x^2 + 12x^4 + 10x + 13x^4 + x^3 + 4x^5 + 14x^2 + 2x^5 + 5x^4 + 8x^6 + 11x^5 + 14x^8 + 17x^7 + 5x^9$

$= 3 + 10x + 8x^2 + 7x^3 + 2x^5 + 8x^6 + 17x^7 + 14x^8 + 5x^9 \in M.$

This is the way '*' operation is performed on M.

Example 2.94: Let

$$V = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{10}, *, (2, 2), 0 \leq i \leq \infty\right\}$$

be a Smarandache non associative semilinear algebra over the S-ring Z_{10}.

Clearly V is a commutative semilinear algebra.

Example 2.95: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{33}, *, (8, 0)\right\}$$

be a Smarandache non associative semilinear algebra over the S-ring Z_{33}. M is a non commutative structure.

Example 2.96: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; ai = (x_1, x_2, x_3, x_4);\, x_j \in Z_{15}, *, (5, 3), 1 \leq j \leq 4\right\}$$

be a Smarandache non associative semilinear algebra of row matrix coefficient polynomials over the S-ring Z_{15}.

Example 2.97: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (y_1, y_2, \ldots, y_{15});\, y_j \in Z_{10}, *, (2, 2), 1 \leq j \leq 15\right\}$$

be a Smarandache non associative semilinear algebra of matrix coefficient polynomials over the S-ring Z_{10}, P is a commutative structure.

Example 2.98: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (t_1, t_2, t_3);\, t_j \in Z_{38}, *, (0, 19), 1 \leq j \leq 3\right\}$$

be a Smarandache non associative semilinear algebra over Z_{38}.

Example 2.99: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{20} \end{bmatrix}; d_j \in Z_{34}, *, (3, 7), 1 \leq j \leq 20 \right\}$$

be a Smarandache non associative semilinear algebra of column matrix coefficient polynomial over a S-ring Z_{34}.

Example 2.100: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{10} \end{bmatrix}, m_j \in Z_6, *, (2, 2), 1 \leq j \leq 10 \right\}$$

be a Smarandache non associative column matrix coefficient polynomial semilinear algebra over S-ring Z_6.

Example 2.101: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}, d_j \in Z_{14}, *, (6, 7), 1 \leq j \leq 4 \right\}$$

be a Smarandache non associative column matrix coefficient polynomial semilinear algebra over S-ring Z_{14}.

Example 2.102: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & \ldots & d_{10} \\ d_{11} & d_{12} & \ldots & d_{20} \\ d_{21} & d_{22} & \ldots & d_{30} \end{bmatrix}, d_j \in Z_{15}, *, (3, 7), 1 \leq j \leq 30 \right\}$$

be a Smarandache non associative matrix coefficient polynomial semilinear algebra over S-ring Z_{15}.

Example 2.103: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & \ldots & d_{18} \\ d_{19} & d_{20} & \ldots & d_{36} \end{bmatrix}, d_j \in Z_6, *, (5, 5), 1 \leq j \leq 36 \right\}$$

be a Smarandache non associative matrix coefficient polynomial semilinear algebra over S-ring Z_6.

Example 2.104: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{bmatrix}, d_j \in Z_{21}, *, (3, 0), 1 \leq j \leq 9 \right\}$$

be a Smarandache non associative matrix coefficient polynomial semilinear algebra over S-ring Z_{21}.

Now we proceed onto define doubly Smarandache non associative semilinear algebra and give examples of them.

DEFINITION 2.6: *Let $M = \{a_i \mid a_i \in Z_n, *, (t, u)\}$ be a Smarandache groupoid. If M is a Smarandache non associative semilinear algebra over the S-ring Z_n then we define M to be a* *double Smarandache non associative semilinear algebra (DS - non associative semilinear algebra) over the S-ring Z_n.*

We give examples of them.

Example 2.105: Let V = {Z_{19}, *, (0, 5)} be a double Smarandahe non associative semilinear algebra over the S-ring Z_{10}.

Example 2.106: Let G = {Z_6, *, (4, 5)} be a S-non associative semilinear algebra over the S-ring Z_6. Clearly G is a double Smarandache non associative semilinear algebra over the S-ring Z_6.

Example 2.107: Let G = {Z_{12}, *, (4, 3)} be a DS non associative semilinear algebra over the S-ring Z_{12}.

Example 2.108: Let G = {Z_{14}, *, (7, 8)} be a DS non associative semilinear algebra over the S-ring Z_{14}.

Example 2.109: Let

$$P = \{(a_1, a_2, a_3, a_4) \mid a_i \in Z_{12}, *(3, 4), *, 1 \leq i \leq 4\}$$

be a DS non associative semilinear algebra over the S-ring Z_{12}.

Example 2.110: Let

$$R = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in Z_{14}, *, (7, 8), *, 1 \leq i \leq 10\}$$

be a DS non associative semilinear algebra over the S-ring Z_{14}.

Example 2.111: Let

$$V = \{(a_1, a_2, ,\ldots, a_{40}) \mid a_i \in Z_{46}, *, (1, 2), *, 1 \leq i \leq 40\}$$

be a DS non associative semilinear algebra over the S-ring Z_{46}.

THEOREM 2.8: *Let V = {$(a_1, \ldots, a_r) \mid a_i \in Z_n, 1 \leq i \leq r, (t, u), *$} be a Smarandache non associative semilinear algebra over the S-ring Z_n; V is a DS non associative semilinear algebra over the S-ring Z_n if $(t, u) = 1$ and $t \neq u$; $t + u \equiv 1 \pmod n$.*

THEOREM 2.9: *Let Z_n be a S-ring,*
$P = \{(a_1, \ldots, a_r) \mid a_i \in Z_n, *, t+u \equiv (\text{mod } n)\ (t, u)\}$
be a DS non associative semilinear algebra over the S-ring Z_n. P is a DS non associative strong Bol semilinear algebra if and only if $t^3 = t \pmod n$ and $u^2 = u \pmod n$.

Proof: We just recall a non assocative semilinear algebra P is strong Bol semilinear if for all x, y, z ∈ V

((x*y(*z)*x = x * ((y*z)*x).

Let $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r)$ and $z = (z_1, z_2, \ldots, z_r) \in P$.

$$
\begin{aligned}
&((x*y)*z) * x \\
&= [(tx + uy) * z] * x \\
&= (t^2x + tuy + uz) * x \\
&= t^3x + t^2uy + tuz + ux \\
&= (t^3x_1 + t^2uy_1 + tuz_1 + ux_1, t^3x_2 + t^2uy_2 + tuz_2 + ux_2, \ldots, t^3x_r \\
&\quad + t^2uy_r + tuz_r + ux_r) \qquad \text{I}
\end{aligned}
$$

Now consider x* ((y*z)*x)

$$
\begin{aligned}
&= x * (t^2y + tuz + ux) \\
&= tx + t_2uy + tu_2z + u_2x \\
&= (tx_1 + t^2uy_1 + tu^2z_1 + u^2x_1, tx_2 + t^2uy_2 + tu^2z_2 + u^2x_2, \ldots, tx_r \\
&\quad + t^2uy_r + tu^2z_r + u^2x_r) \qquad \text{II}
\end{aligned}
$$

I = II if and only if $t^3 = t \pmod n$ and $u^2 = u \pmod n$; hence the claim.

THEOREM 2.10: *Let Z_n be a S-ring.*
$V = \{(a_1, a_2, \ldots, a_r) \mid a_i \in Z_n;\ 1 \le i \le r, (t, u);\ \ t + u \equiv 1 \pmod n\}$ is a DS strong non associative Moufang semilinear algebra if and only if $t^2 = t \pmod n$ and $u^2 = u \pmod n$.

Proof: We say a DS non associative semilinear algebra is strong Moufang if

(x*y) * (z*x) = (x*(y*z))*x for all x, y ∈ V.

Let $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r)$ and $z = (z_1, z_2, \ldots, z_r) \in V$.

Consider $(x*y) * (z*x)$

$$= [(x_1, x_2, \ldots, x_r) * (y_1, y_2, \ldots, y_r)] * [(z_1, z_2, \ldots, z_r) * (x_1, x_2, \ldots, x_r)]$$

$$= (tx_1 + uy_1, tx_2 + uy_2, \ldots, tx_r + uy_r) * (tz_1 + ux_1, tz_2 + ux_2, \ldots, tz_r + ux_r)$$

$$= (t^2x_1 + uty_1 + tuz_1 + u^2x_1, t^2x_2 + uty_2 + utz_1 + u^2x_2, \ldots, t^2x_r + tuy_r + utz_r + u^2xr) \qquad \text{I}$$

Consider $(x*(y*z)) * x$

$$= (x* ((y_1, y_2, \ldots, y_r) * (z_1, z_2, \ldots, z_r))) * x$$

$$= (x * (ty_1 + uz_1, ty_2 + z_2u, \ldots, ty_r + uz_r)) * x$$

$$= ((x_1, x_2, \ldots, x_r) * (ty_1 + uz_1, ty_2 + uz_2, \ldots, ty_r + uz_r)) * x$$

$$= (tx_1 + uty_1 + u^2z_1, tx_2 + uty_2 + u^2z_2, \ldots, tx_r + tuy_r + u^2z_r) * (x_1, x_2, \ldots, x_r)$$
$$= (t^2x_1 + ut^2y_1 + u^2tz_1 + ux_1, t^2x_2 + ut^2y_2 + tu^2z_2 + ux_2, \ldots, t^2x_r + ut^2y_r + t^2uz_r + u^2x_r) \qquad \text{II}$$

I and II are equal if and only if $t^2 \equiv t \pmod n$. Thus V is a DS strong non associative strong Moufang semilinear algebra if and only if $t^2 \equiv t \pmod n$ and $u^2 \equiv u \pmod n$.

THEOREM 2.11: *Let Z_n be a S-ring $V = \{(a_1, a_2, \ldots, a_r) \mid a_i \in Z_n, 1 \leq i \leq r, (t, u), t + u \equiv 1 \pmod n, *\}$ be a DS non associative semilinear algebra over the S-ring Z_n. V is an alternative DS non associative semilinear algebra if and only if $t^2 \equiv t \pmod n$ and $u^2 \equiv u \pmod n$.*

Proof: Recall a DS non associative semilinear algebra V is alternative if (x*y)*y = x* (y*y) and (x*y) * y = x * (x*y) for all x, y $\in$ V.

Let $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r) \in V$.

$(x * y) * y = [(x_1, x_2, \ldots, x_r) * (y_1, y_2, \ldots, y_r)] * y$

$= (x_1 * y_1, x_2 * y_2, \ldots, x_r * y_r) * (y_1, y_2, \ldots, y_r)$
$= (tx_1 + uy_1, tx_2 + uy_2, \ldots, tx_r + uy_r) * (y_1, y_2, \ldots, y_r)$
$= t^2x_1 + tuy_1 + uy_1, t^2x_2 + tuy_2 + uy_2, \ldots, t^2x_r + tuy_r + uy_r)$ I

Consider $x * (y * y) =$

$x ((y_1, y_2, \ldots, y_r) * (y_1, y_2, \ldots, y_r))$
$= x * (ty_1 + uy_1, ty_2 + uy_2, \ldots, ty_r + uy_r)$
$= (x_1, x_2, \ldots, x_r) (ty_1 + uy_1, ty_2 + uy_2, \ldots, ty_r + uy_r)$
$= (tx_1 + uty_1 + u^2y_1, tx_2 + tuy_2 + u^2y_2, \ldots, tx_r + tuy_r + u^2y_r)$ II

I and II are equal if and only if $u^2 = u \pmod n$ and $t^2 = t \pmod n$.

Consider $(x * x) * y = [(x_1, x_2, \ldots, x_r) * (x_1, x_2, \ldots, x_r)] * (y_1, y_2, \ldots, y_r)$

$= (tx_1 + ux_1, tx_2 + ux_2, \ldots, tx_r + ux_r) * (y_1, y_2, \ldots, y_r)$
$t^2x_1 + tux_1 + uy_1, t^2x_2 + tux_2 + uy_2, \ldots, t^2x_r + tux_r + uy_r)$ (a)

Now $x * (x *y) = x * ((x_1, x_2, \ldots, x_r) * (y_1, y_2, \ldots, y_r)$

$= x * (tx_1 + uy_1, tx_2 + uy_2, \ldots, tx_r + uy_r)$
$= (tx_1 + tux_1 + u^2 y_1, tx_2 + tux_2 + u^2y_2, \ldots, tx_r + tux_r + u^2 y_r)$ (b)

a and b are equal if and only if $t^2 = t \pmod n$ and $u^2 = u \pmod n$. Thus V is a alternative DS non associative semilinear algebra if and only if $t^2 = t \pmod n$ and $u^2 = u \pmod n$.

THEOREM 2.12: *Let Z_n be a S-ring*

$$V = \{(a_1, a_2, \ldots, a_r) \mid a_i \in Z_n, 1 \leq i \leq r; (t, u), *\}$$

be a DS non associative semilinear algebra over the S-ring Z_n. If $t+u \equiv 1 \pmod n$ then V is an idempotent DS non associative semilinear algebra over the S-ring Z_n.

Proof: To prove V is an idempotent DS non associative semilinear algebra we have to show if $t + u \equiv 1 \pmod n$ then for every $x \in V$, $x * x = x$.

Consider $x = (x_1, x_2, \ldots, x_r) \in V$,

$x * x = (x_1, x_2, \ldots, x_r) * (x_1, x_2, \ldots, x_r)$

$= (tx_1 + ux_1, \ tx_2 + ux_2, \ldots, tx_r + ux_r)$

$= ((t+u)x_1, (t+u)x_2, \ldots, (t+u)x_r)$

(if $(t+u) \equiv 1 \bmod n$)

$= (x_1, x_2, \ldots, x_r)$. Hence the claim.

THEOREM 2.13: *Let Z_n be a S-ring.*

$V = \{(x_1, x_2, \ldots, x_r) \mid x_i \in Z_n, 1 \leq i \leq r, (t, u), t+u \equiv 1 \pmod n, *\}$ be a DS non associative semilinear algebra over the S-ring Z_n. V is a P-DS non associative semilinear algebra if and only if $t^2 \equiv t \pmod n$ and $u^2 \equiv u \pmod n$.

Proof: To show V is a P-DS non associative semilinear algebra over Z_n it is enough if we show $(x * y) * y = x * (y*y)$ for all $x, y \in V$.

Consider $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r) \in V$.

$(x * y) * y = ((x_1, x_2, \ldots, x_r) * (y_1, y_2, \ldots, y_r)) * (y_1, y_2, \ldots, y_r)$

$= (tx_1 + uy_1, tx_2 + uy_2, \ldots, tx_r + uy_r) * (y_1, y_2, \ldots, y_r)$

$= (t^2x_1 + tuy_1 + uy_1, t^2x_2 + tuy_2 + uy_2, \ldots, t^2x_r + tuy_r + uy_r)$ I

Consider $x * (y * y) =$

$x* (y_1, y_2, \ldots, y_r) * (y_1, y_2, \ldots, y_r))$

$= x * (ty_1 + uy_1, ty_2 + uy_2, \ldots, ty_r + uy_r)$
$= (x_1, x_2, \ldots, x_r) * (ty_1 + uy_1, ty_2 + uy_2, \ldots, ty_r + uy_r)$
$= (tx_1 + tuy_1 + u^2y_1, tx_2 + tuy_2 + u^2y_2, \ldots, tx_r + tuy_r + u^2y_r)$
II

I and II are equal if and only if $t^2 = t \pmod n$ and $u^2 \equiv u \pmod n$. Thus V is a DS non associative semilinear algebra is a P-semilinear algebra if and only if $t^2 \equiv t$ and $u^2 \equiv u \pmod n$.

THEOREM 2.14: *Let Z_n be a S-ring.*

$$V = \{(x_1, x_2, \ldots, x_r) \mid x_i \in Z_n, 1 \leq i \leq r, *, (m, m)\}$$

be a Smarandache non associative semilinear algebra over the S-ring Z_n.

If $m + m \equiv 1 \pmod n$ and $m^2 = m \pmod n$ then

(i) V is a DS Strong idempotent non associative semilinear algebra.
(ii) V is a DS strong P-non associative semilinear algebra.
(iii) V is a DS strong non associative semilinear algebra.
(iv) V is a DS strong Moufang non associative semilinear algebra.
(v) V is a DS strong non associative alternative non linear algebra.

The proof is direct and hence left as an exercise to the reader.

Now it is pertinent to mention here that if V is replaced by a column matrix that is

$$V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \right| a_i \in Z_n, Z_n \text{ a S-ring}, *, (t, u), 1 \leq i \leq m \right\} \text{ or}$$

if $\left\{\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}\right|$ $a_{ij} \in Z_n$, $1 \leq i \leq m$, $1 \leq j \leq n$, Z_n a S-ring, *,

(t, u)}

or if V = $\left\{\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}\right|$ $a_{ij} \in Z_n$, Z_n a S ring, $1 \leq i, j \leq n$, *,

(t, u)}

then all the above theorems 2.9 to 2.16 hold good without any difficulty.

We will give some examples of the situations described in the theorems.

Example 2.112: Let

$$M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}\right| a_i \in Z_6, (3, 4), *\}$$

be a Double Smarandache non associative semilinear algebra over the S-ring Z_6. M is a strong DS non associative Bol semilinear algebra over the S-ring Z_6.

Example 2.113: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in Z_6, (10, 6), *, 1 \le i \le 9 \right\}$$

be a SD non associative semilinear algebra over the S-ring Z_{15}.

(i) M is a DS strong P-non associative semilinear algebra.
(ii) M is a DS strong moufang non associative semilinear algebra.
(iii) m is a DS strong Bol non associative semilinear algebra.
(iv) M is a DS strong alternative non associative semilinear algebra
(v) M is a DS idempotent non associative semilinear algebra.

Example 2.114: Let

$$P = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \right| a_i \in Z_{12}, (4, 9), *, 1 \le i \le 30 \right\}$$

be a DS non associative semilinear algebra over the S-ring Z_{12}.
(i) P is a DS strong P-non associative semilinear algebra.
(ii) P is a DS-strong Bol non associative semilinear algebra.
(iii) P is a DS strong Moufang non associative semilinear algebra.
(iv) P is a DS strong alternative non associative semilinear algebra and
(v) P is DS strong idempotent non associative semilinear algebra.

Example 2.115: Let

M = {(a_1, a_2, a_3) where $a_i \in Z_8$, $1 \leq i \leq 3$, *, (3, 6)} be a DS-strong Bol (or Moufang or alternative or idempotent or P) non associative semilinear algebra over Z_8. Z_8 is not a S-ring.

Example 2.116: Let

$$V = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{array} \right] \middle| \; a_i \in Z_{10}, (6, 5), * \right\}$$

be a DS strong Bol non associative semilinear algebra, DS strong Moufang non associative semilinear algebra, DS strong idempotent non associative semilinear algebra, DS strong alternative non associative semilinear algebra and DS strong P non associative semilinear algebra.

Next we proceed onto define structure similar to DS non associative semilinear algebras as well as Smarandache non associative semilinear algebras.

For this we use the concept of 'groupoid ring' by varying the ring we can build such structure.

DEFINITION 2.7: *Let G be a groupoid. F a field. FG be the groupoid ring. FG is defined as the strong non associative semilinear algebra over F.*

(For definition and properties of groupoid rings please refer [33]).

We now give examples of them.

Example 2.117: Let G = (Z, *, (3, 7)) be a groupoid and R be a field. RG is the groupoid ring which is a strong non associative semilinear algebra over R.

For

$$RG = \left\{ \sum_{i=0}^{\infty} a_i g^i \,\middle|\, g_i \in Z, *, a_i \in R, (3, 7) \right\}$$

is a strong non associative semilinear algebra over the field R. Thus we can create also a different way strong non associative semilinear algebras.

Example 2.118: Let G ={Z_8, *, (7, 2)} be a groupoid. Q be the field of rationals, QG be the groupoid ring QG is a strong non associative semilinear algebra over Q.

$$QG = \left\{ \sum_{i=0}^{7} a_i g^i \,\middle|\, g_i \in Z_8 = \{0 = g_0, g_1 = 1, g_2 = 2, \ldots, g_7 = 7\}; \right.$$

$0 \leq i \leq 7$, *, (7, 2); $a_i \in Q$}. It is easily verifield; QG is also a strong non associative semilinear algebra over Q.

Example 2.119: Let G = {Z_{10}, *, (9, 2)} be a groupoid. Z_2 = {0, 1} be the field of characteristic two.

$$Z_2G = \left\{ \sum_{i=0}^{9} a_i g^i \,\middle|\, g_i \in Z_{10} = \{g_0 = 0, g_1 = 1, \ldots, g_9 = 9\}; \right.$$

$$\left. 0 \leq i \leq 9, a_i \in Z_2, *, (9, 2) \right\}$$

is a groupoid ring as well as a strong non associative semilinear algebra over Z_2.

Example 2.120: Let G = {$Z^+ \cup \{0\}$, *, (10, 9)} be a groupoid. F = Z_3 = {0, 1, 2} be the field of characteristic three. FG be the groupoid ring.

$$FG = \left\{ \sum_{i=0}^{\infty} a_i g^i \,\middle|\, a_i \in Z_3 = \{0, 1, 2\}, *, g_i \in Z = \{0, g_0, 1 = g_1, \right.$$

$$\left. 2 = g_2, \ldots, n = g_n, \ldots\} \right.$$

$g_\infty * g_i = g_\infty = g_i * g_\infty$ is the defined '*' operation on $\infty = g_\infty$.

Now FG can also be realized as a strong non associative semilinear algebra over the field $F = Z_3$.

We will define Smarandache strong non associative semilinear algebra if the field is replaced by a S-ring in these groupoid ring. We give examples of them.

Example 2.121: Let $M = \{Z_9, *, (7, 2)\}$ be a groupoid. $F = Z_6$ be the S-ring.

FM be the groupoid S-ring $FM = \left\{ \sum_{i=0}^{8} a_i g^i \,\middle|\, a_i \in Z_6, g_i \in Z_9 \right.$ $= \{0 = g_0, 1 = g_1, 2 = g_2, \ldots, g_8 = 8), *, (7, 2)\}$. Clearly FM is a Smarandache non associative semilinear algebra over the S-ring Z_6.

It is pertinent to mention here that we can have different modulo integers Z_n that can be used for the groupoid and rings which is clear from Example 2.121.

Example 2.122: Let $G = \{Z_{12}, *, (10, 3)\}$ be the groupoid. $F = Z_{10}$ be the S-ring. FG be the groupoid ring.

$$FG = \left\{ \sum_{i=0}^{11} a_i g^i \,\middle|\, a_i \in Z_{10}, g_i \in Z_{12} = \{0 = g_0, 1 = g_1, 2 = g_2, \right.$$

$$\left. \ldots, g_{11} = 11\}, *, (10, 3)\} \right.$$

Clearly FG is a Smarandache non associative semilinear algebra over the S-ring Z_{10}.

THEOREM 2.15: *Let $G = \{Z_n, *, (t, u)\}$ be a groupoid and Z_m be a S-ring. Let Z_mG be the groupoid ring. Z_mG the Smarandache non associative semilinear algebra is a strong Moufang Smarandache non associative semilinear algebra if and only if $t + u \equiv (\text{mod } n)$ and $t^2 = t \ (\text{mod } n)$ and $u^2 = u \ (\text{mod } n)$.*

Proof: Straight forward and hence left as an exercise to the reader.

THEOREM 2.16: *Let $G = \{Z_n, *, (t, u)\}$ be a groupoid. Z_m be a S-ring. Z_mG be the groupoid S-ring. Z_mG is a Smarandache strong Bol non associative semilinear algebra if and only if $t + u \equiv 1 \ (\text{mod } n)$, $t^3 = t \ (\text{mod } n)$ and $u^2 \equiv u \ (\text{mod } n)$.*

Proof is also direct as in case of groupoids.

THEOREM 2.17: *Let $G = \{Z_n, *, (t, u)\}$ be a groupoid. Z_m be a S-ring; Z_mG be the groupoid S-ring. Z_mG is a S-non associative idempotent semilinear algebra if $t + u \equiv 1 \ (\text{mod } n)$.*

THEOREM 2.18: *Let $G = \{Z_n, *, (t, u), t + u \equiv 1 \ (\text{mod } n)\}$ be a groupoid. Z_m be a S-ring. The groupoid ring Z_mG is a S-non associative P-semilinear algebra if and only if $t^2 \equiv t \ (\text{mod } n)$ and $u^2 \equiv u \ (\text{mod } n)$.*

THEOREM 2.19: *Let $G = \{Z_n, *, (t, u), t+u \equiv 1 \ (\text{mod } n)\}$ be a groupoid. Z_m be a S-ring. The groupoid ring Z_mG is a Smarandache alternative non associative semilinear algebra if and only if $t^2 \equiv t \ (\text{mod } n)$ and $u^2 \equiv u \ (\text{mod } n)$.*

Now if in these theorems the groupoid G is such that it is a row matrix with elements from $\{Z_n, *, (t, u) = 1, t+u \equiv 1 \text{ mod } n\}$ and '*' is the operation on G or a column matrix with elements from Z_n and '*' as its binary operation or a rectangular matrix with elements from Z_n and * operation on G or a square matrix with entries from Z_n and * operation G then also for a S-ring Z_m the groupoid ring Z_mG satisfies all the results of the theorem where Z_m G is also a S-non associative semilinear algebra over the S-ring Z_m.

We will only illustrate these situations by some examples.

Example 2.123: Let

$V = \{(a_1, a_2, a_3) \mid a_i \in Z_n, (t, u); t+u \equiv 1 \pmod{n}; 1 \leq i \leq 3\}$

be a groupoid. R be the field of reals. RV be the groupoid ring of the groupoid V over R. RV is a strong non associative semilinear algebra over R. RV is a P-strong non associative semilinear algebra if and only if $t^2 \equiv t \pmod{n}$ and $u^2 \equiv u \pmod{n}$. So RV is a strong P-non associative semilinear algebra.

Example 2.124: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \right| a_i \in Z_n, (t, u), t+u \equiv 1 \pmod{n}\ 1 \leq i \leq 7 \right\}$$

be a groupoid. Q the field, QV be the groupoid ring of the groupoid V over the field Q. QV is a Bol strong non associative semilinear algebra over Q if and only if $t^3 \equiv t \pmod{n}$ and $u^2 \equiv u \pmod{n}$.

Example 2.125: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \right| a_i \in Z_{22}, 1 \leq i \leq 12, *, (12, 11) \right\}$$

be a groupoid. Q be a field. QM be the groupoid ring. QM is a strong non associative semilinear algebra which is Bol, Moufang and alternative.

Example 2.126: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right| a_i \in Z_{15}, 1 \leq i \leq 19, *, (6, 10)\}$$

be a groupoid. Q be the field. QP be the groupoid ring. QP is a strong non associative semilinear algebra which is Bol, P, Moufang and alternative and idempotent.

Example 2.127: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right| a_i \in Z_{14}, (7, 8), *, 1 \leq i \leq 18\}$$

be a groupoid. R be the field. RM the groupoid ring. RM is a strong non associative semilinear algebra which is Bol, Moufang, P alternative and idempotent. Suppose G is a P-groupoid when F = R or Q or Z_n then the groupoid ring; RG or QG or the S-groupoid ring Z_nG is a strong non associative P-semilinear algebra over R or Q or a Smarandache strong non associative P-semilinear algebra over the S-ring Z_n.

The same type of result happens to be true if G is a Bol groupoid or a Moufang groupoid or an idempotent groupoid or an alternative groupoid. Inview of this we have the following theorems.

THEOREM 2.20: *Let*
$G = \{Z_n, *, (t, u), t+u \equiv 1 \pmod n \; t^2 \equiv t \pmod n, u^2 \equiv u \pmod n\}$ *be a P-groupoid (or Bol groupoid or Moufang groupoid or alternative groupoid), F = R (or Q) be the field. FG be the groupoid ring. FG the strong non associative semilinear algebra is a P-semilinear algebra (or Bol semilinear algebra or Moufang semilinear algebra or alternative semilinear algebra) over the field F = R (or Q).*

Proof is straight forward and hence left as an exercise to the reader.

Now having seen properties of non associative semilinear algebra we can as a matter of routine study their substructures and other related properties.

Chapter Three

NON ASSOCIATIVE LINEAR ALGEBRAS

In this chapter we define a new class of non associative linear algebras using loops. We give several interesting properties associated with them. Infact these are generalized to non associative quasi linear algebras also. We now proceed onto define them and give examples of them.

DEFINITION 3.1: *Let $L_n(m)$ be a loop n a prime, $n > 3$ be a loop. Suppose $F = Z_n$ be the field (n a prime) then we see $L_n(m)$ is a linear algebra over Z_n called or defined as the non associative linear algebra over Z_n if the following conditions are satisfied.*

(i) *For every $g \in L_n(m)$ and $\alpha \in Z_n$ $g\,\alpha = \alpha\,.\,g \in L_n(m)$*
(ii) *$g = 1.g \in L_n(m)$ for all $g \in L_n(m)$*
(iii) *$0.g = 0 \in L_n(m)$ $(0 = g_n)$*
(iv) *$\alpha(g + h) = \alpha g + \alpha h$,*

for all $\alpha \in Z_n$, and $g, h \in L_n(m)$.

For more about loop $L_n(m)$ please refer [37].

We will first illustrate this situation by some examples.

Example 3.1: Let V = {e, g_1, g_2, …, g_7} given by the following table.

*	e	g_1	g_2	g_3	g_4	g_5	g_6	g_7
e	e	g_1	g_2	g_3	g_4	g_5	g_6	g_7
g_1	g_1	e	g_5	g_2	g_6	g_3	g_7	g_4
g_2	g_2	g_5	e	g_6	g_3	g_7	g_4	g_1
g_3	g_3	g_2	g_6	e	g_7	g_4	g_1	g_5
g_4	g_4	g_6	g_3	g_7	e	g_1	g_5	g_2
g_5	g_5	g_3	g_7	g_4	g_1	e	g_2	g_6
g_6	g_6	g_7	g_4	g_1	g_5	g_2	e	g_3
g_7	g_7	g_4	g_1	g_5	g_2	g_6	g_3	e

F = Z_7 be the field. For any x ∈ V and a ∈ F we have the following product.

$$ax = ag_i = g_{ai \text{ (mod 7)}}$$

(if e ∈ V, ae = e for all a ∈ F).

For instance if a = 0 x = g_3 a.g_3 = $g_{0.3 \text{ (mod 7)}}$.

If a = 1 and x = g_3 then a.x = 1g_3 = $g_{3.1}$ = $g_{3 \text{ (mod 7)}}$ = g_3.
If a = 2 and x = g_3 then a.x = 2g_3 = $g_{2.3}$ = $g_{6 \text{ (mod 7)}}$ = g_6.
If a = 3 and x = g_3 then a.x = 3.g_3 = $g_{3.3 \text{ (mod 7)}}$ = g_2.
If a = 4 and x = g_3 then a.x = 4g_5 = $g_{4.3 \text{ (mod 7)}}$ = g_5.
If a = 5 and x = g_3 then a.x = 5g_3 = $g_{5.3 \text{ (mod 7)}}$ = g_1.
If a = 6 and x = g_3 then a.x = 6.g_3 = $g_{6.3 \text{(mod 7)}}$ = $g_{4 \text{ (mod 7)}}$ = g_4.

Thus we see V is a non associative linear algebra over the field Z_7.

Clearly number of elements in V is eight.

Example 3.2: Let V = {e, g_1, g_2, g_3, g_4, g_5} be a given by the following table.

*	e	g_1	g_2	g_3	g_4	g_5
e	e	g_1	g_2	g_3	g_4	g_5
g_1	g_1	e	g_3	g_5	g_2	g_4
g_2	g_2	g_5	e	g_4	g_1	g_3
g_3	g_3	g_4	g_1	e	g_5	g_2
g_4	g_4	g_3	g_5	g_2	e	g_1
g_5	g_5	g_2	g_4	g_1	g_3	e

V is a loop of order 6. Consider F = Z_5. V is a non associative linear algebra over the field Z_5.

Example 3.3: Let V = $L_{19}(4)$ be a loop; V is a non associative linear algebra over the field F = Z_{19}.

We can define dimension of V over Z_p as the number of linearly independent elements needed to generate V.

We see in all the three examples {e, g_i} serves as the set which can generate V over the respective fields.

Example 3.4: Let M = $L_{11}(3)$ be a loop. M is a non associative linear algebra over the field Z_{11} = F. Take {e, g_1} $\subseteq$ M generates M over Z_{11} = F. Infact {e, g_i} $\subseteq$ M (i $\neq$ 11) generates M over Z_{11}. Thus M has 10 and only 10 subsets which are distinct can generate M. Further dimension of M over Z_{11} is two.

Inview of this we have the following theorem.

THEOREM 3.1: *Let V = $L_p(m)$ ($m \neq p$ or $m \neq 1$ or $1 < m < p$) be a loop, p a prime. F = Z_p be the field FV is the non associative linear algebra of dimension two. FV has (p–1) distinct subsets of the form {e, g_i}; i = 1, 2, …, p–1, i $\neq g_p$. Each of the subsets {e, g_i} can generate V over F.*

Proof is direct and hence is left as an exercise to the reader. It is important and interesting to note that by varying m we can get several non associative linear algebras of same cardinality. We will first describe in case of Z_5. We see we have 3 distinct non associative linear algebras of order six given by $L_5(2)$, $L_5(3)$ and $L_5(4)$ defined over the field Z_5.

We say a non associative linear algebra $L_p(m)$ is commutative if $L_p(m)$ is a commutative loop.

We shall denote the collection of all non associative linear algebras of same order by $L_p = \{L_p(m) \mid 1 < m < p\}$. Thus we can say for a given $L_p(m) \in L_p$. $L_p(m)$ is a non associative linear algebra over $Z_p = F$.

It is pertinent to mention here that as in case of usual linear algebras we cannot say that non associative linear algebras of same dimension defined over the same field are isomorphic. This is illustrated by the following examples.

Example 3.5: Let $M = L_7(3)$ be a loop of order 8. $F = L_7$ be the field of characteristic seven. M is a non associative linear algebra over Z_7. Clearly dimension of M over Z_7 is two.

Example 3.6: Let $V = L_7(4)$ be a loop of order 8. $F = Z_7$ be the field of characteristic seven. V is a non associative linear algebra over Z_7 of dimension two.

It is interesting to note both M and V are non associative linear algebras over the same field Z_7 and the same dimension two but however M and V are not isomorphic. Thus even if two non associative linear algebras are defined over the same field and of same dimension they are not isomorphic.

Inview of this we have the following theorem.

THEOREM 3.2: *Let $V = L_p(m)$ and $W = L_p(p+1/2)$ be two non associative linear algebras of dimension two defined over the field Z_p. Clearly $V \not\cong W$.*

Proof: Clearly both V and W are of dimension two over Z_p. However we see W is a commutative linear algebra over Z_p where as V is a non commutative linear algebra over Z_p. So V is not isomorphic with W. Hence the result.

Now we can define substructure in them provided, $H \subseteq L_n(m)$ and H itself is a proper loop such that H is a non associative linear algebra over Z_n (n prime) then H is a non associative linear subalgebra of $L_n(m)$ over Z_n. If $L_n(m)$ has no proper non associative linear subalgebras we define $L_n(m)$ to be simple.

Example 3.7: Let $V = L_{23}(12)$ be a non associative linear algebra over the field $F = Z_{23}$. V has no proper non associative linear subalgebras as the loop $L_{23}(12)$ has no subloops.

Example 3.8: Let $M = L_{29}(20)$ be a non associative linear algebra over the field $F = Z_{29}$. V has no proper non associative linear subalgebras, that is V is simple.

Inview of this we have the following theorem.

THEOREM 3.3: *Let $V = L_n(m)$ (n a prime) be a non associative linear algebra over $Z_n = F$ (n a prime). Clearly V is simple.*

Proof: Follows from the fact V is a loop such that V has no subloops as n is a prime hence the claim.

Now as in case of linear algebras we can define in case of non associative linear algebras the notion of linear operator and linear functionals.

If V and W are non associative linear algebras defined over the same field $F = Z_p$ we can define $T : V \to W$ to be a linear transformation of V to W.

Interested reader can study these structures.

We see $V_n = L_n = \{L_n (m) \mid n$ a prime, $1 < m < n\}$ contains one and only one non associative linear algebra which is commutative.

THEOREM 3.4: *Let $L_n(m) \in L_n$ be a non associative linear algebra over the field Z_n (n a prime). L_n has one and only one commutative, non associative linear algebra when $m = (n+1)/2$.*

Now we can define some more non associative linear algebras using $L_n(m)$; n a prime. We illustrate them by examples as the definition and formation of them are simple.

Example 3.9: Let

$$V = \{(a_1, a_2, \ldots, a_7, a_8) \mid a_i \in L_{11}(7); 1 \leq i \leq 8, *\}$$

be a non associative matrix linear algebra over the field Z_{11}.

Example 3.10: Let $V = \{(a_1, a_2, a_3) \mid a_i \in L_{29} (2); 1 \leq i \leq 3, *\}$ be a non associative row matrix linear algebra over the field Z_{29}. Take $x = (g_3, g_{10}, g_0)$ and $y = (g_7, g_2, g_{11})$ in V.

$$\begin{aligned} \text{Now } x*y &= (g_3 * g_7, g_{10} * g_2, g_0 * g_{11}) \\ &= (g_{14-3 \pmod{29}}, g_{4-10 \pmod{29}}, g_{22-0 \pmod{29}}) \\ &= (g_{11}, g_{23}, g_{22}) \in V. \end{aligned}$$

$$\begin{aligned} \text{Take } 7x &= (g_{7.3 \pmod{29}}, g_{7.10 \pmod{29}}, g_{8.0 \pmod{29}}) \\ &= (g_{21}, g_{12}, g_0) \in V. \end{aligned}$$

This is the way operations are made on V.

Example 3.11: Let

$V = \{(a_1, a_2, \ldots, a_8) \mid a_i \in Z_7 (3); 1 \leq i \leq 8, *\}$

be a non associative row matrix linear algebra over the field Z_7.

$\{(e, e, \ldots, e), (e, g_1, \ldots, e), (g_1, e, \ldots, e), (e, e, g_1, e, \ldots, e), \ldots, (e, e, \ldots, g_1)\}$ is a basis of V over Z_7.

Example 3.12: Let

$$V = \{(a_1, a_2, a_3, a_4) \mid a_i \in L_5(3), *, (1 \leq i \leq 4)\}$$

be a non associative row matrix linear algebra over the field F = Z_5.

Now we proceed onto give examples of column matrix non associative linear algebras over a field Z_p as the definition of this concept is a matter of routine.

Example 3.13: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \right| a_i \in Z_{43}(3);\ 1 \leq i \leq 7 \right\}$$

be a non associative column matrix linear algebra over the field Z_{43}.

Example 3.14: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right| a_i \in L_5(3);\ 1 \leq i \leq 4 \right\}$$

be a column matrix non associative linear algebra over the field Z_5.

$$\text{Consider } x = \begin{bmatrix} g_1 \\ g_3 \\ g_4 \\ g_0 \end{bmatrix} \text{ and } y = \begin{bmatrix} g_2 \\ g_2 \\ g_3 \\ g_1 \end{bmatrix} \text{ in V.}$$

$$\text{Now } x*y = \begin{bmatrix} g_1 * g_2 \\ g_3 * g_2 \\ g_4 * g_3 \\ g_0 * g_1 \end{bmatrix}$$

$$= \begin{bmatrix} g_{6-2(\text{mod } 5)} \\ g_{6-6(\text{mod } 5)} \\ g_{9-8(\text{mod } 5)} \\ g_{3-0(\text{mod } 5)} \end{bmatrix} = \begin{bmatrix} g_4 \\ g_0 \\ g_1 \\ g_3 \end{bmatrix}.$$

$$\text{Now } y * x = \begin{bmatrix} g_2 * g_1 \\ g_2 * g_3 \\ g_3 * g_4 \\ g_1 * g_0 \end{bmatrix} = \begin{bmatrix} g_{3-4(\text{mod } 5)} \\ g_{9-4(\text{mod } 5)} \\ g_{12-6(\text{mod } 5)} \\ g_{0-2(\text{mod } 5)} \end{bmatrix} = \begin{bmatrix} g_4 \\ g_0 \\ g_1 \\ g_3 \end{bmatrix}.$$

Clearly x*y = y*x.

Thus V is a non associative commutative column matrix linear algebra over the field Z_5.

Take $4 \in Z_5$.

$$4x = \begin{bmatrix} g_{4.1(\text{mod } 5)} \\ g_{4.3(\text{mod } 5)} \\ g_{4.4(\text{mod } 5)} \\ g_{4.0(\text{mod } 5)} \end{bmatrix} = \begin{bmatrix} g_4 \\ g_2 \\ g_1 \\ g_0 \end{bmatrix} \in V.$$

V is a commutative non associative linear algebra of finite order.

Example 3.15: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle|\ a_i \in L_7(3);\ 1 \le i \le 3, * \right\}$$

be a non associative linear algebra over the field Z_7.

Consider $B = \left\{ \begin{bmatrix} e \\ e \\ e \end{bmatrix}, \begin{bmatrix} g_1 \\ e \\ e \end{bmatrix}, \begin{bmatrix} e \\ g_1 \\ e \end{bmatrix}, \begin{bmatrix} e \\ e \\ g_1 \end{bmatrix} \middle|\ g_1 \in L_7(3) \right\}$ is a basis of B over Z_7.

THEOREM 3.5: *Let $V = \{(a_1, \ldots, a_q) \mid a_i \in L_n(m);\ 1 \le i \le q, *\}$ be a non associative row matrix linear algebra over the field Z_n (n a prime). V is a commutative non associative row matrix linear algebra if and only if $m = (n+1)/2$.*

Proof is straight forward for more refer [38].

THEOREM 3.6: *Let*

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix} \middle|\ a_i \in L_n(m);\ 1 \le i \le t, * \right\}$$

be a non associative column matrix linear algebra over the field Z_n (n a prime). V is a commutative non associative column matrix linear algebra if and only if $m = (n+1)/2$.

This proof is left as an exercise to the reader.

Example 3.16: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \, a_i \in L_{13}(m);\ 1 \le i \le 12 \right\}$$

be a non associative matrix linear algebra over the field Z_{13}.

Example 3.17: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \, a_i \in L_{53}(m);\ 1 \le i \le 4, * \right\}$$

be a non associative matrix linear algebra over the field Z_{53}.

Example 3.18: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \, a_i \in L_{13}(4);\ 1 \le i \le 15, * \right\}$$

be a non associative matrix linear algebra over the field Z_{13}.

$$\text{Let } x = \begin{bmatrix} g_0 & g_1 & g_2 & g_3 \\ g_7 & g_0 & g_1 & g_2 \\ g_6 & g_1 & g_0 & g_3 \\ g_0 & g_4 & g_1 & g_0 \\ g_1 & g_0 & g_5 & g_1 \end{bmatrix} \text{ and } y = \begin{bmatrix} g_1 & g_2 & g_1 & g_2 \\ g_0 & g_1 & g_0 & g_1 \\ g_2 & g_0 & g_2 & g_0 \\ g_4 & g_1 & g_1 & g_4 \\ g_1 & g_4 & g_4 & g_1 \end{bmatrix}$$

be in W. To find x*y;

$$x*y = \begin{bmatrix} g_{4-0(\text{mod}13)} & g_{8-3(\text{mod}13)} & g_{4-6(\text{mod}13)} & g_{8-9(\text{mod}13)} \\ g_{0-21(\text{mod}13)} & g_{4-0(\text{mod}13)} & g_{0-3(\text{mod}13)} & g_{4-6(\text{mod}13)} \\ g_{8-18(\text{mod}13)} & g_{0-3(\text{mod}13)} & g_{8-0(\text{mod}13)} & g_{0-9(\text{mod}13)} \\ g_{16-0(\text{mod}13)} & g_{4-12(\text{mod}13)} & g_{4-3(\text{mod}13)} & g_{16-0(\text{mod}13)} \\ g_{4-3(\text{mod}13)} & g_{16-0(\text{mod}13)} & g_{16-15(\text{mod}13)} & g_{4-3(\text{mod}13)} \end{bmatrix}$$

$$= \begin{bmatrix} g_4 & g_5 & g_{11} & g_{12} \\ g_5 & g_4 & g_{10} & g_{11} \\ g_3 & g_{10} & g_8 & g_4 \\ g_3 & g_5 & g_1 & g_3 \\ g_1 & g_3 & g_1 & g_1 \end{bmatrix}.$$

This is the way of operation * is performed.

$$7x = \begin{bmatrix} g_0 & g_7 & g_1 & g_8 \\ g_{10} & g_0 & g_7 & g_1 \\ g_3 & g_7 & g_0 & g_8 \\ g_0 & g_2 & g_7 & g_0 \\ g_7 & g_0 & g_9 & g_7 \end{bmatrix} \in V.$$

Example 3.19: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in L_{11}(4); 1 \le i \le 9, * \right\}$$

be a non associative matrix linear algebra over the field Z_{11}.

Let $x = \begin{bmatrix} g_4 & g_0 & g_7 \\ g_8 & g_1 & g_3 \\ g_1 & g_2 & g_4 \end{bmatrix}$ and $y = \begin{bmatrix} g_3 & g_1 & g_2 \\ g_4 & g_7 & g_1 \\ g_2 & g_0 & g_5 \end{bmatrix}$ be in M.

$$x * y = \begin{bmatrix} g_{12-12(\text{mod}11)} & g_{4-0(\text{mod}11)} & g_{8-21(\text{mod}11)} \\ g_{16-24(\text{mod}11)} & g_{28-3(\text{mod}11)} & g_{4-9(\text{mod}11)} \\ g_{8-3(\text{mod}11)} & g_{0-6(\text{mod}11)} & g_{20-12(\text{mod}11)} \end{bmatrix}$$

$$= \begin{bmatrix} g_0 & g_4 & g_7 \\ g_3 & g_3 & g_6 \\ g_5 & g_5 & g_8 \end{bmatrix} \in M.$$

This is the way operation is performed on the linear algebra which is both non associative and non commutative.

Example 3.20: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \middle| \; a_i \in L_{19}(8);\ 1 \leq i \leq 40, * \right\}$$

be a non associative matrix linear algebra which is non commutative.

Example 3.21: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{24} \\ a_{25} & a_{26} & \dots & a_{36} \end{bmatrix} \middle| \; a_i \in L_7(3);\ *,\ 1 \leq i \leq 36 \right\}$$

be a non associative matrix linear algebra over the field Z_7.

Example 3.22: Let

$$P = \left\{ \left[\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{matrix} \right] \middle| \; a_i \in L_{29}(3); *, 1 \leq i \leq 30 \right\}$$

be a non associative matrix linear algebra over the field Z_{29}.

Now we can define linear transformation of two non associative matrix linear algebras, substructures linear operators and linear functionals as in case of usual linear algebras. These are illustrated with examples.

Example 3.23: Let

$$M = \left\{ \left[\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{matrix} \right] \middle| \; a_i \in L_{23}(7); *, 1 \leq i \leq 12 \right\}$$

and

$$V = \left\{ \left[\begin{matrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \end{matrix} \right] \middle| \; a_i \in L_{23}(8), *, 1 \leq i \leq 12 \right\}$$

be two non associative linear algebras defined over the field $F = Z_{23}$.

Define a map T : M → V by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_4 & a_7 & a_{10} & a_2 & a_5 \\ a_8 & a_{11} & a_3 & a_6 & a_9 & a_{12} \end{bmatrix}$$

It is easily verified T is a linear transformation of M into V.

Example 3.24: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in L_{29}(10);\ 1 \leq i \leq 9\right\}$$

be a non associative linear algebra over the field Z_{29}.

Consider

$$W = \left\{\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \middle| \; a_i \in L_{29}(10),\ *\right\} \subseteq M$$

be a non associative linear subalgebra of M over the field Z_{29}.

Example 3.25: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in L_{23}(9);\ *,\ 1 \leq i \leq 16\right\}$$

be a non associative linear algebra over the field Z_{23}.

Let T : W $\to$ W such that

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_5 & a_9 & a_{13} \\ a_2 & a_6 & a_{10} & a_{14} \\ a_3 & a_7 & a_{11} & a_{15} \\ a_4 & a_8 & a_{12} & a_{16} \end{bmatrix}.$$

T is a linear operator on W.

Example 3.26: Let

$$V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \middle| \; a_i \in L_{11}(4); *, 1 \leq i \leq 40\right\}$$

be a non associative linear algebra over the field Z_{11}.

Define f : V $\to$ Z_{11} by

$$f\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix}\right) = 1 + 2 + 3 + 4 \pmod{11}.$$

$$\text{That is if } f\left(\begin{bmatrix} a_8 & a_7 & a_1 & a_0 \\ a_2 & a_3 & a_5 & a_6 \\ \vdots & \vdots & \vdots & \vdots \\ a_{10} & a_1 & a_0 & a_2 \end{bmatrix}\right) = 8 + 7 + 1 + 0 \pmod{11}.$$

= 5.

It is easily verified f is a linear functional on V.

Example 3.27: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in L_5 (3), *, \; 1 \leq i \leq 9\right\}$$

be a non associative linear algebra over the field Z_5.

Consider

$$V = \left\{\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \middle| \, a_i \in L_5 (3), *\right\} \subseteq M;$$

V is a non associative linear sub algebra of M over the field Z_5. Take

T : M → M defined by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix}.$$

It is easily verified T is a linear operator on M.

Consider the map

$$f : M \to Z_5 \text{ defined by } f\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_2 & a_0 \\ a_2 & a_0 & a_2 \end{bmatrix}\right)$$

= sum of the suffix values of a_i which is

$$1 + 2 + 3 + 4 + 2 + 0 + 2 + 0 + 2 = 1 \pmod 5).$$

We will just show for a $v \in M$; let $v = \begin{bmatrix} a_0 & a_2 & a_4 \\ a_3 & a_2 & a_0 \\ a_0 & a_5 & a_1 \end{bmatrix}$ be in M.

$$f\left(\begin{bmatrix} a_0 & a_2 & a_4 \\ a_3 & a_2 & a_0 \\ a_0 & a_3 & a_1 \end{bmatrix}\right) = 0 + 2 + 4 + 3 + 2 + 0 + 0 + 3 + 1 \pmod 5$$

$= 0.$

Thus f is a linear functional on V.

Now we proceed onto define non associative quasi linear algebra of polynomials.

Example 3.28: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in L_7(3); *\right\}$$

be a non associative quasi linear algebra of polynomials over the field Z_7.

Example 3.29: Let

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in L_{23}(8); *\right\}$$

be a non associative quasi linear algebra of polynomials over the field Z_{23}.

Take

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in L_{23}(8); *\right\} \subseteq W,$$

V is a non associative quasi linear subalgebras of W over the field Z_{23}.

Example 3.30: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} \text{ with } d_j \in L_{11}(3); \; *, 1 \leq j \leq 5\right\}$$

be a non associative column matrix coefficient quasi linear algebra over the field Z_{11}.

Take

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a = \begin{bmatrix} d \\ d \\ d \\ d \\ d \end{bmatrix} d \in L_{11}(3); \; *\right\} \subseteq M,$$

W is a non associative quasi linear subalgebra of M over the field Z_{11}.

Example 3.31: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{20} \end{bmatrix} \text{ with } d_j \in L_{29}(10); \; *, 1 \leq j \leq 20\right\}$$

be a non associative column matrix coefficient quasi linear algebra over the field Z_{29}.

Example 3.32: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \text{ with } d_1, d_2 \in L_{31}(8), * \right\}$$

be a non associative column matrix coefficient quasi linear algebra over the field Z_{31}.

Example 3.33: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (d_1, d_2, \ldots, d_{20}) \text{ with } d_j \in L_{13}(3), *, 1 \le j \le 20 \right\}$$

be a non associative column matrix coefficient quasi linear algebra over the field Z_{13}.

Example 3.34: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (d_1, d_2) \text{ with } d_j \in L_{11}(4), *, 1 \le j \le 2 \right\}$$

be a non associative quasi linear algebra of row coefficient polynomials over the field Z_{11}.

Example 3.35: Let

$$M = \left\{ \sum a_i x^i \;\middle|\; a_i = (d_1, \ldots, d_{11}) \text{ with } d_j \in L_5(3), *, 1 \le j \le 11 \right\}$$

be a non associative quasi linear algebra of row matrix coefficient polynomials.

(i) Find dimension of M over Z_5.

(ii) Prove M is commutative.

Example 3.36: Let

$$M = \left\{ \sum a_i x^i \,\middle|\, a_i = \begin{bmatrix} d_1 & d_2 & d_3 & \ldots & d_{15} \\ d_{16} & d_{17} & d_{18} & \ldots & d_{30} \\ d_{31} & d_{32} & d_{33} & \ldots & d_{45} \end{bmatrix} \right.$$

$$\text{with } d_j \in L_{23}(12), *, 1 \leq j \leq 45\}$$

be a non associative quasi linear algebra of row matrix coefficient polynomials.

Example 3.37: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \\ d_9 & d_{10} & d_{11} & d_{12} \\ d_{13} & d_{14} & d_{15} & d_{16} \end{bmatrix} \right.$$

$$\text{with } d_j \in L_{19}(10), *, 1 \leq j \leq 16\}$$

be a non associative quasi linear algebra over the field Z_{19} which is commutative matrix polynomial coefficient quasi linear algebra over Z_{19}.

Take $T : P \to P$ defined by

$$P\left(\sum_{i=0}^{\infty} a_i x^i\right) = \sum_{i=0}^{\infty} a_i^t x^i \text{ where}$$

$$a_i^t = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \\ d_9 & d_{10} & d_{11} & d_{12} \\ d_{13} & d_{14} & d_{15} & d_{16} \end{bmatrix}^t = \begin{bmatrix} d_1 & d_5 & d_9 & d_{13} \\ d_2 & d_6 & d_{10} & d_{14} \\ d_3 & d_7 & d_{11} & d_{15} \\ d_4 & d_8 & d_{12} & d_{16} \end{bmatrix}$$

for every square matrix a_i. T is easily verified to be a linear operator on P.

Example 3.38: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & \dots & d_8 \\ d_9 & d_{10} & \dots & d_{16} \end{bmatrix}, d_j \in L_{11}(6), *, 1 \leq j \leq 16 \right\}$$

be a non associative quasi linear algebra over the field Z_{11}. Clearly P is commutative.

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d & d & \dots & d \\ d & d & \dots & d \end{bmatrix} d \in L_{11}(6), * \right\} \subseteq P$$

is a matrix coefficient polynomial non associative quasi linear subalgebra of P over Z_{11}.

Now we proceed onto define Smarandache non associative linear algebras.

DEFINITION 3.2: *Let V be a loop suppose V is such that for a S-ring R we have the following conditions to be true.*

(i) For all $r \in R$ and $v \in V$ rv and $vr \in V$.

(ii) $1 \in R$ and $\forall v \in V$ $1.v = v \in V$.

(iii) $(a + b)v = av + bv$ and $a(v*u) = (av) * u = v * (au)$ for all $a \in R$ and $u, v \in V$.

Then we define V to be a Smarandache non associative linear algebra over the S-ring R.

We will illustrate this situation by some examples.

Example 3.39: Let V = $\{L_{15}(8), *\}$ be a S-non associative linear algebra over the S-ring Z_{15}.

Example 3.40: Let S = $\{L_{21}(11), *\}$ is a S-non associative linear algebra over the S-ring Z_{21}.

Example 3.41: Let P = $\{L_{33}(14), *\}$ be a S-non associative linear algebra over the S-ring Z_{33}.

We see if $g_5, g_{25} \in P$.

$g_5 * g_{25} = g_{(25 \times 14 - 5 \times 13) \bmod 33} = g_{21}$

$g_{25} * g_5 = g_{(5 \times 14 - 25 \times 13) \bmod 33} = g_{32}$

Clearly $g_{25} * g_5 \neq g_5 * g_{25}$; hence the S-non associative linear algebra is non commutative.

Suppose $10 \in Z_{33}$ now $10g_5 = g_{50 \, (\bmod 33)} = g_{17}$.

Infact $20ge = g_{20}$ that is for $a \in Z_{33}$ $a.ge = ga$.

Also $0.\ g_t = g_{0.t \, (\bmod 33)} = g_0$.

Example 3.42: Let M = $\{L_{57}(23), *\}$ is a S-non associative linear algebra over the S-ring Z_{57}.

Now the study the substructure in S-non associative linear algebras is a matter of routines, hence is left as an exercise to the reader.

However we now built different S-non associative linear algebras over S-rings, which will be illustrated by some examples.

Example 3.43: Let V = $\{(a_1, a_2, a_3) \mid a_i \in L_{57}(32); 1 \leq i \leq 3, *\}$ be a S-non associative row matrix linear algebra over the S-ring Z_{57}. Take P = $\{(a, a, a) \mid a \in L_{57}(32), *\} \subseteq V$ is a S-non associative row matrix linear subalgebra of V over the S-ring Z_{57}.

Example 3.44: Let P = $\{(a_1, a_2, a_3, a_4, a_5)$ where $a_i \in Z_{15}(8)$, $1 \leq i \leq 5, *\}$ be a S-non associative row matrix linear algebra over the S-ring Z_{15}.

Example 3.45: Let M = $\{(a_1, a_2, \ldots, a_{10}) \mid a_i \in L_{57}(23), 1 \leq i \leq 10, *\}$ be a S-non associative row matrix linear algebra over the S-ring Z_{57}.

We see it is not easy to find sublinear algebras such that M is a direct sum.

However we have S non associative sublinear algebras. Further we can define linear transformation in case of two S-non associative sublinear algebras only if they are defined over the same S-ring.

We give illustrations of these.

Example 3.46: Let M = $\{(a_1, a_2, a_3, a_4) \mid a_i \in L_{15}(8), *, 1 \leq i \leq 4\}$ be a S-non associative row matrix linear algebra over the S-ring Z_{15}.

Example 3.47: Let

$$T = \left\{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{10} \end{bmatrix} \text{ where } d_j \in L_{51}(14), 1 \leq j \leq 10, *\right\}$$

be a S-non associative column matrix linear algebra over the S-ring Z_{51}.

Example 3.48: Let

$$W = \left\{ \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_6 \end{bmatrix} \middle| \; m_j \in L_{69}(11),\ 1 \leq j \leq 6,\ * \right\}$$

be a S-non associative column matrix linear algebra over the S-ring Z_{69}.

Example 3.49: Let

$$P = \left\{ \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \middle| \; d_j \in L_{87}(26),\ 1 \leq j \leq 3,\ * \right\}$$

be a S-non associative column matrix linear algebra over the S-ring Z_{87}.

Example 3.50: Let

$$W = \left\{ \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_7 \end{bmatrix} \middle| \; d_j \in L_{21}(20),\ 1 \leq j \leq 7,\ * \right\}$$

be a S-non associative column matrix linear algebra over the S-ring Z_{21}.

Now we give some examples of S-non associative matrix linear algebras over the S-ring.

Example 3.51: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_j \in L_{15}(8), 1 \leq j \leq 16, * \right\}$$

be a S-non associative column matrix linear algebra over the S-ring Z_{15}.

Clearly M is commutative.

Example 3.52: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| \; a_i \in L_{57}(8), 1 \leq i \leq 30, * \right\}$$

be a S-non associative matrix linear algebra over the S-ring Z_{57}.

Example 3.53: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_i \in L_{15}(8), 1 \leq i \leq 30, * \right\}$$

be a S-non associative matrix linear algebra over the S-ring Z_{15}.

Example 3.54: Let

$$V = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}\right| a_i \in L_{57}(8),\ 1 \le i \le 12, *\right\}$$

be a S-non associative matrix linear algebra over the S-ring Z_{57}.

Example 3.55: Let

$$M = \{(a_1, a_2, \ldots, a_9) \text{ where } a_j \in L_{111}(11),\ 1 \le j \le 9, *\}$$

and

$$V = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i \in L_{111}(8);\ 1 \le i \le 9, *\right\}$$

be two S-non associative matrix linear algebra over the S-ring Z_{111}.

Consider $T : M \to V$ be a map such that

$$T((a_1, a_2, \ldots, a_9)) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}.$$

T is a linear transformation from M to V.

Example 3.56: Let

$$V = \{a_i \in L_{15}(8);\ 1 \le i \le 12, *\}$$

be a S-non associative matrix linear algebra over the S-ring Z_{15}.

Consider a map $T : V \to V$ defined by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} a_{12} & a_{11} & a_{10} \\ a_9 & a_8 & a_7 \\ a_6 & a_5 & a_4 \\ a_3 & a_2 & a_1 \end{bmatrix}.$$

T is a linear operator on V.

Example 3.57: Let

$$V = \left\{\begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \middle| \; a_i \in L_{33}(17);\ 1 \leq i \leq 40, *\right\}$$

is a S-non associative matrix linear algebra over the S-ring Z_{33}.

Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{40} \end{bmatrix} \middle| \; a_i \in L_{33}(20);\ 1 \leq i \leq 40, *\right\}$$

be a S-non associative matrix linear algebra over the S-ring Z_{33}.

Define $\eta : V \to W$;

$$\eta\left(\begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{40} \end{bmatrix}.$$

η is a linear transformation from V into W.

Now all other properties associated with non associative linear algebras can be defined and derived with simple operations.

Chapter Four

GROUPOID VECTOR SPACES

In this chapter we introduced some new concepts called groupoid vector spaces, loop vector spaces and quasi loop vector spaces. We study and analyse these notions in this chapter by examples and results.

DEFINITION 4.1: *Let V be a set and G a groupoid under a binary operation *. We say V is a groupoid vector space over the groupoid G if the following conditions are true;*

(i) *For all $v \in V$ and $g \in G$, $gv = vg \in V$.*
(ii) $(g_1 * g_2)v = g_1v * g_2 v$ *for all* $g_1, g_2 \in G$ *and* $v \in V$.

We will illustrate this situation by some examples.

Example 4.1: Let V = {3Z, 5Z, 7Z} be a set. G = {Z, *, (9, 4)} be a groupoid. V is a groupoid vector space over the groupoid G.

Example 4.2: Let W = {5Z, 8Z, 23Z, 11Z} be a set. G = {Z, *, (0, 12)} be a groupoid. V is a groupoid vector space over the groupoid G.

Example 4.3: Let M = $\{18Z^+ \cup \{0\}, 29Z^+ \cup \{0\}, 17Z^+ \cup \{0\}\}$ be a set and G = $\{Z^+ \cup \{0\}, *, (13, 13)\}$ be a groupoid. M is a groupoid vector space over the groupoid G.

Example 4.4: Let S = {3Z, 5Z, 17Z, 29Z, 43Z} be a set.
G = {Z, *, (17, 19)} be a groupoid. S is a groupoid vector space over the groupoid G.

All these groupoid vector spaces are of infinite dimension. We will show now groupoid vector space of matrices.

Example 4.5: Let W = $\{(a_1, a_2, a_3), (a_1, a_2), (a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in 3Z \cup 5Z \cup 7Z\}$ be a set. G = {Z, *, (29, 0)} be groupoid. W is a groupoid vector space of matrices.

Example 4.6: Let

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, (a_1, a_2, \ldots, a_{10}) \mid a_i \in 2Z \cup 5Z \cup 7Z, \right.$$

$$\left. 1 \leq i \leq 10 \right\}$$

be a set and G = {Z, *, (3, 17)} be a groupoid. R is a groupoid vector space of matrices.

Example 4.7: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{25} & a_{26} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & \ldots & \ldots & a_8 \\ a_9 & \ldots & \ldots & a_{12} \\ a_{13} & \ldots & \ldots & a_{16} \end{bmatrix} \right|$$

$$\left. a_i \in Z, 1 \leq i \leq 26 \right\}$$

be a set of some matrices. G = {Z, *, (3, 2)} be a groupoid. W is a groupoid vector space of matrices.

Example 4.8: Let

$$M = \left\{ (a_1, a_2, \ldots, a_{10}), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix}, \begin{bmatrix} a_1 & \ldots & a_{16} \\ a_{17} & \ldots & a_{32} \end{bmatrix} \middle| \; a_i \in Z; 1 \leq i \leq 32 \right\}$$

be a set of matrices. G = {Z, *, (10, 11)} be a groupoid; M is groupoid vector space over the groupoid G. We can define subgroupoids vector subspaces and groupoid vector subspaces, this is a matter of routine.

We will illustrate this situation by some examples.

Example 4.9: Let

$$P = \left\{ (a_1, a_2, \ldots, a_{16}), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{18} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \begin{pmatrix} a_1 & \ldots & a_9 \\ a_{10} & \ldots & a_{18} \\ a_{19} & \ldots & a_{27} \end{pmatrix} \middle| \; a_i \in Z; 1 \leq i \leq 27 \right\}$$

be a set of matrices. G = {Z, *, (2, 13)} be a groupoid. P is groupoid vector space over G.

$$X = \left\{ (a_1, a_2, \ldots, a_{16}), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{18} \end{bmatrix} \middle| \; a_i \in Z; 1 \leq i \leq 18 \right\} \subseteq P$$

is a groupoid vector subspace of P over G.

Example 4.10: Let

$$M = \left\{(a_1, a_2), \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix} \middle| \; a_i \in Z, 1 \leq i \leq 20\right\}$$

be a groupoid vector space over the groupoid G = {Z, *, (7, 0)}.

Consider

$$Y = \left\{(a_1, a_2), \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix} \middle| \; a_i \in Z, 1 \leq i \leq 20\right\} \subseteq M,$$

Y is a groupoid vector space over the groupoid H = {7Z, *, (7, 0)} $\subseteq$ G.

Infact Y is a subgroupoid vector subspace of M over the subgroupoid H of M.

Example 4.11: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \right.$$

$$\left. (a_1, a_2, \ldots, a_{11}) \;\middle|\; a_i \in Z; 1 \leq i \leq 18 \right\}$$

be a groupoid vector space of matrices over the groupoid G = {Z, *, (17, 17)}.

Consider

$$M = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, (a_1, a_2, \ldots, a_{11}) \mid a_i \in Z; 1 \leq i \leq 11 \right\} \subseteq P;$$

M is a subgroupoid vector subspace of P over the subgroupoid $H = \{3Z, *, (17, 17)\} \subseteq G$.

Now we proceed onto give example of finite groupoid vector space defined over the groupoid G.

Example 4.12: Let

$$V = \left\{ (a_1, a_2), \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in Z_9; 1 \leq i \leq 6 \right\}$$

be a groupoid vector space of matrices over the groupoid $G = \{Z_9, *, (7, 2)\}$.

Example 4.13: Let

$$M = \left\{ (a_1, a_2, \ldots, a_7), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \right.$$

$$\left. a_i \in Z_{19}; 1 \leq i \leq 27 \right\}$$

be a groupoid vector space of matrices over the groupoid

$$G = \{Z_{19}, *, (12, 9)\}.$$

$$P = \left\{ (a_1, a_2, \ldots, a_7), \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in Z_{19}; 1 \leq i \leq 7 \right\}$$

be a groupoid vector subspace of matrices over the groupoid $G = \{Z_{19}, *, (12, 9)\}$.

Example 4.14: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_7 \\ a_8 & a_9 & \dots & a_{14} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right| a_i \in Z_{240};\ 1 \leq i \leq 30 \}$$

be a groupoid vector space of matrices over the groupoid $G = \{Z_{240}, *, (11, 19)\}$.

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_7 \\ a_8 & a_9 & \dots & a_{14} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right| a_i \in Z_{240};$$

$$1 \leq i \leq 25\} \subseteq W;$$

be a groupoid vector subspace of matrices over the groupoid $G = \{Z_{240}, *, (11, 19)\}$.

Example 4.15: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \end{bmatrix} \right| a_i \in Z_{43};\ 1 \leq i \leq 40\}$$

be a groupoid vector space of matrices over the groupoid $G = \{Z_{43}, *, (20, 23)\}$.

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \end{bmatrix} \right| a_i \in Z_{43};\ 1 \leq i \leq 43\}$$

$\subseteq$ M; be a groupoid vector subspace of matrices over the groupoid G.

We see M = P + T and P $\cap$ T = ϕ that is M is a direct sum of subspaces over the groupoid G.

Example 4.16: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \right.$$

$$\left.\left[\begin{array}{cccc} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{array}\right], \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_{40} \end{array}\right], (a_1, a_2, \ldots, a_{15}) \right| a_i \in Z_{45}; 1 \leq i \leq 40\}$$

be a groupoid vector space of matrices over the groupoid $G = \{Z_{45}, *, (11, 0)\}$.

Take

$$P_1 = \left\{\left.\left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{array}\right], \left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}\right] \right| a_i \in Z_{45}; 1 \leq i \leq 12\right\} \subseteq M;$$

$$P_2 = \left\{\left.\left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}\right], (a_1, a_2, \ldots, a_{15}) \right| a_i \in Z_{45}; 1 \leq i \leq 15\right\} \subseteq M;$$

$$P_3 = \left\{\left.\left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}\right], \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_{40} \end{array}\right] \right| a_i \in Z_{45}; 1 \leq i \leq 40\right\} \subseteq M$$

and

$$P_4 = \left\{\left.\left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array}\right], \left[\begin{array}{cccc} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{array}\right] \right| a_i \in Z_{45}; 1 \leq i \leq 40\right\} \subseteq M$$

be a groupoid vector subspaces of M over the groupoid.

$$\text{We see } P_i \cap P_j = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, 1 \leq i, j \leq 4; i \neq j.$$

Clearly $M \subseteq P_1 + P_2 + P_3 + P_4$. Thus M is a pseudo direct sum of groupoid vector subspaces of M over the groupoid G.

Example 4.17: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix}, (a_1, a_2, \ldots, a_{12}) \right| a_i \in Z_{425};$$

$$1 \leq i \leq 30\}$$

be a groupoid vector space over the groupoid G = $\{Z_{425}, *, (11, 19)\}$.

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \right| a_i \in Z_{425}; 1 \leq i \leq 6\} \subseteq V,$$

$$M_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \right| a_i \in Z_{425}; 1 \leq i \leq 30\} \subseteq V,$$

and $M_3 = \{(a_1, a_2, \ldots, a_{12}) \mid a_i \in Z_{425}; 1 \leq i \leq 12\} \subseteq V$ be groupoid vector subspaces of V over the groupoid G.

Clearly $M_i \cap M_j = \phi$; $1 \leq i, j \leq 3$ if $i \neq j$.

Further $M_1 + M_2 + M_3 = V$. Thus V is a direct sum of groupoid vector subspaces of V over the groupoid G.

We now as in case of usual structures define groupoid linear algebra over the groupoid G.

We say a groupoid vector space V over the groupoid G is a groupoid linear algebra if V is closed under some operation.

We illustrate this by the following examples.

Example 4.18: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in Z_{45}; *, (12, 0), 1 \leq i \leq 12 \right\}$$

be a groupoid vector space over the groupoid G = $\{Z_{45}, *, (12, 0)\}$. Since V is a groupoid under *; so (V, *) is a groupoid linear algebra over the groupoid G.

Example 4.19: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_i \in Z_{25}; *, (21, 4), 1 \leq i \leq 30 \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{25}, *, (21, 4)\}$.

Example 4.20: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{19} \end{bmatrix} \middle| a_i \in Z_{18}; *, (11, 7), 1 \leq i \leq 19 \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{18}, *, (11, 7)\}$.

Example 4.21: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \, a_i \in Z_{182}; *, (3, 13), 1 \leq i \leq 4 \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{182}, *, (3, 13)\}$.

Example 4.22: Let
W = $\{(a_1, a_2, \ldots, a_{19}) \mid a_i \in Z_{53}; *, (22, 31), 1 \leq i \leq 19\}$ be a groupoid linear algebra over the groupoid G = $\{Z_{53}, *, (22, 31)\}$.

We have the following theorem.

THEOREM 4.1: *Let V be a groupoid linear algebra over the groupoid G. V is a groupoid vector space over the groupoid G. But a groupoid vector space V over the groupoid G in general is not a groupoid linear algebra over G.*

The proof is direct and hence is left as an exercise to the reader. We now give examples of groupoid linear subalgebra of a groupoid linear algebra.

Example 4.23: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \, a_i \in Z_{16}; *, (3, 7), 1 \leq i \leq 4 \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{16}, *, (3, 7)\}$.

$$W = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| \, a \in Z_{16}; *, (3, 7) \right\} \subseteq V,$$

W is a groupoid linear algebra over the groupoid G = $\{Z_{16}, *, (3, 7)\}$.

Example 4.24: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \middle| \, a_i \in Z_{27}, *, \ (17, 0), 1 \leq i \leq 40 \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{27}, *, (17, 0)\}$.

Consider

$$M = \left\{ \begin{bmatrix} a & a & a & a \\ a & a & a & a \\ \vdots & \vdots & \vdots & \vdots \\ a & a & a & a \end{bmatrix} \middle| \, a \in Z_{27}, *, \ (17, 0) \right\} \subseteq V,$$

is a groupoid linear subalgebra of V over the groupoid G.

Example 4.25: Let

V = $\{(a_1, a_2, \ldots, a_{16}) \mid a_i \in Z_{29}, *, \ (3, 3), 1 \leq i \leq 16\}$ be a groupoid linear algebra over the groupoid G = $\{Z_{29}, *, (3, 3)\}$.

Take W = $\{(a, a, \ldots, a) \mid a \in Z_{29}, *, \ (3, 3)\}$ be a groupoid linear algebra over the groupoid G.

We can define transformation and linear operator on groupoid vector spaces (linear algebras).

Example 4.26: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, (a_1, a_2, \ldots, a_{10}) \, \middle| \, a_i \in Z_{18}, \right.$$

$$\left. 1 \leq i \leq 10 \right\}$$

be a groupoid linear algebra over the groupoid G = {Z_{18}, *, (3, 7)}.

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}, (a_1, a_2, a_3, a_4), \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in Z_{18},\ 1 \leq i \leq 10 \right\}$$

be a groupoid vector space over the groupoid G = {Z_8, *, (3, 7)}.

Define T : V $\rightarrow$ W;

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix},$$

$$T\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = (a_1, a_2, a_3, a_4)$$

$$T((a_1, a_2, \ldots, a_{10})) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix}.$$

T is a linear transformation from V into W.

Example 4.27: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, (a_1, a_2, \ldots, a_{18}), \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix} \right|$$

$$a_i \in Z_{45},\ 1 \leq i \leq 40\}$$

be a groupoid vector space over the groupoid G = $\{Z_{45}, *, (7, 3)\}$.

Define $T : V \rightarrow V$ by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = (a_1, a_1, a_2, a_2, a_3, a_3, \ldots, a_9, a_9).$$

$$T\left(\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_{12} & a_{15} \\ a_{30} & a_{20} & a_{10} \\ a_{25} & a_{35} & a_{40} \end{bmatrix}$$

and

$$T((a_1, a_2, \ldots, a_{18})) = \begin{bmatrix} a_1 & a_3 & a_5 \\ a_7 & a_9 & a_{11} \\ a_{13} & a_{15} & a_{17} \end{bmatrix}.$$

T is a linear operator on V.

Example 4.28: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in Z_{40}, 1 \leq i \leq 9, *, (11, 11) \right\}$$

and

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_9 \end{bmatrix} \middle| a_i \in Z_{40}, 1 \leq i \leq 9, *, (11, 11) \right\}$$

be two groupoid linear algebras over the groupoid G = $\{Z_{40}, *, (11, 11)\}$.

Define a map T : M → W by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_9 \end{bmatrix}.$$

T is a groupoid linear transformation of V into W over G.

Example 4.29: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_4 & a_7 & a_{10} \\ a_2 & a_5 & a_8 & a_{11} \\ a_3 & a_6 & a_9 & a_{12} \end{bmatrix} \middle| a_i \in Z_{143}, 1 \leq i \leq 12, *, (11, 15) \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{143}, *, (11, 15)\}$.

$T : V \to V$ be defined as

$$T \left(\begin{bmatrix} a_1 & a_4 & a_7 & a_{10} \\ a_2 & a_5 & a_8 & a_{11} \\ a_3 & a_6 & a_9 & a_{12} \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_2 & a_1 & a_2 \\ a_4 & a_5 & a_4 & a_5 \\ a_7 & a_8 & a_7 & a_8 \end{bmatrix};$$

T is a linear operator on V.

Example 4.30: Let

$$P = \{(a_1, a_2, \ldots, a_{40}) \mid a_i \in Z_{45}, 1 \leq i \leq 40, *, (3, 0)\}$$

be a groupoid linear algebra over the groupoid $G = \{Z_{45}, *, (3, 0)\}$. Define $f : P \to G$ by

$$f((a_1, a_2, \ldots, a_{40})) = a_1 * a_{40}.$$

f is a linear functional on P.

Example 4.31: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{58} & a_{59} & a_{60} \end{bmatrix} \middle| a_j \in Z_{125}, 1 \leq j \leq 60, *, (23, 0) \right\}$$

be a groupoid linear algebra over the groupoid $G = \{Z_{125}, *, (23, 0)\}$.

Define $T : V \to V$ by

$$T \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{58} & a_{59} & a_{60} \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_1 & a_1 \\ a_4 & a_4 & a_4 \\ \vdots & \vdots & \vdots \\ a_{58} & a_{58} & a_{58} \end{bmatrix}.$$

T is a linear operator on V.

Define f : V → G.

$$f\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{58} & a_{59} & a_{60} \end{bmatrix}\right) = a_1 * a_{60}.$$

F is a linear functional on V.

Example 4.32: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, (a_1, a_2, \ldots, a_{14}), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \, a_i \in Z_{49}, \right.$$

$$\left. 1 \leq i \leq 18 \right\}$$

be a groupoid vector space over the groupoid $\{Z_{49}, *, (7, 11)\} = G$.

Define $f : V \to Z_{49}$ by

$$f\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = a_1 * a_4$$

$$f((a_1, a_2, \ldots, a_{11})) = a_{11}$$

$$f\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix}\right) = a_{10} * a_{12}$$

and

$$f\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix}\right) = a_1 * a_{18};$$

f is a linear functional on V.

Take

$$W_1 = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in Z_{49}, 1 \leq i \leq 4\right\} \subseteq V,$$

$$W_2 = \{(a_1, a_2, \ldots, a_{11}) \mid a_i \in Z_{49}, 1 \leq i \leq 11\} \subseteq V,$$

$$W_3 = \left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix} \middle| a_i \in Z_{49}, 1 \leq i \leq 15\right\} \subseteq V$$

and

$$W_4 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| a_i \in Z_{49}, 1 \leq i \leq 18\right\} \subseteq V$$

be groupoid vector subspaces of V over the groupoid G.

Clearly $V = W_1 + W_2 + W_3 + W_4$ and $W_i \cap W_j = \phi$; $i \neq j$, $1 \leq i, j \leq 4$.

Now we see V is the direct sum of groupoid vector subspaces of V over the groupoid G.

We can also define the notion of groupoid vector spaces and groupoid linear algebras over the groupoid G. This is a matter of routine and hence left as an exercise to the reader. We supply only examples.

Example 4.33: Let

$$V = \left\{ \sum_{i=0}^{7} a_i x^i, \sum_{i=12}^{45} a_i x^i, \sum_{i=40}^{200} a_i x^i, \sum_{i=208}^{300} a_i x^i \;\middle|\; a_i \in Z_{47}, 0 \leq i \leq 300 \right\}$$

be a groupoid vector space over the groupoid G = $\{Z_{47}, *, (3, 4)\}$.

This is known as the groupoid vector space of polynomials.

Example 4.34: Let

$$M = \left\{ \sum_{i=0}^{20} a_i x^i, \sum_{i=40}^{49} a_i x^i \;\middle|\; a_i \in Z_{73}, 0 \leq i \leq 49 \right\}$$

be a groupoid vector space over the groupoid G = $\{Z_{73}, *, (0, 49)\}$ = G.

Example 4.35: Let

$$T = \left\{ \sum_{i=2}^{7} a_i x^i, \sum_{i=9}^{27} a_i x^i \;\middle|\; a_i \in Z_{48}, 0 \leq i \leq 27 \right\}$$

be a groupoid vector space of polynomials over the groupoid G = $\{Z_{48}, *, (29, 0)\}$.

Example 4.36: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{19}, *, (7, 2) \right\}$$

be a groupoid linear algebra of polynomials over the groupoid G = $\{Z_{19}, *, (7, 2)\}$.

Example 4.37: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_7, *, (4, 0)\right\}$$

be a groupoid linear algebra polynomial over the groupoid $G = \{Z_7, *, (4, 0)\}$.

We now proceed onto give examples of substructures.

Example 4.38: Let

$$V = \left\{ \sum_{i=0}^{20} a_i x^i, \sum_{i=40}^{90} a_i x^i, \sum_{i=98}^{200} a_i x^i \;\middle|\; a_i \in Z_{17}\right\}$$

be a groupoid vector space over the groupoid $G = \{Z_{17}, *, (11, 7)\}$.

Take

$$P_1 = \left\{ \sum_{i=0}^{20} a_i x^i \;\middle|\; a_i \in Z_{17}\right\} \subseteq V,$$

$$P_2 = \left\{ \sum_{i=40}^{90} a_i x^i \;\middle|\; a_i \in Z_{17}\right\} \subseteq V$$

and

$$P_3 = \left\{ \sum_{i=98}^{200} a_i x^i \;\middle|\; a_i \in Z_{17}\right\} \subseteq V$$

be a groupoid vector subspaces of polynomials over the groupoid $G = \{Z_{17}, *, (11, 7)\}$. Clearly $V = P_1 + P_2 + P_3$ is a direct sum of subspaces of V.

Example 4.39: Let

$$V = \left\{\sum_{i=0}^{3} a_i x^i, \sum_{i=7}^{12} a_i x^i, \sum_{i=20}^{26} a_i x^i, \sum_{i=100}^{120} a_i x^i \;\middle|\; a_i \in Z_{420}\right\}$$

be a groupoid vector space over the groupoid G = $\{Z_{420}, *, (3, 43)\}$.

Take

$$M_1 = \left\{\sum_{i=0}^{3} a_i x^i, \sum_{i=7}^{12} a_i x^i \;\middle|\; a_i \in Z_{420}\right\} \subseteq V,$$

$$M_2 = \left\{\sum_{i=0}^{3} a_i x^i, \sum_{i=20}^{26} a_i x^i \;\middle|\; a_i \in Z_{420}\right\} \subseteq V,$$

$$M_3 = \left\{\sum_{i=0}^{3} a_i x^i, \sum_{i=100}^{120} a_i x^i \;\middle|\; a_i \in Z_{420}\right\} \subseteq V$$

be groupoid vector subspaces of V over the groupoid G.

$$M_i \cap M_j = \left\{\sum_{i=0}^{3} a_i x^i \;\middle|\; a_i \in Z_{420}\right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 3.$$

Further $V \subseteq M_1 + M_2 + M_3$ is the pseudo direct sum of groupoid vector subspaces of V over G.

Example 4.40: Let

$$M = \left\{\sum_{i=0}^{13} a_i x^i, \sum_{i=20}^{43} a_i x^i, \sum_{i=50}^{70} a_i x^i, \sum_{i=76}^{100} a_i x^i, \sum_{i=120}^{140} a_i x^i \;\middle|\; a_i \in Z_{421}\right\}$$

be a groupoid vector space over the groupoid G = $\{Z_{421}, *, (143, 0)\}$.

The reader is left with the task of finding direct sum, pseudo direct sum, linear operator and linear functional on M.

Now we can define in a similar way groupoid linear algebras over a groupoid G.

Example 4.41: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{47}, *, (19, 0) \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{47}, *, (19, 0)\}$.

Example 4.42: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{14}, *, (2, 2) \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{14}, *, (2, 2)\}$.

Example 4.43: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{23}, *, (7, 6) \right\}$$

be a groupoid linear algebra over the groupoid G = $\{Z_{23}, *, (7, 6)\}$.

Now we can use instead of these groupoid G = $\{Z_n, *, (t, u)\}$, the complex groupoids

C(G) = $\{C(Z_n), *, (t, u)$ with $i_F^2 = n-1$ where $C(z_n) = \{a + bi_F \mid a, b \in Z_n\}\}$.

We will only illustrate this situation by a few examples. When complex groupoids are used we call these groupoid vector spaces as complex groupoid vector spaces.

Example 4.44: Let V = {C (Z_{29}, *, (7, 3)} be a strong complex groupoid linear algebra C (Z_{29}) = {a + bi_F | i_F^2 = 28, *, (7, 3)}.

Note if C (Z_{29}) is replaced just by the groupoid G = (Z_{29}, *, (7, 3)} we call V only a complex linear algebra over the groupoid G.

Example 4.45: Let

$$V = \left\{ (a_1, a_2, a_3), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in C\,(Z_{24}),\ i_F^2 = 23, \; 1 \leq i \leq 9 \right\}$$

be a groupoid complex vector space over the groupoid G = {Z_{24}, *, (3, 8)}.

It is interesting to note that we can get for the same set of matrices V with entries from C(Z_{24}) several groupoids G where (t, u) can take any value from $Z_{24} \times Z_{24}$. This feature will be very much helpful when one practically works on a problem.

Example 4.46: Let

$$V = \left\{ (a_1, a_2, a_3), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in C\,(Z_{24}),\ i_F^2 = 23, \; 1 \leq i \leq 9 \right\}$$

be a groupoid complex vector space over the groupoid G = {Z_{24}, *, (13, 0)}.

Example 4.47: Let

$$V = \left\{ (a_1, a_2, a_3), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in C(Z_{24}),\ i_F^2 = 23, \right.$$

$$\left. 1 \leq i \leq 9 \right\}$$

be a complex groupoid vector space over the groupoid G = {Z_{24}, *, (0, 19)}.

Example 4.48: Let

$$W = \left\{ (a_1, a_2, a_3), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in C(Z_{24}),\ i_F^2 = 23, \right.$$

$$\left. 1 \leq i \leq 9 \right\}$$

be a complex groupoid vector space over the groupoid G = {Z_{24}, *, (12, 12)}.

Example 4.49: Let

$$M = \left\{ (a_1, a_2, a_3), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in C(Z_{24}),\ i_F^2 = 23, \right.$$

$$\left. 1 \leq i \leq 9 \right\}$$

be a complex groupoid vector space over the groupoid G = {Z_{23}, *, (13, 19)}.

Example 4.50: Let

$$P = \left\{ (a_1, a_2, a_3), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in C(Z_{24}),\; i_F^2 = 23, \right.$$

$$\left. 1 \leq i \leq 9 \right\}$$

be a strong groupoid complex vector space (strong complex groupoid vector space) over the complex groupoid G = {C (Z_{24}), *, (10, 12) where C (Z_{24}) = {a + bi_F | i_F^2 = 23, a, b ∈ Z_{24}}.

Example 4.51: Let

$$M = \left\{ (a_1, a_2, a_3), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in C(Z_{24}),\; 1 \leq i \leq 9 \right\}$$

be a strong groupoid complex groupoid vector space over the complex groupoid

G = {C (Z_{24}) = {a + i_Fb | i_F^2 = 23, a, b ∈ Z_{24}, *, (20, 20)}.

Now we can give one or two examples of strong complex groupoid linear algebras and complex groupoid linear algebras.

Example 4.52: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in C(Z_{43}),\; 1 \leq i \leq 6,\; i_F^2 = 42, (0, 3), * \right\}$$

be a groupoid complex linear algebra over the groupoid $G = \{Z_{43}, *, (0, 3)\}$.

Example 4.53: Let

$$M = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{array} \right] \middle| \; a_i \in C(Z_{30}),\ 1 \le i \le 40, \right.$$

$$\left. i_F^2 = 29, (5,5), * \right\}$$

be a complex groupoid linear algebra over the groupoid $G = \{Z_{30}, *, (5, 5)\}$.

Example 4.54: Let

$$P = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{array} \right] \middle| \; a_i \in C(Z_{26}),\ 1 \le i \le 9, (10,8), * \right\}$$

be a complex groupoid linear algebra over the groupoid $G = \{Z_{26}, *, (10, 8)\}$.

Example 4.55: Let

$M = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in C(Z_{48}), 1 \le i \le 10, (1,7), *\}$
be a complex groupoid linear algebra over the groupoid $G = \{Z_{48}, *, (1, 7)\}$.

Example 4.56: Let

$$M = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{array} \right] \middle| \; a_i \in C(Z_{15}),\ 1 \le i \le 15, (10,5), *, i_F^2 = 14 \right\}$$

be a groupoid complex linear algebra over the groupoid $G = \{Z_{15}, *, (10, 5)\}$.

Now we can also have strong groupoid complex linear algebras.

Example 4.57: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \, a_i \in C\,(Z_{12}),\ 1 \leq i \leq 9,\ (3,7),\ *,\ i_F^2 = 1 \right\}$$

be a strong complex groupoid linear algebra over the groupoid $C\,(G) = \{C\,(Z_{12}) = \{a + i_F b \mid a, b \in Z_{12},\ i_F^2 = 11\}, *, (3, 7)\}$.

Example 4.58: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \, a_i \in C\,(Z_{40}),\ 1 \leq i \leq 16, \right.$$

$$\left. (10,10),\ *,\ i_F^2 = 39 \right\}$$

be a strong complex groupoid linear algebra over the groupoid $C\,(G) = \{C\,(Z_{40}), *, (10, 10)\}$.

Example 4.59: Let
$W = \{(a_1, a_2, \ldots, a_{20}) \mid a_i \in C\,(Z_{10}),\ 1 \leq i \leq 20,\ (5,5),\ *,\ i_F^2 = 9\}$
be a strong complex groupoid linear algebra over the groupoid $C\,(G) = \{C\,(Z_{10}), *, (15, 5),\ i_F^2 = 9\}$.

Now we can define substructures in them which is a matter of routine.

Example 4.60: Let

$$M = \left\{ (a_1, a_2, a_3, a_4), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} \end{bmatrix} \right|$$

$$a_i \in C(Z_{25}),\ 1 \leq i \leq 21,\ i_F^2 = 24\}$$

be a complex groupoid vector space over the complex groupoid $G = \{Z_{25}, *, (10, 11)\}$.

Take

$$P_1 = \{(a_1, a_2, a_3, a_4) \mid a_i \in C(Z_{25}),\ 1 \leq i \leq 4,\ i_F^2 = 24\} \subseteq M,$$

$$P_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right| a_i \in C(Z_{25}),\ 1 \leq i \leq 4,\ i_F^2 = 24\} \subseteq M$$

and

$$P_3 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} \end{bmatrix} \right| a_i \in C(Z_{25}),$$

$$1 \leq i \leq 21,\ i_F^2 = 24\} \subseteq M,$$

P_1, P_2 and P_3 are complex groupoid vector subspaces of M over the groupoid $G = \{Z_{25}, *, (10, 11)\}$. Further $P_i \cap P_j = \phi$, $1 \leq i, j \leq 3$ and $M = P_1 + P_2 + P_3$; that is M a direct sum of subspaces.

Example 4.61: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} . \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{20} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} (a_1, a_2, \ldots, a_{16}), \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right| a_i \in$$

$$C(Z_{47}),\ 1 \leq i \leq 20,\ i_F^2 = 46\}$$

be a complex groupoid vector space.

Take

$$S_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{20} \end{bmatrix} \right| a_i \in C\ (Z_{47}),\ 1 \leq i \leq 20,$$

$$i_F^2 = 46\} \subseteq B,$$

$$S_2 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{20} \end{bmatrix} \right| a_i \in C\ (Z_{47}),\ 1 \leq i \leq 20,\ i_F^2 = 46\} \subseteq B$$

and

$$S_3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{20} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right| a_i \in C(Z_{47}),$$

$$1 \leq i \leq 20,\ i_F^2 = 46\} \subseteq B;$$

S_1, S_2 and S_3 are complex groupoid vector subspaces of B over the groupoid G.

Clearly $S_i \cap S_j = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{20} \end{bmatrix}$ for $i \neq j$, $1 \leq i, j \leq 3$ and $B \subseteq S_1 + S_2 + S_3$ that is B is the pseudo direct sum of complex groupoid vector subspaces of B over the groupoid G.

Example 4.62: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{17} \end{bmatrix}, (a_1, a_2, \ldots, a_{14}), \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \right| a_i \in C(Z_{50}),$$

$$1 \leq i \leq 17,\ i_F^2 = 49\}$$

be a groupoid complex vector space over the groupoid G = $\{Z_{50}, *, (10, 12)\}$.

Consider

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{17} \end{bmatrix} \middle| \; a_i \in Z_{50}, 1 \le i \le 17 \right\}$$

to be the groupoid vector space over the groupiod G. We say M is a pseudo complex groupoid vector subspace of V over the groupoid G.

Example 4.63: Let

$$V = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{pmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{16} \end{bmatrix}, (a_1, a_2, \dots, a_{20}) \; \middle| \; a_i \in C\,(Z_{120}), 1 \le i \le 20, i_F^2 = 119 \right\}$$

be a groupoid complex vector space over the groupoid $G = \{Z_{120}, *, (26, 0)\}$.

Consider

$$P = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{pmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{16} \end{bmatrix} \middle| \; a_i \in \right.$$

$$\left. C(Z_{120}), 1 \le i \le 20 \right\} \subseteq T,$$

P is a pseudo complex groupoid vector subspace of T over the groupoid G.

Consider a map f : T $\to$ T defined by

$$f\left(\begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{pmatrix}\right) = (a_1, a_2, \ldots, a_{20})$$

$$f\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{16} \end{bmatrix}.$$

$$f((a_1, a_2, \ldots, a_{20})) = \begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{pmatrix}$$

and

$$f\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{16} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}.$$

F is clearly a linear operator on T.

Let η : T $\to$ Z_{120} defined by

$$\eta\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{16} \end{bmatrix}\right) = \text{sum of real part of } (a_1, a_2, \ldots, a_{16}) \pmod{120}.$$

$\eta((a_1, \ldots, a_{20}))$ = sum of real part of a_{10} and a_{20} (mod 120)

$$\eta\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right)$$

= sum of real part of $a_1 + a_6 + a_{11} + a_{16}$ (mod (120))

$$\eta\left(\begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{pmatrix}\right)$$

= sum of real part of $(a_5 + a_{10} + a_{15} + a_{20})$ (mod 120).

In this way η is a linear functional on T.

Example 4.64: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix}, \right.$$

$$(a_1, a_2, \ldots, a_{10}), \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \left| \; a_i \in C(Z_{10}),\ 1 \le i \le 12, \right.$$

$$\left. i_F^2 = 9 \right\}$$

be a complex groupoid vector space over the groupoid $G = (Z_{10}, *, (5, 2)\}$.

$$V = \left\{ (a_1, a_2, \ldots, a_9), \begin{pmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \end{pmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_9 & a_{10} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \right| \; a_i \in C(Z_{10}),\ 1 \leq i \leq 10\}$$

be a complex groupoid vector space over the groupoid G = $\{Z_{10}$, *, (5, 2).

Define $T : W \rightarrow V$ by

$$T \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right) = (a_1, a_2, \ldots, a_9)$$

$$T \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}$$

$$T((a_1, a_2, \ldots, a_{10})) = \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \end{bmatrix}$$

$$T \left(\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \right) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_9 & a_{10} \end{bmatrix}.$$

Clearly T is a linear transformation of W to V.

Example 4.65: Let

$$M = \left\{(a_1, a_2, a_3), \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \;\middle|\; a_i \in C\,(Z_6),\ 1 \le i \le 4,\ i_F^2 = 5\right\}$$

be a complex groupoid vector space over the groupoid $\{Z_6, *, (3, 0)\} = G$.

We define a map $f : M \to G$ as follows:

$f (a_1, a_2, a_3)$ = sum of real part of a_i (mod 6).

That is if $f (3 + 4i_F, 2i_F, 5i_F + 5)) = 3 + 5 \text{ (mod 6)} = 2$

$$f \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \text{real part of } a_2 \text{ that is}$$

$$f \left(\begin{bmatrix} 3+5i_F \\ 5+2i_F \end{bmatrix}\right) = 5$$

and

$$f \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = \text{ sum of real part of } (a_1 + a_4) \text{ (mod 6)}.$$

$$\text{That is } f \left(\begin{bmatrix} 3i_F+5 & 2i_F \\ 2i_F+4 & 4+4i_F \end{bmatrix}\right) = 5+4 \text{ (mod 6)} = 3.$$

Thus f is a linear functional on M.

Now we can get similar results in case of strong complex groupoid vector spaces / linear algebras over a complex groupoid C (G). We see every strong complex groupoid vector space contains a pseudo strong complex subgroupoid vector space.

Example 4.66: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix}, (a_1, a_2, \ldots, a_8), \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right| a_i \in C\ (Z_{13}),$$

$$1 \leq i \leq 8\}$$

be a strong complex groupoid vector space over the complex groupoid C (G) = $\{a + bi_F \mid a, b \in Z_{13},\ i_F^2 = 12, *, (3, 10)\}$.

Take

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right| a_i \in C\ (Z_{13}),\ 1 \leq i \leq 7,$$

$$i_F^2 = 12\} \subseteq V,$$

P is a pseudo strong complex subgroupoid vector subspace over the subgroupoid G = $\{a \mid a \in Z_{13}, (3, 10), *\}$.

Example 4.67: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \right| a_j \in C\ (Z_{45}), *, (13, 0)\ 1 \leq j \leq 8\}$$

be a strong complex groupoid vector space over the complex groupoid C(G) = $\{a + bi_F \mid a, b \in Z_{45}, *, (13, 0)\ i_F^2 = 44\}$.

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| \; a_j \in C\,(Z_{45}), *, (13, 0)\ 1 \le i \le 8, \right.$$

$$\left. i_F^2 = 44 \right\} \subseteq M$$

is a pseudo strong complex subgroupoid linear algebra over the subgroupoid $G = \{Z_{45}, *, (13, 0)\}$.

Next we proceed onto define loop vector spaces, strong loop vector spaces, quasi loop vector spaces and strong quasi loop vector spaces and describe them with examples.

DEFINITION 4.2: *Let $S = \{L_n (m) \mid \{e, 1, 2, \ldots, n\} \in Z_n \cup \{e\}, n > 3$, n odd, $km < n$ with $(m-1, n) = 1$, and $(n, m) = 1, *\}$ be a loop of order n. V be a set; if for all $v \in V$ and $s \in S$*

(i) $sv, vs \in V$

(ii) $s_1 * s_2 (v) = s_1 v * s_2 v$ for all $s_1, s_2 \in S$ and $v \in V$ then we define V to be a loop vector space defined over the loop S.

If V is itself a loop and V is a loop vector space we define V to be strong loop linear algebra.

If on V is defined a closed binary operation and V is a loop vector space over S then we define V to be a loop linear algebra over S.

We will give examples of them.

Example 4.68: Let

$$P = \left\{ (a_1, a_2, a_3), \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix} \middle| \right.$$

$$\left. a_i \in L_{13}(7),\ 1 \le i \le 20 \right\}$$

be a loop vector space over the loop $S = L_{13}(7)$.

Example 4.69: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix} \right| a_i \in L_{13}(7),$$

$$1 \leq i \leq 20\}$$

be a loop vector space over the loop S = L_{13} (7).

Example 4.70: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right|$$

$$a_i \in L_{15}(8),\ 1 \leq i \leq 24\}$$

be a loop vector space over the loop S = L_{15} (8).

Example 4.71: Let

$$J = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, (a_1, a_2, \ldots, a_{20}) \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ \vdots & \vdots & & \vdots \\ a_{91} & a_{92} & \ldots & a_{100} \end{bmatrix} \right|$$

$$a_i \in L_{43}(8),\ 1 \leq i \leq 100\}$$

be a loop vector space over the loop S = L_{43} (8).

Consider

$$P_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in L_{43}(8),\ 1 \le i \le 9\right\},$$

P_1 is a loop vector subspace of J over the loop S = L_{43} (8).

P_2 = {(a_1, a_2, …, a_{20}) | $a_i \in L_{43}(8)$, $1 \le i \le 20$} is a loop vector subspace of J over the loop S = L_{43} (8).

Finally

$$P_3 = \left\{\begin{bmatrix} a_1 & a_2 & \cdots & a_{10} \\ a_{11} & a_{12} & \cdots & a_{20} \\ \vdots & \vdots & & \vdots \\ a_{91} & a_{92} & \cdots & a_{100} \end{bmatrix} \middle| \; a_i \in L_{43}(8),\ 1 \le i \le 100\right\}$$

is a loop vector subspace of J over the loop S.

Clearly $P_i \cap P_j = \phi$ if $i \neq j$; $1 \le i, j \le 3$ and J = $P_1 + P_2 + P_3$. Thus J is a direct sum of loop vector subspace of J over S.

Example 4.72: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}, \begin{pmatrix} a_1 & a_2 & \cdots & a_9 \\ a_{10} & a_{11} & \cdots & a_{18} \\ a_{19} & a_{20} & \cdots & a_{27} \end{pmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_8 \\ a_9 & a_{10} & \cdots & a_{16} \\ a_{17} & a_{18} & \cdots & a_{24} \\ a_{25} & a_{26} & \cdots & a_{32} \end{bmatrix} \right|$$

$$a_i \in L_{53}(19),\ 1 \le i \le 32\}$$

be a loop vector space over the loop S = L_{53} (19).

$$V_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in L_{53}(19),\ 1 \leq i \leq 10 \right\} \subseteq M,$$

$$V_2 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \end{pmatrix} \middle| \; a_i \in L_{53}(19),\ 1 \leq i \leq 18 \right\}$$

$$V_3 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \end{pmatrix}, \middle| \; a_i \in L_{53}(19), \right.$$

$$\left. 1 \leq i \leq 27 \right\} \subseteq M$$

and

$$V_4 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{32} \end{bmatrix} \middle| \; a_i \in L_{53}(19), \right.$$

$$\left. 1 \leq i \leq 32 \right\} \subseteq M$$

are loop vector subspace of M over the loop L_{53} (19).

Clearly $V_i \cap V_j = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$; $i \neq j$, $1 \leq i, j \leq 4$ and $M \subseteq V_1 + V_2 + V_3 + V_4$; thus M is a pseudo direct sum of loop vector subspace of M.

Example 4.73: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in L_{19}(8),\ 1 \leq i \leq 9\}$$

and

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}, (a_1, a_2, \ldots, a_7), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \right| a_i \in L_{19}(8),\ 1 \leq i \leq 9\}$$

be two loop vector spaces over the loop $S = L_{19}$ (8).

Define $T : V \rightarrow W$ by

$$T \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}$$

$$T \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \right) = (a_1, a_2, \ldots, a_7) \text{ and}$$

$$T \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix}.$$

T is a linear transformation of V into W.

Example 4.74: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, (a_1, a_2, \ldots, a_6), \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \\ a_9 \end{bmatrix} \right|$$

$$a_i \in \mathrm{I}\ L_{59}(25),\ 1 \leq i \leq 9\}$$

be a loop vector space over the loop $S = L_{59}$ (25).

Define $T : V \to V$ as follows.

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}\right) = (a_1, a_2, a_3, a_4, a_5, a_6)$$

$$T((a_1, a_2, \ldots, a_6)) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$$

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \\ a_9 \end{bmatrix}$$

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \\ a_9 \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix};$$

T is a linear operator on V.

Define f : M → L_{59} (25)

$$f\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}\right) = (a_1+a_2+a_3) \text{ mod } 59$$

$$\text{That is } f\left(\begin{bmatrix} 40 & 29 & 31 \\ 6 & 2 & 17 \end{bmatrix}\right) = (40 + 29 + 31) \text{ (mod 59)}$$

$$= 41.$$

$f((a_1, a_2, \ldots, a_6)) = \sum a_i$ (mod 59)

that is

f ((20, 4, 8, 16, 7, 1)) = (20 + 4+ 8 + 16 + 7 + 1) (mod 59)

$$= 56.$$

$$f\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = a_1 + a_5 + a_9 \text{ (mod 59)}$$

that is

$$f\left(\begin{bmatrix} 40 & 0 & 2 \\ 7 & 26 & 19 \\ 6 & 43 & 29 \end{bmatrix}\right) = (40 + 26 + 29) \text{ (mod 59)} = 36$$

and

$$f\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}\right) = a_1 + a_9 \ (\text{mod } 59)$$

$$f\left(\begin{bmatrix} 20 \\ 22 \\ 40 \\ 31 \\ 21 \\ 10 \\ 11 \\ 26 \\ 52 \end{bmatrix}\right) = (20 + 52 \ (\text{mod } 59)) = 13.$$

Thus f is a linear functional on M.

By using these linear functional concept is an extended way to any structure over which the vector space or linear algebra is defined.

Example 4.75: Let

$$T = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \text{ where } a_i \in L_{23}(7), *, \ \ 1 \leq i \leq 6\right\}$$

be a loop linear algebra over the loop $L_{23}(7)$.

Example 4.76: Let

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in L_{29}(8), * \right\}$$

be a loop linear algebra over the loop L_{29} (7).

If p (x) = $7 + 3x^2 + 5x^3 \in W$ and $4 \in L_{29}$ (8) then

$$\begin{aligned} 4p(x) &= 4\,(7 + 3x^2 + 5x^3) \\ &= 4 * 7 + 4 * 3x^2 + 4 * 5x^3 \\ &= (56 - 28) + (24 - 28)x^2 + (40 - 28)x^3 \\ &= (27+1) + 25x^2 + 12x^3 \end{aligned}$$

Suppose $s(x) = 6 + 2x + 9x^4 \in W$

$$\begin{aligned} \text{Now } s(x) * p(x) &= (6 + 2x + 9x^4)\,(7 + 3x^2 + 5x^3) \\ &= 6 * 7 + (2 * 3)x^3 + (9*5)x^7 \\ &= (56–42) + (24–14)\,x^3 + (40-63)x^7 \\ &= 14 + 10x^3 + 6x^7 \in W. \end{aligned}$$

This is the way operations on W, the loop linear algebra is performed.

Suppose instead the polynomial has matrix coefficients we perform the natural product of matrices and get the loop linear algebra of polynomials.

Recall if

$$x = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \text{ and } y = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

be column matrices with entries from L_{19} (3) that is
$a_j, d_i \in L_{19}(3)$; $1 \le i, j \le 4$.

$$\text{Then } x * y = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} * \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} d_1 * a_1 \\ d_2 * a_2 \\ d_3 * a_3 \\ d_4 * a_4 \end{bmatrix}.$$

This is way the natural product of column matrices are defined.

$$\text{Suppose } x = \begin{bmatrix} 3 \\ 8 \\ 1 \\ 5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 4 \end{bmatrix} \text{ then}$$

$$x * y = \begin{bmatrix} 3 \\ 8 \\ 1 \\ 5 \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \\ 7 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3*2 \\ 8*3 \\ 1*7 \\ 5*4 \end{bmatrix} = \begin{bmatrix} 6-6 \\ 9-16 \\ 21-2 \\ 12-10 \end{bmatrix} = \begin{bmatrix} 19 \\ 12 \\ 19 \\ 2 \end{bmatrix}$$

This is the way natural product which is inherited from the loop $L_{19}(3)$ is obtained.

$$\text{Likewise if } x = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 7 \\ e & 3 & 1 \\ 5 & 7 & e \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 2 \\ 1 & e & 2 \\ 4 & e & 5 \end{bmatrix}$$

$$x*y = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 7 \\ e & 3 & 1 \\ 5 & 7 & e \end{bmatrix} * \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 2 \\ 1 & e & 2 \\ 4 & e & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6-6 & 3-4 & 12-2 \\ 9-2 & 15-10 & 6-14 \\ 1 & 3 & 6-2 \\ 12-10 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 18 & 10 \\ 7 & 5 & 11 \\ 1 & 3 & 4 \\ 2 & 7 & 5 \end{bmatrix}.$$

This is the way natural product of 4×3 matrices with entries from the loop are performed.

Finally as in case of complex groupoid vector spaces / linear algebras we can also in case of complex loops and complex quasi loops define the notion of complex loop vector space or complex loop linear algebra over a complex loop.

This is a matter of routine and hence is left as an exercise to the reader.

Now we can define special loop linear algebras and Smarandache special loop linear algebras as follows.

DEFINITION 4.3: *Let*

$$FL = \left\{ \sum_{i=1}^{+} a_i g_i \;\middle|\; a_i \in F \text{ where } F \text{ is a field and } g_i \in L_n(m) \right.$$

a loop of order $n+1$, $n > 3$, n odd with $(m, n)= 1$ and $(m-1, n) = 1$, $1 < m < n$, t finite} be the loop ring of the loop L over the field F. We call FL the special non associative linear algebra over the field F. If the field F is replaced by a S-ring we define FL to be a Smarandache special non associative linear algebra over the S-ring.

These special non associative linear algebra as well as Smaranache special linear algebras are finite dimensional some commutative and a few of them satisfy special identities.

These will be just illustrated by some examples.

Example 4.77: Let Q be the field of rationals. $L = L_{19}(10)$ be a loop. QL the loop ring is a special non associative linear algebra which is commutative and finite dimensional over Q.

Example 4.78: Let R be the field of reals. $L = L_7(3)$ be a loop. RL the loop ring which is a strong non associative linear algebra which is non commutative.

Example 4.79: Let F = Q be the field. $L = L_{19}(2)$ be a loop. FL is the strong non associative linear algebra which satisfies the right alternative law.

Example 4.80: Let R be the field of reals. $L = L_{23}(22)$ be a loop. RL is a strong non associative linear algebra which satisfies the left alternative two.

We have the following theorem the proof of which is direct using the properties of the loops $L_n(m)$.

THEOREM 4.2: *Let*
$L_n(m) \mid n > 3$, n odd, $1 < m < n$, $(n, m) = 1$ and $(n, m-1) = 1$} be the collection of all loops. F be a field. FL the strong non associative linear algebra.

(i) *There exist only one loop $L \in L_n$ with $L = L_n$ (n+1/2) such that FL is a commutative strong non associative linear algebra.*

(ii) *There exists only one loop $L \in L_n$ with $L = L_n$ (2) which is such that FL is a strong non associative linear algebra which satisfies the right alternative identity (that is right alternative strong non associative linear algebra)*

(iii) *There exists one and only one loop $L \in L_n$ with $L = L_n$ (n–1) such that the strong non associative linear algebra satisfies the left alternative identity, that is a left alternative strong non associative algebra.*

(iv) *There exists no loop $L \in L_n$ such that FL the strong non associative linear algebra satisfies the alternative identity, that is FL for no L is an alternative strong non associative linear algebra.*

(v) *For every $L \in L_n$, the strong non associative linear algebra FL is always non associative.*

(vi) *For no $L \in L_n$ we have the strong non associative linear algebra FL is a Bol strong non associative linear algebra.*

(vii) *No strong non associative linear algebra FL is a Moufang strong non associative linear algebra.*

We will give examples of S-strong non associative linear algebras over the S-ring.

Example 4.81: Let P be a S-ring $L = L_n$ (m) be a loop. PL is the loop ring of the loop L over the S-ring P. PL is defined as the S-strong non associative linear algebra over P.

Example 4.82: Let $P = Z_6$ a S-ring $L = L_9$ (8) be a loop PL is a S-strong non associative linear algebra over P. Infact PL has only finite number of elements in them.

Example 4.83: Let $P = Z_{15}$ be a S-ring. $L = L_{19}(3)$ be a loop. PL is a S-strong non associative linear algebra over P.

Example 4.84: Let P = Z_{216} be a S-ring. L = $L_{19}(18)$ be a loop. PL is a S-strong non associative linear algebra over P. Infact L is a left alternative S-strong non associative linear algebra over P of finite order.

Example 4.85: Let P = Z_{177} be a S-ring. L = L_{23} (2) be a loop. PL is a S-strong non associative linear algebra over the S-ring. Infact P is a right alternative S-strong non associative linear algebra over P.

Example 4.86: Let P = Z_{371} be a S-ring. L = L_{25} (13) be a loop. PL is a commutative S-strong non associative linear algebra over P.

The theorem stated for strong non associative linear algebras over a field hold good in case of S-strong non associative linear algebras over the S-ring.

Chapter Five

APPLICATION OF NON ASSOCIATIVE VECTOR SPACES / LINEAR ALGEBRAS

We see the operations or working of the models in general are not associative. We have situations in which the functioning of the model is non associative. Thus these non associative vector spaces / linear algebras will cater to the needs of the researcher when the operations on them are non associative.

Certainly these new structures will be much useful and most utilized one in due course of time, when researchers scientists and technologists become familiar with these algebraic structures. Further it is needless to say the polynomials as linear algebras / vectors spaces are non associative finding solutions or solving equations happens to be a challenging research.

These non associative linear algebras / vector spaces using matrices will be useful when the working field happens to be a non associative one. It is pertinent to mention only when this study becomes familiar among researchers more and more applications would be found.

Chapter Six

SUGGESTED PROBLEMS

In this chapter we suggest two hundred and fourteen number of problems, some of which are simple, some difficult and some at research level.

1. Find a groupoid of column matrices built using Z_{14}.

2. Let $G = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right| a_i \in Z_{15}, 1 \leq i \leq 4, *, (6, 10) \right\}$ be a groupoid.
   (i) Is G a S-groupoid?
   (ii) Find order of G?
   (iii) Is G commutative?
   (iv) Is G an alternative groupoid?
   (v) Is G a Bol groupoid?
   (vi) Is G a Moufang Groupoid?

3. Give an example of a square matrix groupoid which is not idempotent.

4. Let

$$M = \left\{\left.\begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{24} \\ a_{25} & a_{26} & \dots & a_{32} \end{bmatrix}\right| a_i \in Z_{25},\ 1 \le i \le 32,\ *,\right.$$

(2, 3)} be a groupoid?

(i) Does M satisfy any one of the special identities?

(ii) Can M be a S-groupoid?

(iii) Does M contain subgroupoids?

5. Let $T = \left\{\left.\sum_{i=0}^{\infty} a_i x^i \right| a_i \in Z_{22},\ *, (11, 33)\right\}$ be a groupoid.

(i) Prove T is infinite.

(ii) Find subgroupoids if any.

(iii) Is T a S-groupoid?

(iv) Can T satisfy any of the special identities?

6. Let $W = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i \in Z_{43}, 1 \le i \le 9, *, (3, 0)\right\}$ be a groupoid.

(i) Prove W is finite.

(ii) Is W a S-groupoid?

(iii) Find S-subgroupoids if any in W.

(iv) Does W satisfy any of the special identities?

(v) Prove W is non associative.

(vi) Prove W is non commutative.

7. Let S = $\left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (t_1, t_2, t_3, t_4, t_5) \text{ with } t_j \in Z_{20},\right.$ $1 \leq j \leq 5$, $*$, $(5, 16)\}$ be a row matrix coefficient polynomial groupoid.

(i) Prove S satisfies all special identities like Bol, Moufang, alternative and P.

(ii) Is S a Smarandache groupoid?

(iii) Prove S is non commutative.

(iv) Find substructures in S.

(v) Prove these special polynomial groupoids are non associative.

8. Let P = $\left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{10} \end{bmatrix} \text{ with } m_j \in Z_{19},\ 1 \leq j \leq 10,\right.$ $*$, $(5, 4)\}$ be a groupoid.

(i) Is P a S-Moufang groupoid?

(ii) Prove P is non commutative.

(iii) Is P a S-groupoid?

(iv) Find S-Bol subgroupoids if any in P?

(v) Is P an idempotent groupoid?

(vi) Find in P subgroupoids which are not S subgroupoids.

9. Let $G = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \right.$ where $a_i \in Z_{12}$, $1 \leq i \leq 40$, $*$, $(6, 7)\}$ be a matrix coefficient polynomial groupoid.

(i) Prove G is a S-strong Bol groupoid.

(ii) Is G a S-strong P-groupoid?

(iii) Prove G is a S-strong alternative groupoid.

(iv) Is G a S-strong Moufang groupoid?

10. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right.$ where $a_i \in Z_{24}$, $1 \leq i \leq 9$, $*$, $(3, 22)\}$ be a matrix coefficient polynomial groupoid.

(i) Is M a P-groupoid?

(ii) Is M a S-groupoid?

(iii) Does M satisfy any of the special identities?

(iv) Find S-subgroupoids if any in M.

(v) Does M contain a subgroupoid which is not a S-subgroupoid?

11. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_9 \end{bmatrix} \right.$ with $m_j \in Z_{29}$, $*$, $(11, 26)\}$ be a column matrix coefficient polynomial groupoid.

(i) Is M a S-groupoid?

(ii) Is M a S-Bol groupoid?

(iii) Is M a S-moufang groupoid?

(iv) Find a subgroupoid in M which is not a S-subgroupoid.

12. Find any interesting properties about matrix groupoid.

13. What are the special features enjoyed by polynomial groupoids?

14. Determine some nice properties associated with matrix coefficient polynomial groupoid.

15. Describe some new features of non associative semilinear algebras.

16. Let M = $\{Q^+ \cup \{0\}, *, (0, 4)\}$ be the non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

(i) Find non associative semilinear subalgebras of M over S.

(ii) Is it possible to write M as a direct sum of subsemilinear algebras? Justify your claim.

17. Let $P = \{(a, b, c, d) \mid a, b, c, d \in Z^+ \cup \{0\}, *, (7, 7)\}$ be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

18. Let $P = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{10} \end{array}\right] \middle| a_i \in Z^+ \cup \{0\}, 1 \leq i \leq 10, *, (7, 9) \right\}$ be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

(i) Can P be written as a direct sum of semilinear subalgebras?

(ii) Can P be written as a pseudo direct sum of semilinear subalgebras?

(iii) Is P quasi simple?

19. Let $M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in R^+ \cup \{0\}, 1 \leq i \leq 9, *, (24, 7)\right\}$ be a non associative semilinear algebra over the semifield $Q^+ \cup \{0\}$?

(i) Is $T = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \end{bmatrix} \middle| a_i \in Q^+ \cup \{0\}, 1 \leq i \leq 3, *, (24, 7)\right\} \subseteq M$; a non associative semilinear algebra of M over the semifield $Q^+ \cup \{0\}$.

(ii) Prove M is not quasi simple.

(iii) Can M be written as a direct sum?

20. Let $M = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| a_i \in Z^+ \cup \{0\}, 1 \leq i \leq 30, *, (0, 9)\right\}$ be a non associative semilinear algebra over the semifield $Z^+ \cup \{0\} = F$.

(i) Is S quasi simple?

(ii) Can S be written a direct sum of non associative semilinear algebras over $F = Z^+ \cup \{0\}$?

(iii) Define a semilinear operator T on S, so that T^{-1} does not exist.

(iv) Suppose Hom_F (S, S) is the collection of all semilinear operators. What is the algebraic structure enjoyed by Hom_F (S, S)?

21. Let T = M = $\left\{\left.\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{12} \end{bmatrix}\right| m_j \in Z^+ \cup \{0\}, 1 \leq i \leq 12, *, (3, 2)\right\}$ be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

(i) Write non associative semilinear subalgebras of T over S.

(ii) Write T as a direct sum.

(iii) Is T quasi simple?

(iv) Is W = $\left\{\left.\begin{bmatrix} m_1 \\ m_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| m_1, m_2 \in Z^+ \cup \{0\}, *, (3, 2)\right\} \subseteq T$ a non associative semilinear subalgebra of T over S.

22. Let M = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i \in Z^+ \cup \{0\}, 1 \leq i \leq 9, *, (3, 10)\right\}$ be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

(i) Find non associative semilinear subalgebras of M over S.

(ii) Is T = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right|$ $a_i \in Z^+ \cup \{0\}$, $1 \leq i \leq 3$, *, (3, 10)} $\subseteq$ M a semilinear subalgebra of M over S?

23. Let V = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in Z^+ \cup \{0\}$, *, (3, 0)} be a non associative semilinear algebra over the semifield S = $Z^+ \cup \{0\}$.

(i) Find non associative semilinear subalgebras of V over S.

(ii) Is V quasi simple?

24. Let M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in R^+ \cup \{0\}$, *, (2, 2)} be a non associative semilinear algebra over the semifield S = $Z^+ \cup \{0\}$.

(i) Prove M is a commutative non associative semilinear algebra.

(ii) Can M be written as a direct sum of semilinear subalgebras?

25. Let M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in Z^+ \cup \{0\}$, *, (3, 21)} be a non associative semilinear algebra over the semifield S = $Z^+ \cup \{0\}$.

(i) Find some interesting properties enjoyed by M. Distinguish between usual associative semilinear algebra of polynomials with coefficients from $Z^+ \cup \{0\}$ and M.

26. Let $T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in (d_1, d_2, \ldots, d_{10});\ d_j \in Z^+ \cup \{0\},\right.$ $\left. 1 \leq j \leq 10,\ *,\ (3, \sqrt{3}) \right\}$ be a non associative semilinear algebra over the semifield $S = Q^+ \cup \{0\}$.

(i) Find the special features enjoyed by row matrix coefficient polynomials.

(ii) Find non associative sublinear algebras of T over S.

(iii) Is T quasi simple?

(iv) Can T be written as a direct sum of non associative semilinear subalgebras?

27. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{15} \end{bmatrix},\ t_j \in Z^+ \cup \{0\},\ *,\ (7, 2),\right.$ $\left. 1 \leq j \leq 15 \right\}$ be a non associative semilinear algebra over the semifield $S = Z^+ \cup \{0\}$.

(i) Find all interesting features enjoyed by M.

(ii) Can M be written as a pseudo direct sum of non associative semilinear subalgebras?

28. Let $W = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \text{ where } t_j \in Z^+ \cup \{0\},\ *,\right.$ $\left. (3, 20),\ 1 \leq j \leq 4 \right\}$ be a non associative semilinear algebra of square matrix coefficient polynomials.

(i) Is W quasi simple?

(ii) Write W as a direct sum! (Is it possible).

(iii) Define T : W $\to$ W so that T^{-1} exists. (T a semilinear operator on W).

29. Let $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \\ m_{10} & m_{11} & m_{12} \\ m_{13} & m_{14} & m_{15} \\ m_{16} & m_{17} & m_{18} \end{bmatrix} m_j \in Q^+ \cup \{0\}, *, (0, 19), 1 \leq j \leq 18 \right\}$ be a matrix coefficient polynomial non associative semilinear algebra over the semifield $S = Q^+ \cup \{0\}$.

(i) Prove V is not quasi simple.

(ii) Find semilinear subalgebras of V over S.

(iii) Can V be written as a direct sum?

30. Obtain some special and interesting features enjoyed by strong non associative semilinear algebra defined over the field F.

31. Distinguish between strong non associative semilinear algebras and non associative semilinear algebras.

32. Let V = {Q, *, (3, 0)} be a strong non associative semilinear algebra over Q.

(i) Is it possible to write V as direct sum?

(ii) Can V be written as a direct sum?

33. Let W = {R, *, (8, 8)} be a strong non associative semilinear algebra over Q.

(i) Is W quasi simple?

(ii) Can W be written as a pseudo direct sum of semilinear subalgebras?

(iii) Find strong non associative semilinear subalgebras of W.

(iv) Find T : W → W so that T^{-1} does not exist.

34. Let P = {R, *, (13, 0)} be a strong non associative semilinear algebra over the field R.

(i) Is P quasi simple?

(ii) Prove P is non commutative?

(iii) Does P contain strong non associative semilinear subalgebras?

35. Let S = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in R, 1 \leq i \leq 5, *, (3, 2)\}$ be a strong non associative semilinear algebra over the field R.

(i) Can R be written as direct sum of subsemilinear algebras?

36. Let M = $\left\{\left.\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{15} \end{bmatrix} \right| p_j \in Q, 1 \leq i \leq 15, *, (5, 5)\right\}$ be a strong non associative semilinear algebra over the field Q.

(i) Show M is quasi simple.

(ii) Is $P = \left\{\left.\begin{bmatrix} a \\ a \\ \vdots \\ a \end{bmatrix}\right| a \in Q, *, (5, 5)\right\} \subseteq M$ a strong non associative semilinear subalgebra of M over Q?

37. Let

$$P = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right| a_i \in Q,\ 1 \leq i \leq 16, *, (3, 2)\right\}$$

be a strong non associative square matrix semilinear algebra over the field Q.

(i) Find all special features enjoyed by P.

(ii) Is P quasi simple?

(iii) Can P be written as a direct sum?

(iv) Can P be written as a pseudo direct sum?

38. Let

$$W = \left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix}\right| a_i \in Q,\ 1 \leq i \leq 40, *, (7, 0)\right\}$$

be a strong non associative semilinear algebra over the field Q.

(i) Is W quasi simple?

(ii) Can W be expressed as a direct sum of strong non associative semilinear subalgebras?

(iii) Find a subsemilinear subalgebras of W.

39. Let

$$T = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{32} \\ a_{33} & a_{34} & \ldots & a_{40} \\ a_{41} & a_{42} & \ldots & a_{48} \\ a_{49} & a_{50} & \ldots & a_{56} \\ a_{57} & a_{58} & \ldots & a_{64} \\ a_{65} & a_{66} & \ldots & a_{72} \end{array} \right] \middle| \; a_j \in R,\ 1 \leq j \leq 72,\ *,\ (3, 3) \right\}$$

be a strong non associative semilinear algebra over Q.

(i) Prove T is commutative.

(ii) Prove T is non associative.

(iii) Prove T is quasi simple.

(iv) Find a subsemilinear algebra of T over Q.

40. Let $W = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Q,\ *,\ (2, 0) \right\}$ be a strong non associative semilinear algebra over Q.

(i) Is W quasi simple?

(ii) Find a semilinear subalgebra of W over Q.

(iii) Can W be written as pseudo direct sum of semilinear subalgebras?

41. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (d_1, d_2, d_3, \ldots, d_7) \text{ where } d_j \in Q, 1 \leq j \leq 7,\ *,\ (7, 7) \right\}$ be a strong row matrix coefficient polynomial non associative semilinear algebra over the field Q.

(i) Enumerate all the special features enjoyed by M.

(ii) Prove M is commutative.

(iii) Find M as a direct sum.

42. Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_{20} \end{bmatrix}, m_j \in Q, 1 \leq j \leq 20, *, (10, 12) \right\}$$

be a strong non associative semilinear algebra over the field Q.

(i) Prove S is non commutative and non associative.

(ii) Can S be written as a direct sum?

(iii) Does S satisfy any of the special identities?

(iv) Find T : S $\rightarrow$ Q so that T is a linear functional.

(v) Find T : S $\rightarrow$ S so that T^{-1} exist (T a semilinear operator).

43. Let $$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \\ d_{10} & d_{11} & d_{12} \\ d_{13} & d_{14} & d_{15} \\ d_{16} & d_{17} & d_{18} \end{bmatrix}, d_j \in Q, 1 \leq j \leq 18, *, (10, 0) \right\}$$ be a strong non associative semilinear algebra over the field Q.

(i) Find strong non associative semilinear subalgebras of M.

(ii) Prove M is non commutative.

(iii) Find f : M → Q a linear functional.

(iv) Find the structure of Hom_Q (M, Q).

(v) Find the structure of Hom_Q (M, M).

44. Let $P = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \\ d_9 & d_{10} & d_{11} & d_{12} \\ d_{13} & d_{14} & d_{15} & d_{16} \end{bmatrix}, d_i \in R, 1 \leq i \leq 16, *, (3, 3)\right\}$ be strong non associative semilinear algebra over the field R.

(i) Find substructures of P.

(ii) Is P quasi simple?

(iii) Prove P is commutative.

(iv) Find f : P → R a linear functional.

(v) Find T : P → P; T a linear operator on P.

45. Let $S = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \\ \vdots & \vdots & \vdots & \vdots \\ d_{37} & d_{38} & d_{39} & d_{40} \end{bmatrix}, d_j \in Q, 1 \leq j \leq 40, *, (20, 1)\right\}$ be strong non associative semilinear algebra over the field Q.

(i) Show S is quasi simple.

(ii) Find Hom_Q (S, Q), (that is L(S,Q)).

(iii) Find Hom_Q (S, S).

46. Let P = $\{a_i \in Z_{11}, (3, 7), *\}$ be a strong non associative semilinear algebra over the field Z_{11}.

(i) Prove P is finite.

(ii) Does P have subsemilinear algebras?

(iii) Can P be written as direct sum?

47. Let S = $\{a_i \mid a_i \in Z_{23}, *, (3, 0)\}$ be a strong non associative semilinear algebra over the field Z_{23}.

(i) Find order of S. (Is o(S) = 23?).

(ii) Is S commutative?

(iii) Find subsemilinear algebras of S.

(iv) Find f : S $\rightarrow$ Z_{23} a linear functional.

(v) Find T : S $\rightarrow$ S a linear operator on S so that T^{-1} exist.

(vi) Find $\text{Hom}_{Z_{23}}(S, Z_{23})$.

48. Let M = $\{a_i \mid a_i \in Z_{19}, *, (3, 13)\}$ be a strong non associative semilinear algebra over the field Z_{19}.

(i) Find $\text{Hom}_{Z_{19}}(M, M)$.

(ii) Find $\text{Hom}_{Z_{19}}(M, Z_{19})$.

(iii) Find substructures in M.

(iv) Does M satisfy any of the special identities?

49. Let P = $\{a_i \mid a_i \in Z_{43}, *, (7, 0)\}$ be a strong non associative semilinear algebra over the field Z_{43}.

(i) Find $\text{Hom}_{Z_{43}}(P, P)$.

(ii) Prove P is non commutative.

(iii) Find a subsemilinear algebra of P (Is it possible?)

(iv) Does P satisfy any of the special identities?

50. Let P = $\{(a_1, a_2, a_3, \ldots, a_8) \mid a_i \in Z_7, *, (4, 4)\}$ be a strong non associative semilinear algebra over the field Z_7.

(i) Find any of the special properties enjoyed by P.

(ii) Is P commutative?

51. Let S = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z_{11}, *, (9, 0)\}$ be a strong non associative semilinear algebra over the field Z_{11}.

(i) Prove S is non commutative.

(ii) Find a strong non commutative semilinear subalgebra of S over Z_{11}.

52. Let M = $\{(a_1, a_2, a_3) \mid a_i \in Z_{71}, 1 \leq i \leq 3, *, (43, 29)\}$ be a strong non associative semilinear algebra over the field Z_{71}.

(i) Prove M is non commutative.

(ii) Is T = $\{(a, a, a) \mid a \in Z_{71}, *, (43, 29)\} \subseteq$ M a string non associative semilinear subalgebra of M over Z_{71}?

53. Obtain some nice properties enjoyed by strong non associative semilinear algebras built over the prime field Z_p.

54. Let W = $\left\{\left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{array}\right] \middle| \; a_i \in Z_{13}, 1 \leq i \leq 15, *, (9, 5)\right\}$ be a strong non associative semilinear algebra defined over the field Z_{13}.

(i) Does W satisfy any of the special identities?

(ii) Find a strong non associative semilinear subalgebra of W over Z_{13}.

(iii) Is M = $\left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_i \in Z_{13}, 1 \leq i \leq 5, *, (9, 5)\right\}$ be a strong non associative semilinear subalgebra of W? Justify your claim.

55. Let S = $\left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix}\right| a_i \in Z_2, *, (1, 0)\right\}$ be a strong non associative semilinear algebra defined over the field Z_2.

(i) Find number of elements in S.

(ii) Can S have strong non associative semilinear subalgebras?

(iii) Does S satisfy any of the special identities?

(iv) Is T = $\left\{\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\\vdots\\0\end{bmatrix}\right|$ *, (1, 0)} $\subseteq$ S a strong non associative semilinear subalgebra? Justify.

56. Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{24} \end{bmatrix}\right| a_i \in Z_3, 1 \le i \le 24, *, (2, 3)\}$$

be a strong non associative semilinear algebra defined over the field Z_3.

(i) Find the order of S.

(ii) Find strong non associative semilienar subalgebras of S over Z_3.

(iii) Does S satisfy any of the special identities?

(iv) Find Hom_{Z_3} (S, S).

57. Let

$$P = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{13} \\ a_{14} & a_{15} & \ldots & a_{26} \\ a_{27} & a_{28} & \ldots & a_{39} \\ a_{40} & a_{41} & \ldots & a_{52} \end{bmatrix}\right| a_i \in Z_5, 1 \le i \le 52, *, (3, 3)\}$$

be a strong non associative semilinear algebra defined over the field Z_5.

(i) Prove P is commutative.

(ii) Find subsemilinear algebras of P.

(iii) What is the order of P?

(iv) Find L (P, Z_5).

(v) Find a semilinear operator T : V → V such that T^{-1} exists.

(vi) If T, S, R are semilinear operators on S will be composition (T o S) o R = To (S o R)?

58. Let T = $\left\{\left[\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & \cdots & \cdots & a_{10} \\ a_{11} & a_{12} & \cdots & \cdots & a_{15} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{96} & a_{97} & a_{98} & a_{99} & a_{100} \end{array}\right] \middle| \; a_i \in Z_3, 1 \leq i \leq 100, *, (2, 2)\right\}$ be a strong non associative semilinear algebra defined over the field Z_3.

(i) Find order of T.

(ii) Is T a strong non associative idempotent semilinear algebra over Z_3?

59. Give an example of a strong non associative semilinear idempotent algebra built using matrices over the field Z_5.

60. Does there exist a strong non associative Bol semilinear algebra of matrices built using Z_{13}?

61. Does there exist a strong non associative Moufang semilinear algebra of matrices built using Z_7?

62. Does there exist a strong non associative P-semilinear algebra of matrices built using Z_{19}?

63. Does there exists a strong non associative alternative semilinear algebra built using matrices over Z_{11}?

64. Does there exist a strong non associative left alternative semilinear algebra built using Z_{43}?

65. Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix} \middle| \; a_i \in Z_{41}, 1 \leq i \leq 25, *, (3, 29) \right\}$$

be a strong non associative semilinear algebra over the field Z_{41}.

(i) Is M finite?

(ii) Is M commutative?

(iii) Find strong non associative semilinear linear subalgebras of M.

(iv) Find $\mathrm{Hom}_{Z_{41}}$ (M, M).

(v) Find L (M, Z_{41}).

66. Let T = $\left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{11}, *, (3, 7) \right\}$ be a strong non associative semilinear algebra over the field Z_{11}.

(i) Prove T is infinite.

(ii) Prove T is non commutative.

67. Find all interesting properties associated with S = $\left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_p \text{ (p a prime), (t, u); } *, t, u \in Z_p \right\}$, the strong non associative field Z_p.

68. Let W = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_3 \right.$ *, (2, 1)} be the strong non associative polynomial semilinear algebra over Z_3.

(i) Is W commutative?

(ii) Find semilinear subalgebras of W over Z_3.

69. Let S = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_2, \right.$ *, (1, 0)} be the strong non associative semilinear algebra over the field Z_2.

(i) Show S is non commutative.

(ii) Does S have strong non associative semilinear subalgebras?

70. Let M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (m_1, m_2, m_3, m_4), m_i \in Z_5, \right.$ *, (2, 2), $1 \leq i \leq 4\}$ be a strong non associative semilinear algebra over the field Z_5.

(i) Prove M is commutative.

(ii) Is M quasi simple?

(iii) Find subsemilinear algebra of M.

71. Let V = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = (p_1, p_2, \ldots, p_{10}); p_j \in Z_3, 1 \leq j \leq 10, \right.$ *, (2, 1)} be a strong non associative semilinear algebra over the field Z_3.

(i) Show V is non commutative.

(ii) Find Hom_{Z_3} (V, V) = T.

(iii) Is T a non associative algebraic structure?

72. Obtain some interesting properties enjoyed by strong non associative semilinear algebras defined over Z_p, p a prime.

73. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (m_1, m_2, m_3);\ m_i \in Z_{71},\ 1 \le i \le 3, *,\right.$ $\left.(0, 8)\right\}$ be a strong non associative semilinear algebra over the field Z_{71}.

(i) Find l $(S, Z_{71}) = P$.

(ii) Is P a non associative structure?

74. Let W = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{20} \end{bmatrix};\ m_j \in Z_{23},\ 1 \le j \le 20, *,\right.$ $\left.(11, 13)\right\}$ be a strong non associative semilinear algebra over the field $F = Z_{23}$.

(i) Find substructures of W.

(ii) Show W is non commutative.

(iii) Find $L(W, Z_{23})$.

75. Let P = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_8 \end{bmatrix};\ t_j \in Z_7,\ 1 \le j \le 8, *, (2, 6)\right\}$ be a strong non associative semilinear algebra over the field $F = Z_7$.

(i) Show P satisfies the idemponent identity.

(ii) Find subsemi linear algebra of P.

(iii) Find $\mathrm{Hom}_{Z_7}(P, P)$.

76. Let $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}; m_j \in Z_5, 1 \le j \le 4, *, (2, 0) \right\}$ be a strong non associative semilinear algebra over the field Z_5.

(i) Show P is non commutative.

(ii) Does P satisfy any of the special identities?

(iii) Find Hom_{Z_5} (P, P).

77. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix}; m_j \in Z_3, 1 \le j \le 9, *, (2, 2) \right\}$ be a strong non associative semilinear algebra over the field Z_3.

(i) Prove M is commutative.

(ii) Prove M is a strong non associative idempotent semilinear algebra over the field Z_3.

(iii) Find L_{Z_3} (M, Z_3).

78. Let $S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & \dots & d_9 \\ d_{10} & \dots & d_{18} \\ \vdots & \vdots & \vdots \\ d_{73} & \dots & d_{81} \end{bmatrix}; d_j \in Z_{11}, 1 \le j \le 81, *, (9, 3) \right\}$ be a strong non associative semilinear algebra over the field Z_{11}.

(i) Does S satisfy any of the special identities?

(ii) Is S commutative?

(iii) Find the algebraic structure enjoyed by $\text{Hom}_{Z_{11}}$ (S, S).

79. Let T = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ \vdots & \vdots & \vdots \\ m_{28} & m_{29} & m_{30} \end{bmatrix}; m_j \in Z_{23}, 1 \le j \right.$ ≤ 30, *, (3, 7)} be a strong non associative semilinear algebra over the field Z_{23}.

(i) Does T satisfy any of the special identities?

(ii) Is T commutative?

(iii) Find the algebraic structure of L (T, Z_{23}).

80. Let P = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & \ldots & d_{20} \\ d_{21} & d_{22} & \ldots & d_{40} \end{bmatrix}; d_j \in Z_{19}, 1 \le j \right.$ ≤ 40, *, (3, 17)} be a strong non associative semilinear algebra over the field Z_{19}.

(i) Is P commutative?

(ii) Does P satisfy any of the special identities?

(iii) Can P have subsemilinear algebras?

(iv) Find the algebraic structure enjoyed by L(T, Z_{23}).

81. Let V = {a_i | $a_i \in Z_6$, (3, 5), *} be a S-non associative semilinear algebra over the S-ring Z_6.

(i) Find order of V.

(ii) Can V satisfy any of the special identities?

82. Let V = {a_i | $a_i \in Z_{12}$, *, (7, 6)} be a S-non associative semilinear algebra.

(i) Find the special properties enjoyed by V.

(ii) Prove V is an S-non associative P-semilinear algebra.

(iii) Prove V is a S-string non associative Bol semilinear algebra.

(iv) Does V contain S non associative semilinear algebras?

83. Let V = {a | a ∈ Z_{22}, *, (7, 16)} be a S-non associative semilinear algebra over the S-ring Z_{22}.

(i) Does V satisfy any of the special identities?

(ii) Can S have subsemilinear algebras which does not satisfy any of the special identities?

(iii) Find the algebraic structure of $\text{Hom}_{Z_{22}}$ (V, V).

(iv) If W = {a | a ∈ Z_{22}, *, (3, 20)} a S-non associative semilinear algebra over the S-ring find $\text{Hom}_{Z_{12}}$ (V, W).

84. Let
V = {a | a ∈ Z_{10}, (3, 8), *} and W = {a | a ∈ Z_{10}, *, (7, 4)} be two S-non associative semilinear algebras over the S-ring Z_{10}.

(i) Find $\text{Hom}_{Z_{10}}(V, W) = T$.

(ii) Find $\text{Hom}_{Z_{10}}(W, V) = S$.

(iii) Can T and S be related in any way?

(iv) Can V and W satisfy the same special identity? Justify.

85. Let W = {a | a ∈ Z_{86}, *, (7, 12)} be a S-non associative semilinear algebra over the S-ring Z_{86}.

(i) Find $\text{Hom}_{Z_{86}}$ (W, W).

(ii) Find L(W, Z_{86}).

(iii) Does W satisfy any of the special identities?

(iv) Prove o(W) = 86.

86. Let $P = \{(a_1, a_2, a_3, a_4) \mid a_i \in Z_{12}, 1 \leq i \leq 4, *, (6, 7)\}$ be a S-non associative semilinear algebra over the S-ring Z_{12}.

(i) Prove P satisfies all the five 5 special identities, Bol, Moufang, P, idempotent and alternative.

(ii) Find $\text{Hom}_{Z_{12}}(P, P)$.

(iii) Find order of P.

87. Let $M = \{(a_1, a_2, \ldots, a_{12}) \mid a_i \in Z_{10}, *, 1 \leq i \leq 12, (5, 6)\}$ be a S-non associative semilinear algebra over the S-ring Z_{10}.

(i) Find the special identities satisfies by M.

(ii) Find L (M, Z_{10}).

88. Let $R = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in Z_{15}, 1 \leq i \leq 10, *, (3, 3)\}$ be a S-non associative semilinear algebra over the S-ring Z_{15}.

(i) Is R commutative?

(ii) Does R satisfy any of the known special identities?

(iii) Find the algebraic structure enjoyed by $\text{Hom}_{Z_{15}}(M, M)$.

89. Let $T = \{(a_1, a_2, \ldots, a_7) \mid a_i \in Z_6, *, (4, 3)\}$ be a non associative semilinear algebra over the S-ring Z_6.

(i) Is T commutative?

(ii) Does T satisfy any of the special identities?

(iii) Find the algebraic structure enjoyed by L (T, Z_6).

90. Let S = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}\right|$ $a_i \in Z_{12}$, $1 \leq i \leq 12$, $*,(4, 9)\}$ be a S-non associative smilinear algebra over the S-ring Z_{12}.

(i) Find the order of S.

(ii) Does S satisfy any of the special identities?

(iii) Find the algebraic structure enjoyed by $\text{Hom}_{Z_{12}}$ (S, S).

(iv) Can S be written as a direct sum of S-non associative semilinear subalgebras of S over Z_{12}?

91. Let P = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix}\right|$ $a_i \in Z_{15}$, $1 \leq i \leq 15$, $*,(12, 4)\}$ be a S-non associative semilinear algebra over the S-ring Z_{15}.

(i) Find the special identities that are true on P.

(ii) Prove P is a S-non associative idempotent semilinear algebra over Z_{15}.

(iii) Find L (P, Z_{15}).

92. Let W = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right|$ $a_i \in Z_6$, $1 \leq i \leq 4$, $*,(3, 4)\}$ be a S-non associative semilinear algebra over the S-ring Z_6.

(i) Find Hom_{Z_6} (W, W).

(ii) Is W a S-non associative strong Bol semilinear algebra?

(iii) Can W be written as a direct sum?

93. Let M = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{11} \\ a_{12} & a_{13} & a_{14} & \ldots & a_{22} \\ a_{23} & a_{24} & a_{25} & \ldots & a_{33} \end{bmatrix}\right| a_i \in Z_{12}, 1 \leq i \leq 33,\right.$ *, (3, 10)} be a S-non associative semilinear algebra over the S-ring Z_{12}.

(i) Prove M is an S-non associative idempotent semilinear algebra.

(ii) Does M satisfy any of the special identities?

(iii) Will M be a S-non associative strong Bol semilinear algebra over Z_{12}?

94. Let M = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{96} & a_{97} & a_{98} & a_{99} & a_{100} \end{bmatrix}\right| a_i \in Z_{15}, 1 \leq i \leq\right.$ 100, *, (12, 4)} be a non associative semilinear algebra over the S-ring Z_{15}.

(i) Does M satisfy any of the special identities?

(ii) Find $\text{Hom}_{Z_{15}}$ (M, M).

(iii) Find the algebraic structure enjoyed by L (M, Z_{15}).

95. Let $M = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right| a_i \in Z_{42}, 1 \leq i \leq 16, *, (20, 23)\right\}$ be a S-non associative semilinear algebra over the S-ring Z_{42}.

(i) Does M satisfy any of the special identities?

(ii) Prove M is an idempotent S-non associative semilinear algebra over the S-ring Z_{42}.

(iii) Find the algebraic structure enjoyed by $\text{Hom}_{Z_{42}}$ (M, M).

(iv) Find the algebraic structure enjoyed by L (M, Z_{42}).

96. Let $M = \left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right| a_i \in Z_{10}, 1 \leq i \leq 4, *, (5, 6)\right\}$ be a S-non associative semilinear algebra over the S-ring Z_{10}.

(i) Find the order of M.

(ii) Can M have subsemilinear algebras?

(iii) Prove M is a S-non associative idempotent semilinear algebra.

(iv) Can M satisfy any special identity?

(v) Can M be written as a direct sum of substructures?

(vi) Find f : M → Z_{10}, f is non trivial.

97. Let $P = \left\{ \left[\begin{matrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{matrix} \right] \right|$ $a_i \in Z_{33}$, $1 \leq i \leq 25$, $*$, (16, 18)} be a S-non associative semilinear algebra of square matrices.

(i) Is P an idempotent S-non associative semilinear algebra?

(ii) Find $\text{Hom}_{Z_{33}}$ (P, P).

(iii) Find L (P, Z_{33}).

(iv) What is the special features enjoyed by P?

98. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in Z_{12}$, $*$, (6, 7)} be the S-non associative semilinear algebra over the S-ring Z_{12}.

(i) Prove M satisfies special identities.

(ii) Find $T : M \to M$ so that T^{-1} exists.

(iii) Find the algebraic structure enjoyed by L (M, Z_{12}).

(iv) Find any special difference between M and usual polynomial Z_{12} [x].

99. Let $T = \left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in Z_{46}$, $*$, (2, 2)} be the S-non associative semilinear algebra over the S-ring Z_{46}.

(i) Prove T is commutative.

(ii) Find $\text{Hom}_{Z_{46}}$ (T, T).

(iii) What is the algebraic structure enjoyed by L (T, Z_{46})?

(iv) Does T satisfy any of the special identities?

100. Let V $= \left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in Z_{25}$, *, (10, 5)} be the S-non associative semilinear algebra over the S-ring Z_{25}.

(i) Find all the special identities satisfied by V.

(ii) Prove V is not a S-non associative semilinear idempotent algebra over Z_{25}.

(iii) Find $\text{Hom}_{Z_{25}}$ (V, V).

101. Let P = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in Z_{12}$, *, (4, 9)} be a S-non associative semilinear algebra over the S-ring Z_{12}.

(i) Find the algebraic structure enjoyed by $\text{Hom}_{Z_{12}}$ (P, P).

(ii) Prove P satisfies several special identities.

102. Let T = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in Z_{15}$, *, (11, 0)} be a S-non associative semilinear algebra over the S-ring Z_{15}.

(i) Find $L_{Z_{15}}$ (T, T).

(ii) Which of the special identities are satisfied by T?

(iii) Prove T is non commutative.

103. Let S = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ $a_i = (d_1, d_2, d_3, d_4)$; $d_j \in Z_{12}$, *, (8, 0), $1 \leq j \leq 4$} be a S-non associative semilinear algebra over the S-ring Z_{12}.

(i) Prove S is non commutative.

(ii) Find $\text{Hom}_{Z_{12}}(S, S)$.

(iii) Find the special identities satisfied by S.

104. Let W = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (d_1, d_2, \ldots, d_q); d_j \in Z_{15}, *, (10, 6)\right\}$ be a S-non associative semilinear algebra over the S-ring Z_{15}. Prove W satisfies several of the special identities.

105. Let M = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{20} \end{bmatrix}; t_j \in Z_6, 1 \leq j \leq 20, *, (4, 3)\right\}$ be a S-non associative semilinear algebra over the S-ring Z_6.

(i) Is M a S-strong bol semilinear algebra?

(ii) Find $\text{Hom}_{Z_6}(M, M)$.

(iii) Is M an S-strong idempotent semilinear algebra?

106. Let P = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}; t_j \in Z_{21}, 1 \leq j \leq 5, *, (10, 12)\right\}$ be a S-non associative semilinear algebra over the S-ring P.

(i) Find $\text{Hom}_{Z_{21}}(P, P) = S$, what is the algebraic structure enjoyed by S.

(ii) Prove P is non commutative.

(iii) Does P satisfy any of the special identities?

107. Let $S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ \vdots & \vdots & \vdots \\ d_{28} & d_{29} & d_{30} \end{bmatrix}; d_j \in Z_{15}, *, (4, 12), 1 \leq j \leq 30 \right\}$ be a S-non associative semilinear algebra over the S-ring Z_{15}.

(i) Is S-commutative?

(ii) Does S satisfy any of the special identities?

(iii) Find L (S, Z_{15}).

(iv) Study the algebraic structure enjoyed by $\text{Hom}_{Z_{15}}$ (S, S).

108. Let $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 & m_2 & \ldots & m_{10} \\ m_{11} & m_{12} & \ldots & m_{20} \\ m_{21} & m_{22} & \ldots & m_{30} \end{bmatrix}; m_j \in Z_{12}, *, (6, 4), 1 \leq j \leq 30 \right\}$ be a S-non associative semilinear algebra over the S-ring Z_{12}.

(i) Does P satisfy any of the special identities?

(ii) Find the algebraic structure enjoyed by $\text{Hom}_{Z_{12}}$ (P, P).

(iii) Is P commutative?

(iv) Does P enjoy any other special features?

109. Let $T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{bmatrix}; d_j \in Z_{15}, *, (2, 0), 1 \leq j \leq 9 \right\}$ be a S-non associative semilinear algebra over the S-ring Z_{15}. Study and describe all the special features enjoyed by T.

110. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} m_1 & m_2 & \cdots & m_6 \\ m_7 & m_8 & \cdots & m_{12} \\ \vdots & \vdots & & \vdots \\ m_{31} & m_{32} & \cdots & m_{36} \end{bmatrix}\right.$; $m_j \in Z_6$, *, (3, 4), $1 \leq j \leq 36\}$ be a S-non associative semilinear algebra over the S-ring Z_6.

(i) Prove S satisfies several of the special identities.

(ii) Find Hom_{Z_6} (S, S).

(iii) Find L (S, Z_6).

(iv) Is S commutative?

111. Obtain the special features enjoyed by double Smarandache non associative semilinear algebras.

112. Give examples of double Smarandache non associative semilinear algebras.

113. Give an example of a double Smarandache strong Moufang non associative semilinear algebra.

114. Give an example of a double Smarandache strong non associative Bol semilinear algebra.

115. Give examples of double Smarandache strong non associative P-semilinear algebras.

116. Give examples of double Smarandache strong non associative alternative semilinear algebra.

117. Give examples of groupoid rings which are strong non associative linear algebras.

118. Obtain some special, distinct and interesting properties enjoyed by strong non associative linear algebras which are groupoid rings.

119. Obtain some special properties of groupoid rings which satisfy the special identities and which are DS non associative semilinear algebras.

120. Mention the special features enjoyed by matrix groupoid rings FG which are DS non associative semilinear algebras where F is a S-ring.

121. Study the special properties enjoyed by polynomial groupoid rings which are DS non associative semilinear algebras.

122. Find applications of strong non associative semilinear algebras.

123. Find applications of DS non associative semilinear algebras.

124. Let G = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \\ a_{28} & a_{29} & \ldots & a_{36} \end{bmatrix}\right.$ where $a_i \in Z_{12}$, $1 \leq i \leq 36$, $*, (4, 7)\}$ be a matrix groupoid. $S = Z_{12}$ be the S-ring. SG be the groupoid ring that is SG is also a DS non associative semilinear algebra over the S-ring Z_{12}.

(i) Does SG satisfy any of the special identities?

(ii) Find Hom_S (SG, SG).

(iii) Is SG commutative?

(iv) Find any special features enjoyed by SG.

125. Let $G = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \\ a_{15} & a_{16} \end{bmatrix} \right| a_i \in Z_{15}, *, 1 \leq i \leq 16, (11, 5), * \}$ be a groupoid and Q be the field. QG the groupoid ring is a strong non associative semilinear algebra over Q.

(i) Is QG commutative?

(ii) Find L(QG, Q).

(iii) Find Hom_S (QG, QG).

(iv) Does QG satisfy any of the special identities?

(v) Prove QG is a strong non associative idempotent semilinear algebra over Q.

(vi) Can QG have substructures?

126. Obtain any interesting properties about groupoid rings which are

(i) Strong non associative semilinear algebras over a field.

(ii) DS non associative semilinear algebras over $S = Z_n$.

(iii) S - non associative semilinear algebras over a S-ring.

127. Determine some special features enjoyed by non associative linear algebras.

128. Find some interesting properties enjoyed by the non associative linear algebra L_n (m), n prime over the field Z_n.

129. Let $V = \{L_{17}(8)\}$ be a non associative linear algebra over the field Z_{17}.

(i) Does V enjoy any special properties?

(ii) Does V satisfy any special identity?

(iii) Prove V is simple.

130. Let $V = \{L_{11}(6)\}$ be a non associative linear algebra over the field.

(i) Prove $L_{11}(6)$ is commutative.

(ii) Find the number of basis of V.

131. Let $V = L_{43}(40)$ be a non associative linear algebra over the field Z_{43}.

(i) Find the number of basis in V.

(ii) Is V commutative?

(iii) Can V have any substructures?

(iv) Is V simple?

132. Let $V = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in L_{23}(3); 1 \leq i \leq 10\}$ be a non associative linear algebra over the field Z_{23}.

(i) How many sets of basis are in V over Z_{23}?

(ii) Is V commutative?

(iii) Can V be non associative linear subalgebras over the field Z_{23}?

(iv) Find a linear operator T on V so that T^{-1} exists.

(v) Obtain any other interesting properties associated with V.

133. Let $M = \{(a_1, a_2, \ldots, a_7) \mid a_i \in L_7(3), 1 \leq i \leq 7, *\}$ be a non associative row matrix linear algebra over the field Z_7.

(i) Find a basis of M over Z_7.

(ii) What is the dimension of M over Z_7?

(iii) Find non associative linear subalgebras of M over Z_7.

(iv) Does M satisfy any of the special identities?

(v) Find $\text{Hom}_{Z_7}(M, M)$.

134. Let $P = \{(a_1, a_2, a_3) \mid a_i \in L_{47}(3), 1 \leq i \leq 3, *\}$ be a non associative row matrix linear algebra over Z_{47}.

(i) Find a basis of P.

(ii) Find dimension of P.

(iii) Find the algebraic structure enjoyed by L (M, Z_7).

135. Let $T = \left\{\left.\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}\right| d_i \in Z_{29}(3)\ *, 1 \leq i \leq 3\right\}$ be a non associative column matrix linear algebra over the field Z_{29}.

(i) Find dimension of T over Z_{29}.

(ii) Is T commutative?

(iii) What is the algebraic structure enjoyed by $\text{Hom}_{Z_{29}}(T, T)$?

136. Let $P = \left\{ \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{12} \end{bmatrix} \middle| d_j \in Z_{23}\ (7),\ *,\ 1 \leq j \leq 12 \right\}$ be a non associative column matrix linear algebra over the field Z_{23}.

(i) What is dimension of P over Z_{23}?

(ii) Does P satisfy any of the special identities?

(iii) Does P contain more than one non associative column matrix linear subalgebra over Z_{23}? Justify your claim.

137. Let $W = \left\{ \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{12} \end{bmatrix} \middle| d_j \in L_{19}\ (10),\ *,\ 1 \leq j \leq 12 \right\}$ be a non associative column matrix linear algebra over Z_{19}.

(i) What is dimension of W over Z_{19}?

(ii) Is W commutative?

(iii) Is W finite?

(iv) Find $T : W \to W$ so that T^{-1} exists.

138. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{32} \\ a_{33} & a_{34} & \dots & a_{48} \end{bmatrix} \middle| a_i \in L_7\ (4),\ *,\ 1 \leq i \leq 48 \right\}$ be a non associative matrix linear algebra over the field Z_7.

(i) Find a basis of V over Z_7.

(ii) Find the number of elements in Z_7.

(iii) Does V contain sublinear algebras?

(iv) Find L (V, Z_7).

139. Let $M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{17} & a_{18} & \ldots & a_{20} \\ \vdots & \vdots & & \vdots \\ a_{71} & a_{72} & \ldots & a_{80} \end{bmatrix} \right| a_i \in L_5 (3), *, 1 \leq i \leq 8 \right\}$ be a non associative matrix linear algebra over the field Z_5.

(i) Find a basis of M over Z_5.

(ii) Find the number of elements in M.

(iii) Find Hom (M, M).

140. Let $S = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in L_{11} (7), *, 1 \leq i \leq 9 \right\}$ be a non associative matrix linear algebra over the field Z_{11}.

(i) Find dimension of S over Z_{11}.

(ii) Is S commutative?

(iii) Does S satisfy any of the special identities?

141. Let

$P = \left\{ \left. \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \\ \vdots & \vdots & & \vdots \\ d_{37} & d_{38} & d_{39} & d_{40} \end{bmatrix} \right| d_j \in L_{17}(3), *, 1 \leq j \leq 40 \right\}$ be a non associative matrix linear algebra over the field Z_{17}.

(i) Find a basis of P over Z_{17}.

(ii) Is P commutative?

(iii) Does P satisfy any of the special identities?

(iv) Can P have more than one sublinear algebra?

142. Let T = $\left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right| a_i \in L_{59}(30), *, 1 \leq i \leq 4\right\}$ be a non associative matrix linear algebra over the field Z_{59}.

(i) Prove T is commutative.

(ii) Find dimension of T over Z_{59}.

143. Let M = $\left\{\left.\sum_{i=0}^{\infty} a_i x^i \right| a_i \in L_{29}(15), *\right\}$ be a non associative polynomial linear algebra.

(i) Prove M is commutative.

(ii) Find a basis of M over Z_{29}.

(iii) Can M have linear subalgebras?

144. Let V = $\left\{\left.\sum_{i=0}^{\infty} a_i x^i \right| a_i \in L_7(4), *\right\}$ be a non associative polynomial linear algebra over the field Z_7.

(i) Find dimension of V over Z_7.

(ii) Is V commutative?

(iii) Can V have substructures?

145. Give any interesting properties enjoyed by the non associative polynomial linear algebra.

146. Let M = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (d_1, d_2, d_3),\ d_j \in L_{11}(5),\ 1 \le j \le 3, *\right\}$ be a non associative row matrix coefficient polynomial linear algebra over the field Z_{11}.

(i) What is the dimension of M over Z_{11}?

(ii) Is M commutative?

(iii) Find the algebraic structure of $\text{Hom}_{Z_{11}}(M, M)$.

147. Let T = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{12} \end{bmatrix},\ d_j \in L_{19}(8),\ 1 \le j \le 3, *\right\}$

and V = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_j = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \\ d_9 & d_{10} & d_{11} & d_{12} \end{bmatrix},\ d_p \in L_{19}(7),\ 1 \le p \le 12, *\right\}$ be two non associative matrix coefficient linear algebras over the field Z_9.

(i) Find the algebraic structure enjoyed by $\text{Hom}_{Z_{19}}(T, V)$.

(ii) Find a linear transformation f : V $\to$ T so that f^{-1} does not exist.

(iii) Compare the algebraic structure $\text{Hom}_{Z_{19}}(V, V)$ and $\text{Hom}_{Z_{19}}(T, T)$.

148. If V is a non associative linear algebra over the field Z_p (p a prime, p $\ge$ 5).

(i) Study the algebraic structure enjoyed by $\text{Hom}_{Z_p}(V, V)$.

(ii) What is the algebraic structure enjoyed by L (V, Z_p)?

149. Obtain some interesting properties about non associative linear algebra of matrices over a field Z_p, $p \geq 5$.

150. Obtain some interesting properties enjoyed by non associative linear algebra of matrix coefficient polynomials.

151. Let $S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 & d_2 & \ldots & d_{10} \\ d_{11} & d_{12} & \ldots & d_{20} \\ \vdots & \vdots & & \vdots \\ d_{61} & d_{62} & \ldots & d_{70} \end{bmatrix}, d_j \in L_{43}(17), \right.$

$1 \leq j \leq 70, *\}$ be a non associative linear algebra over the field Z_{43}.

(i) Find $\mathrm{Hom}_{Z_{43}}(S, S)$.

(ii) Find $\mathrm{Hom}_{Z_{43}}(S, Z_{43})$.

(iii) Find $T : S \to S$ so that T^{-1} does not exist.

(iv) Find $T_1 : S \to S$ so that T_1^{-1} exists.

152. Let $P = \{L_{33}(17), *\}$ be a S-non associative linear algebra over the S-ring Z_{33}.

(i) Find order of P.

(ii) Does P satisfy any of the special identities?

(iii) Does P have substructures?

(iv) Find $\mathrm{Hom}_{Z_{33}}(P, P)$.

153. Let $M = \{(a, b) \mid a, b \in L_{35}(12), *\}$ be a S-non associative linear algebra over the S-ring Z_{35}.

(i) Find the number of elements in M.

(ii) Can M be written as direct sum of linear subalgebras?

154. Let $P = \{(a_1, a_2, a_3, \ldots, a_7) \mid a_i \in Z_{15}\,(14), 1 \le i \le 7, *\}$ be a S-non associative linear algebra over the S-ring Z_{15}.

(i) Find $\text{Hom}_{Z_{15}}(P, P)$.

(ii) Can P be written as pseudo direct sum of S-non associative linear subalgebras over Z_{15}?

155. Let $V = \left\{ \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{10} \end{bmatrix} \middle| d_j \in L_{77}\,(10), *, 1 \le j \le 10 \right\}$ be a S-non associative linear algebra over the S-ring Z_{77}.

(i) What is the dimension of V over Z_{77}?

(ii) Find (V, Z_{77}).

(iii) Find the number of elements in V.

156. Let $P = \left\{ \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{10} \end{bmatrix} \middle| d_j \in L_{77}\,(10), *, 1 \le j \le 10 \right\}$ be a S-non associative linear algebra over the S-ring Z_{77}.

(i) What is the dimension of V over Z_{77}?

(ii) Find (V, Z_{77}).

(iii) Find the number of elements in V.

157. Let $P = \left\{\left[\begin{array}{cc} d_1 & d_2 \\ d_3 & d_4 \\ \vdots & \vdots \\ d_{19} & d_{20} \end{array}\right] \middle| d_j \in L_{129}(5), *, 1 \leq j \leq 20\right\}$ be a S-non associative linear algebra over the S-ring Z_{129}.

(i) Prove P is finite dimensional.

(ii) Find $\text{Hom}_{Z_{129}}(P, P)$.

(iii) Will $\text{Hom}_{Z_{129}}(P, P)$ be a non associative structure?

(iv) Find order of P.

158. Let W = $a_i = \left[\begin{array}{cccc} d_1 & d_2 & \ldots & d_5 \\ d_6 & d_7 & \ldots & d_{10} \\ d_{11} & d_{12} & \ldots & d_{15} \end{array}\right]$ $d_j \in L_{15}(8), *, 1 \leq j \leq 15\}$ be a S-non associative linear algebra over the S-ring Z_{15}.

(i) Prove V is infinite dimensional.

(ii) Find $\text{Hom}_{Z_{15}}(V, V)$.

159. Let

$V = \left\{\sum a_i x^i \,\middle|\, a_i = (d_1, \ldots, d_{15}), d_j \in L_{159}(47), *, 1 \leq j \leq 15\right\}$

be a S-non associative linear algebra over the S-ring Z_{159}.

(i) Find L (W, Z_{159}).

160. Let V be S-non associative polynomial linear algebra over the S-ring Z_n.

(i) Find $\text{Hom}_{Z_n}(V, V)$.

(ii) Find $L(V, Z_n)$.

161. Let V be a non associative 5×5 matrix linear algebra over the S-ring Z_{55}.

(i) Find $\text{Hom}_{Z_{55}}(V, V)$.

(ii) Find $L(V, Z_{55})$.

162. Find the marked difference between S-non associative linear algebras over a S-ring and a linear algebra over a field.

163. Find some applications of S-non associative linear algebras over a S-ring.

164. What are the benefits of using S-non associative linear algebras over a S-ring?

165. What are the special features enjoyed by groupoid vector spaces?

166. What are the differences between groupoid vector spaces and non associative linear algebras?

167. Give an example of a finite groupoid vector space.

168. Describe finite groupoid linear algebras and illustrate them by examples.

169. Let V = {5Z, 3Z, 11Z} be a set. G = {Z, *, (13, 7)} be a groupoid vector space over the groupoid G.

(i) Write V a direct sum of groupoid vector subspaces over G.

(ii) Write V as a pseudo direct sum of groupoid vector subspaces over G.

(iii) Find Hom_G (V,V).

170. Let V = {10Z, 19Z, 3Z} be a groupoid vector space over the groupoid G.

(i) Find L (V, G).

171. Let $V = \left\{(a_1, a_2, \ldots, a_{10}), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in Z; 1 \leq i \leq 10\right\}$ be a groupoid vector space over the groupoid G = {Z, *, (3, 13)}.

(i) Write V as a direct sum.

(ii) Find subgroupoid vector subspace (Does it exist?).

(iii) Find the algebraic structure enjoyed by L (V, Z).

172. Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{pmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \middle| a_i \in Z, 1 \leq i \leq 40 \right\}$$ be a groupoid vector space over the groupoid G = {Z, *, (11, 11)}.

(i) Find $\text{Hom}_G (V, V)$.

(ii) Write V as a pseudo direct sum.

(iii) Write V as a direct sum.

173. Let $V = \left\{\left.\begin{bmatrix} a_1 & a_2 & ... & a_{10} \\ a_{11} & a_{12} & ... & a_{20} \\ a_{21} & a_{22} & ... & a_{30} \end{bmatrix}\right| a_i \in Z, 1 \leq i \leq 30\right\}$ be a groupoid linear algebra over the groupoid G = {Z, *, (3, 13)}.

(i) Find Hom (V, V).

(ii) Write V as a direct sum.
(Is it possible?)

(iii) Can V be written as a pseudo direct sum?

174. Give some nice results about groupoid vector spaces / linear algebras over the groupoid G = {Z, *, (t, u)}.

175. Let $V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & ... & a_{20} \\ a_{21} & a_{22} & a_{23} & ... & a_{40} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix},\right.$

$\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix}\right| a_i \in Z_{28}, 1 \leq i \leq 40\}$ be a groupoid vector space over the groupoid $G = \{Z_{28}, *, (7, 0)\}$.

(i) Find number of elements in V.

(ii) Write V as a direct sum.

(iii) Find $\text{Hom}_{Z_{28}}$ (V, V).

176. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix} \right.$, where $a_i \in$ Z_{43}, *, (7, 0), $1 \leq i \leq 36\}$ be a groupoid linear algebra over the groupoid G = {Z_{43}, *, (7, 0)}

(i) Find Hom (M, M).

(ii) Find L (M, G).

(iii) Is it possible to write M as a direct sum?

(iv) Does M satisfy any of the special identities as a groupoid?

(v) Does M contain any proper groupoid linear subalgebras?

177. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{21} & a_{22} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, (a_1, a_2, \ldots, a_{13}) \,\right|\, a_i \in Z_{143},\ 1 \leq i \leq 22\}$$ be a groupoid vector space over the groupoid G = {Z_{143}, *, (3, 140)}.

(i) Does V contain any substructures?

(ii) Will V satisfy any of the special identities?

(iii) Write V as a direct sum.

(iv) Write V as a pseudo direct sum.

178. Let

$$V = \left\{ (a_1, a_2, \ldots, a_{12}), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{40} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{32} \end{pmatrix} \right|$$

$a_i \in Z_{148}$, $1 \leq i \leq 40\}$ be a groupoid vector space over the groupoid $G = \{Z_{148}, *, (7, 142)\}$.

(i) Write P as a direct sum.

(ii) Write P as a pseudo direct sum.

(iii) Find Hom (P, P).

(iv) Find the algebraic structure enjoyed by L (P, Z_{148}).

179. Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \end{pmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{40} \end{bmatrix}, \right.$$

$$\left. (a_1, a_2, \ldots, a_{15}), \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| \; a_i \in Z_7,\ 1 \leq i \leq 40\} \text{ be a}$$

groupoid vector space over the groupoid $G = \{Z_7, *, (6, 0)\}$.

(i) Prove M is a non associative and non commutative structure.

(ii) Is M a direct sum?

(iii) Can M have substructures?

(iv) Find Hom (M, M).

(v) Find L (M, Z_7).

180. Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}, \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ \vdots & \vdots & & \vdots \\ a_{91} & a_{92} & \dots & a_{100} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{46} \end{bmatrix} (a_1, a_2, \dots, a_{25}) \right| a_i \in Z_3,\ 1 \le i \le 100\}$$ be a groupoid vector space over the groupoid G = {Z_3, *, (2, 1)}.

(i) Prove M is finite.

(ii) Write M as a direct sum.

(iii) Can M satisfy any of the special identities?

(iv) What is the algebraic structure enjoyed by Hom (M, M)?

181. Let $W = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{pmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i$ $\in Z_{25}$, $1 \le i \le 20\}$ be a groupoid vector space over the groupoid G = {Z_{25}, *, (20, 6)}.

(i) Find the number of elements in W.

(ii) Write W as a pseudo direct sum.

(iii) Find the algebraic structure enjoyed by Hom_G (W, W).

182. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right|$ $a_i \in Z_{43}$, (40, 4), $1 \le i \le$ 16, *} be a groupoid linear algebra over the groupoid Z_{43}.

(i) Does V satisfy and special identity?

(ii) Prove V is non commutative.

(iii) Can V have substructures?

183. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \right|$ $a_i \in Z_{45}$, (20, 26), $1 \le i \le$ 40, *} be a groupoid linear algebra over the groupoid G = {Z_{45}, *, (20, 26)}.

(i) Find Hom (M, M).

(ii) What is the algebraic structure enjoyed by L (M, Z_{45})?

(iii) Does the groupoid linear algebra satisfy any of the special identities?

184. Let P = $\left\{\sum_{i=0}^{\infty} a_i x^i \right|$ $a_i \in Z_{47}$, *, (3, 46)} be a groupoid linear algebra over the groupoid G = {Z_{47}, *, (3, 46)}.

(i) Find substructures of P.
(ii) Is P a commutative groupoid linear algebra?
(iii) Can P be written as a direct sum?

185. Let M = $\left\{\sum_{i=0}^{4} a_i x^i, \sum_{i=0}^{8} d_i x^i \;\middle|\; a_i = (p_1, p_2, p_3, p_4) \text{ and } d_j = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right.$ where $p_t, b_k \in Z_{50}$, $0 \le i \le 4$, $0 \le j \le 8$, $1 \le t \le 4$, $1 \le k \le 3\}$ be a groupoid vector space over the groupoid G = $\{Z_{50}, *, (10, 0)\}$.

(i) Write M as a direct sum.

(ii) Can M have more than three substructures?

(iii) Find Hom (M, M).

186. Let W $= \left\{\sum_{i=0}^{9} q_i x^i, \sum_{i=0}^{10} b_i x^i, \sum_{i=0}^{11} d_i x^i, \sum_{i=0}^{10} p_i x^i \;\middle|\; q_i = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{10} \end{bmatrix}, b_t = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix}, d_i = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, p_i = (t_1, t_2, \ldots, t_{10})\right.$ with $0 \le j \le 9$, $0 \le t \le 10$, $1 \le r \le 11$, $0 \le 1 \le 10$; $1 \le i \le 10$, $1 \le k \le 10$, $1 \le j \le 10$, $a_i, t_j, y_k \in Z_5\}$ be a groupoid vector space over the groupoid G = $\{Z_5, *, (2, 3)\}$.

(i) The number of elements in W is finite.

(ii) Write W as a direct sum.

(iii) Express W as a pseudo direct sum.

187. Let T = $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{19}, *, (16, 4)\right\}$ be a groupoid linear algebra over the groupoid G = $\{Z_{19}, *, (16, 4)\}$.

(i) Show T is non commutative linear algebra.

(ii) Does T satisfy any of the standard identities?

(iii) Can T be written as a direct sum of groupoid linear subalgebra.

(iv) Find L (T, Z_{19}).

188. Obtain the major difference between groupoid vector spaces and complex groupoid vector spaces.

189. Mention any special property enjoyed by complex groupoid vector spaces built using $C(Z_n)$.

190. Let V = {C (Z_{30}), *, (3, 10)} be a complex groupoid linear algebra over the groupoid G = {Z_{30}, *, (3, 10)}.

(i) Find the number of elements in V.

(ii) Give a basis of V over G.

(iii) Find Hom (V, V).

191. Let W = {(a_1, a_2, a_3) | $a_i \in$ C (Z_{40}), $i_F^2 = 39$, *, (20, 20)} be a strong complex groupoid linear algebra over the complex groupoid C(G) = {C (Z_{40}), *, (20, 20), $i_F^2 = 39$}.

(i) Find a basis of W over C(G).

(ii) Can W have strong complex groupoid linear subalgebras?

(iii) Find Hom (W, W).

192. Let $M = \left\{(a_1, a_2, a_3) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in C(Z_{43}), 1 \leq i \leq 4, i_F^2 = 42\right\}$ be a complex groupoid vector space over the groupoid $\{Z_{43}, *, (10, 3)\}$.

(i) Prove M is finite.

(ii) Write M as a direct sum.

(iii) Find Hom (M, M).

193. Let $P = \left\{\begin{pmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{pmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}, (a_1, a_2, \ldots, a_{25})\right.$ where $a_i \in C(Z_{23})$; $1 \leq i \leq 25$, $i_F^2 = 22\}$ be a strong complex groupoid vector space over the complex groupoid $C(G) = \{C(Z_{23}), *, (3, 14), i_F^2 = 22\}$.

(i) Show P has only finite number of elements.

(ii) Express P as a direct sum of strong complex groupoid vector subspaces.

(iii) Show P has pseudo complex strong groupoid vector subspace over $H = \{Z_{23}, *, (3, 14)\}$.

194. Let W = $\left\{(a_1, a_2, \ldots, a_{10}), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{45} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{45} \end{bmatrix} \middle| a_i \in \right.$ C (Z_{43}), $i_F^2 = 12$, $1 \leq i \leq 45\}$ be complex groupoid linear algebra over the groupoid G = $\{Z_{13}, *, (10, 4)\}$.

(i) Find the algebraic structure enjoyed by
   (a) L (W, Z_{13}). (b) $\text{Hom}_{Z_{13}}$ (W, W).

(ii) Does W have a basis over the groupoid G?

195. Let P = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \\ a_{37} & a_{38} & \ldots & a_{48} \end{bmatrix} \middle| a_i \in \right.$ C (Z_{15}), $i_F^2 = 14$, $1 \leq i \leq 48$, (12, 4), *} be a strong complex groupoid linear algebra over the complex groupoid C(G) = {C (Z_{15}), *, (12, 4)}.

(i) Find the number of elements in P.

(ii) Find a basis of P over C(G).

(iii) Find a strong complex groupoid linear subalgebra of P over the complex groupoid C (G).

196. Let W = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in \right.$ $C(Z_{40})$, $i_F^2 = 39$, $1 \leq i \leq 12$, (3, 19), *} and P = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| a_i \in \right.$ $C(Z_{40})$,

$i_F^2 = 39$, $1 \le i \le 12$, (3, 19), *} be two strong complex groupoid linear algebras over the complex groupoid C(G) =

{C (Z_{40}) = {a + bi_F | $i_F^2 = 39$, a, b ∈ Z_{40}}, *, (3, 19)}

(i) Find T : W → P so that T^{-1} exists (T a linear transformation of W to P).

(ii) Find f : P → W a linear transformation of P into W so that f^{-1} does not exist.

(iii) Find Hom (W, P) and Hom (P, W).

(iv) Does there exist any relation between L (P, C(G)) and L (W, C(G))?

197. Let

$$P = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}, (a_1, a_2, \ldots, a_7), \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \right.$$

where $a_i \in$ C (Z_{14}); $1 \le i \le 10$, $i_F^2 = 13$} be a strong complex groupoid vector space over the complex groupoid C (G) = {C (Z_{14}) = {a + bi_F | $i_F^2 = 13$, a, b ∈ Z_{14}}, *, (3, 12)}.

(i) Find a basis of P over C(G).

(ii) Find the number of elements in P.

(iii) Write P as a direct sum.

(iv) Find L (P, C(G)).

198. Mention some special features enjoyed by loop vector spaces defined over the loop $L_n(M)$.

199. Let V = {(a, b) | a, b ∈ L_{19} (8), *} be a loop linear algebra over the loop L_{19} (8) = L.

(i) Find the number of elements in V.

(ii) Find a basis for V over L.

(iii) What is the dimension of V over L?

(iv) Find Hom_L (V, V).

200. Let S = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in L_{23}(7); 1 \leq i \right.$ $\leq 10\}$ be a loop vector space over the loop L = L_{23} (7).

(i) Find a basis of S over L.

(ii) Write S as a direct sum of loop vector subspaces of S over L.

(iii) Write S as a pseudo direct sum of loop vector subspaces of S over L.

(iv) Find L (S, L).

201. Derive some interesting features enjoyed by strong loop linear algebras.

202. Let P = $\left\{ (a_1, a_2, \ldots, a_{10}), \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \vdots & \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}, \right.$

$\left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| a_i \in L_9(5), 1 \leq i \leq 30 \right\}$ be a loop vector space over the loop $L_9(5)$ = L and

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \\ \vdots & \vdots & \dots & \vdots \\ a_{25} & a_{26} & \dots & a_{30} \end{bmatrix} \right| a_i \in L_9(5),\ 1 \leq i \leq 30 \Bigg\} \text{ be}$$

a loop vector space over the loop $L = L_9(5)$.

(i) Define $T : P \to M$ so that T^{-1} does not exist.

(ii) Study the algebraic structure enjoyed by Hom (M, P).

(iii) What is the algebraic structure of L (M, L)?

203. Give an example of a loop linear algebra over a loop $L = L_n(m)$.

204. Give an example of a strong non associative linear algebra which satisfies Moufang identity.

205. Does there exist a strong non associative linear algebra which satisfies Bol identity?

206. Give an example of a strong non associative linear algebra which satisfies the Burck identity.

207. Obtain some interesting properties enjoyed by S-strong non associative linear algebras over a S-ring.

208. Give an example of a commutative S-strong non associative linear algebra of finite order over a S-ring.

209. Give an example of a right alternative S-strong non associative linear algebra over a S-ring. Can that ever be commutative? (justify).

210. Does there exists an alternative S-strong non associative linear algebra over a S-ring?

211. Does there exist a Burck S-strong non associative linear algebra defined over a S-ring?

212. Does there exist a Moufang S-strong non associative linear algebra defined over a S-ring?

213. Does there exist a Bol S-strong non associative linear algebra defined over a S-ring?

214. Give an example of a S-strong non associative linear algebra which is right alternative over S-ring.

215. Let V = $a_i \in L_{63}(11)$, *, $1 \leq i \leq 9\}$ be a S-non associative linear algebra over the S-ring Z_{63}.

(i) Find dimension of V.

(ii) Find a basis of V.

(iii) Can V have more than one basis?

(iv) Find sublinear algebras W and a basis of that W.

(v) Is M = a $\in$ $L_{63}(11)$, *} $\subseteq$ V a sublinear algebra?

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Dr.W.B.Vasantha Kandasamy is an Associate Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 646 research papers. She has guided over 68 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 65th book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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In this book, we introduce the notion of non-associative linear algebras. We mainly use the concept of loops and groupoids to build these structures. We have also introduced the notion of non-associative semi-linear algebras. In future, non-associative linear algebras will find applications in mathematical models that need not in general be associative.

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