Wave Function for Classical Mechanics

J. M. Karimäki, *Department of Physics, University of Helsinki*

28th April 2000

Abstract

In this paper the relationship of classical physics and quantum physics is studied by introducing a partial differential equation, which describes classical mechanics, but looks very similar to the Schrödinger wave equation of quantum mechanics. This work is largely based on David Bohm's causal interpretation of quantum mechanics, but is in some sense complementary to it. In Bohm's theory the Schrödinger wave equation is used to derive classical looking equations of motion for quantum physics. Here exactly the opposite is done. The equations of classical physics are put into a form resembling the Schrödinger equation of quantum physics.

1 Introduction

There has been considerable debate over the role of the wave function in quantum mechanics and the interpretation of quantum mechanics in general. At the beginning of the quantum era it was not clear whether the wave function would represent matter waves that actually exist or something else. The problem was partly solved by Max Born who proposed that the wave function does not represent any truly existing physical wave, but only gives us the probability of finding the particle at a specific location after measurement.

There were, however, other opinions and different approaches, which did not receive much attention. In 1927 Louis de Broglie suggested that the role of the wave fuction was to guide the particle along a continuous curve [1], but this approach was quickly abandoned after having received harsh criticism. Among other things, it was claimed that the theory would not work for more than one particle.

Nevertheless, independently of de Broglie's ideas, in 1952 David Bohm published a similar theory [3], which, in addition, took account of the many particle case. Bohm's theory, or the causal interpretation of quantum mechanics, starts from the Schrödinger wave equation, which is used to derive two equations: The first one is similar to the equation of classical physics and describes particle trajectories. The second one describes the probability of the particles to be moving along a specific set of trajectories.

In this paper, the equation for the trajectories of the particles is replaced by the corresponding equation of classical mechanics, but the form of the probability equation remains the same as in Bohm's theory. From these two equations a single differential equation is deduced, which is very similar to the Schrödinger wave equation of quantum mechanics, but which produces the results of classical physics.

2 The Causal Interpretation of Quantum Mechanics

In this section we will reproduce the derivation of David Bohm's causal interpretation of quantum mechanics for the case of a single non-relativistic particle. We shall go into more detail than is usual, since the results of this section will be needed in the developments of later sections.

We will start from the Schrödinger wave equation

$$i\hbar \partial\Psi/\partial t = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \tag{1}$$

The wave function $\Psi(\mathbf{x}, t)$ can be written as a product of two parts, corresponding to an amplitude and a phase:

$$\Psi = R\exp(iS/\hbar), \tag{2}$$

where R and S are real functions of $\mathbf{x}$ and t. When this expression is inserted to the wave equation, and the resulting equation is divided by $-\Psi$, we get a complex equation, whose real part is

$$\frac{\partial S}{\partial t} = \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} - \frac{(\nabla S)^2}{2m} + V, \tag{3}$$

and imaginary part

$$-\frac{i\hbar}{R}\frac{\partial R}{\partial t} = \frac{i\hbar}{2m}[\nabla^2 S + 2\frac{\nabla R \cdot \nabla S}{R}]. \tag{4}$$

We can reorder the terms in equation (3) and multiply the equation (4) by $-R/i\hbar$ to get a pair of coupled equations for R and S:

$$\frac{\partial S}{\partial t} = -[\frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}], \tag{5}$$

$$\frac{\partial R}{\partial t} = -\frac{1}{2m}[R\nabla^2 S + 2\nabla R \cdot \nabla S]. \tag{6}$$

Now the original complex wave equation (1) of the complex valued function Ψ has been transformed into two real equations of two real valued functions. These are the sought for equations of the causal interpretation of quantum mechanics in the form written by Bohm [3]. Let us take a closer look at their physical meaning.

The equation (5) is the same as the Hamilton-Jacobi equation of classical mechanics apart from an additional "quantum potential"

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}. \tag{7}$$

According to the causal interpretation of quantum mechanics the particle moves along a continuous curve and its velocity can be calculated from the phase S using the Hamilton-Jacobi-relation:

$$\mathbf{v}(\mathbf{x}, t) = \frac{\nabla S(\mathbf{x}, t)}{m}. \tag{8}$$

It is furthermore assumed that, before any measurements are performed, the actual position of the particle is unknown, but the probability density for the particle to be at $\mathbf{x}$ at time t is given by the square of the amplitude R:

$$P(\mathbf{x},t) = R(\mathbf{x},t)^2. \tag{9}$$

The equation (6) can be written in a more familiar form by multiplying it by $2R$ and using the above definitions of P and $\mathbf{v}$:

$$\frac{\partial P}{\partial t} = -\nabla \cdot (P\mathbf{v}). \tag{10}$$

This is a conservation equation for the probability. It ensures that the assumption (9) is valid for $t > t_0$, if it is valid at t_0 and the system has not been disturbed between t_0 and t.

To find the trajectory of the particle one has to find a solution for the equations (5) and (6). The velocity can then be obtained from S using (8) and the position of the particle. Once the initial position of the particle is known the trajectory can be calculated from the velocity $\mathbf{v}$. In reality the initial position of the particle cannot be known exactly and only the initial probability distribution R is available. However, we can imagine that the initial position exists, even if it is impossible to know it exactly. The initial positions can then be considered to be so called "hidden variables" and the causal interpretation a "hidden variable" theory of quantum mechanics.

Instead of (8) it is also possible to use the following equation of motion to calculate the particle trajectories:

$$m\frac{d^2\mathbf{x}}{dt^2} = -\nabla(V+Q). \tag{11}$$

This differs from the Newtonian equation of motion only in that the quantum potential Q has been added to the classical potential V. The quantum potential can be thought of as exerting an additional "quantum force" $-\nabla Q$ to the particle. The equation (11) is perhaps intuitively the most appealing link between classical and quantum mechanics and its value lies in its philosophical implications rather than its usefulness in calculations.

Finally, it is perhaps worth mentioning that the theory can be formulated without explicit reference to the functions R and S. The formula (8) for the velocity of the particle can be written using Ψ only:

$$\mathbf{v}(\mathbf{x},t) = \frac{\hbar}{m} Im \frac{\nabla\Psi}{\Psi}. \tag{12}$$

Since the probability density can be written in the more familiar form

$$P = |\Psi|^2 \tag{13}$$

and the quantum potential as

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2|\Psi|}{|\Psi|}, \tag{14}$$

we only need to solve the original Schrödinger equation (1) to be able to calculate the particle trajectories.

3 The Classical Limit of Quantum Mechanics

How classical physics can be retrieved as a limit case of quantum physics is a delicate question involving both mathematics and interpretation. In the causal interpretation of quantum mechanics the question becomes easier than in the standard, or Copenhagen, interpretation.

Let us first see how things work in the causal interpretation. If we set $\hbar = 0$ in the equation (5), we get the Hamilton-Jacobi equation of classical mechanics:

$$\frac{\partial S}{\partial t} = -[\frac{(\nabla S)^2}{2m} + V]. \tag{15}$$

This equation describes the behaviour of the exact classical counterpart of the quantum system. A solution for the phase function $S(\mathbf{x}, t)$ can be found using the standard methods of classical mechanics and the velocity of the particle can be determined from the expression (8). There is no more need for the equation (6), since R does not appear in (15).

In the standard interpretation of quantum mechanics we have to start from the Schrödinger equation (1), but here we cannot directly put $\hbar = 0$, for this would lead to the trivial equation

$$0 = V\Psi, \tag{16}$$

which obviously does not describe the above mentioned classical situation.

Thus, it seems that, whereas the standard interpretation of quantum mechanics is concerned, classical physics can only be approached as a limit case by letting $\hbar \to 0$. In practice, we can do this by replacing $\hbar$ in the Schrödinger equation (1) by $\alpha\hbar$, where the factor α is a positive real number:

$$i(\alpha\hbar)\partial\Psi/\partial t = -\frac{(\alpha\hbar)^2}{2m}\nabla^2\Psi + V\Psi. \tag{17}$$

Let us take a classical particle moving in the potential V. For each choice of α we can choose a wave packet Ψ_α which obeys the equation (17). Moreover, these wave packets can be chosen so that their motion approaches the motion of the given classical particle as $\alpha \to 0$.

Nothing prevents us from using the limit value approach in the causal interpretation too. We can replace $\hbar$ by $\alpha\hbar$ in the equations (2) and (5) and use (8) for the velocity. An alternative way, which again uses the wave function Ψ directly and avoids S and R, is to use the equation (17) and the following velocity formula:

$$\mathbf{v}(\mathbf{x}, t) = \frac{\alpha\hbar}{m} Im \frac{\nabla\Psi}{\Psi}. \tag{18}$$

Whether we use Ψ or S and R is more or less a matter of taste at this point. We mention both ways here, because the expressions (17), (8) and (18) are needed later. What is interesting, though, is that by using the causal interpretation we can smoothly change from quantum mechanics to classical mechanics (or vice versa) just by moving the value of α between 1 and 0.

To sum up things: in the causal interpretation the classical limit can be reached either by using a limit value or simply by replacing $\hbar$ by 0, but in the standard interpretation the use of the limit value seems necessary.

4 The Wave Equation for Classical Physics

There is, however, another possible approach to restore classical physics. We can make a small modification to the equation (5) of the causal interpretation and work backwards. To get the classical situation we start directly from the Hamilton-Jacobi equation (15) and the probability equation (10)

$$\frac{\partial S}{\partial t} = -[\frac{(\nabla S)^2}{2m} + V], \tag{19}$$

$$\frac{\partial P}{\partial t} = -\nabla \cdot (P\mathbf{v}). \tag{20}$$

These can be written in the following form:

$$\frac{\partial S}{\partial t} = -[\frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}], \tag{21}$$

$$-\frac{i\hbar}{R}\frac{\partial R}{\partial t} = \frac{i\hbar}{2m}[\nabla^2 S + 2\frac{\nabla R \cdot \nabla S}{R}]. \tag{22}$$

By multiplying these by $-\Psi = -R\exp(iS/\hbar)$ and adding them together we get, after some manipulations, the Schrödinger equation and an additional term:

$$i\hbar\partial\Psi/\partial t = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi + \frac{\hbar^2}{2m}\exp(iS/\hbar)\nabla^2 R. \tag{23}$$

Although this equation looks very much like the Schrödinger equation, it produces exactly the classical behaviour for the particle. To show this the equation (23) must be interpreted the same way as the Schrödinger equation is interpreted in the causal interpretation of quantum mechanics, i.e. we must solve the equation (23) and then fix an initial position for the particle at the initial time and use the Hamilton-Jacobi relation $\mathbf{v}(\mathbf{x}, t) = \nabla S(\mathbf{x}, t)/m$ for the velocity of the particle.

As an additional result we can now also work with initial probability distributions as in the quantum case, since the equation (10) is still valid.

The equation (23) can be put into a more elegant form using Ψ only:

$$i\hbar\partial\Psi/\partial t = -\frac{\hbar^2}{2m}(\nabla^2\Psi - \frac{\Psi}{|\Psi|}\nabla^2|\Psi|) + V\Psi. \tag{24}$$

or, using the definition (7) of the quantum potential Q:

$$i\hbar\partial\Psi/\partial t = -\frac{\hbar^2}{2m}\nabla^2\Psi + (V - Q)\Psi. \tag{25}$$

It is clear from (24) that the new equation is not linear with respect to Ψ. Although this equation should not be considerably more difficult to solve than the original Schrödinger equation, we lose the possibility to create new solutions by linear combination. This is one of the features that make classical and quantum physics distinct.

A second remark to be made from (24) is that it still contains $\hbar$. However, the actual value of $\hbar$ has absolutely nothing to do with the solution of the equation! Any value of $\hbar$ would lead to exactly the same classical physics.

5 Initial Conditions for the One Particle Case

Any wave function $\Psi(\mathbf{x},0)$ that can be used as an initial value in the quantum case can also be used in the classical case. So, the amplitude R and the phase S can be chosen quite freely. For example, it is always possible to choose a constant velocity for the particle at time zero. It is also possible to choose any differentiable velocity field on a given 2-dimensional plane of the three-space. However, this is not possible for the whole 3-dimensional space, since the velocity field must satisfy $\mathbf{v}(\mathbf{x},0) = \nabla S(\mathbf{x},0)/m$, which restricts the available initial velocity fields to those that are non-rotational ($\nabla \times \mathbf{v} = 0$).

The subsequent evolution of the wave function may differ from the quantum case, because there is a possibility of arriving at singularities. This is due to the fact that classical trajectories can cross each other, whereas in the one particle case of the causal interpretation of quantum mechanics it is not possible.

6 Free Particle

As an example, we shall consider a classical free particle moving at the velocity $\mathbf{v}$. The equation (23) now becomes

$$i\hbar\partial\Psi/\partial t = -\frac{\hbar^2}{2m}(\nabla^2\Psi - \exp(iS/\hbar)\nabla^2 R). \tag{26}$$

One possible solution is

$$\Psi = \Psi_0(\mathbf{x} - \mathbf{v}t)\exp[i(m\mathbf{v}\cdot\mathbf{x} - \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}t)/\hbar], \tag{27}$$

where the real function $\Psi_0(\mathbf{x})$ is the initial amplitude of the wave function. This solution represents a wave packet moving with constant velocity $\mathbf{v}$ and maintaining its initial shape during the motion. It should be noted that in this example there is no uncertainty for the momentum $m\mathbf{v}$ and that the wave packet can be confined to an arbitrarily small volume, thus allowing the uncertainty for the particle position to be decreased under any given value.

7 N particles

Let us first take a look at the causal interpretation of quantum mechanics for N non-relativistic particles. The starting point is the Schrödinger equation:

$$i\hbar\partial\Psi/\partial t = -[\sum_{j=1}^{N}\frac{\hbar^2}{2m_j}\nabla_j^2\Psi] + V(\mathbf{x}_1,...,\mathbf{x}_N,t)\Psi, \tag{28}$$

where Ψ is a function of N positions $\mathbf{x}_j$ and time t and ∇_j stands for the gradient respective to $\mathbf{x}_j$. As in the single particle case the wave function can be written as a product of an amplitude and a phase factor:

$$\Psi(\mathbf{x}_1,...,\mathbf{x}_N,t) = R(\mathbf{x}_1,...,\mathbf{x}_N,t)\exp[iS(\mathbf{x}_1,...,\mathbf{x}_N,t)/\hbar]. \tag{29}$$

Inserting this to (28) leads us to the equations of the causal interpretation for N particles:

$$\frac{\partial S}{\partial t} = -\sum_{j=1}^{N}(\frac{(\nabla_j S)^2}{2m_j} + \frac{\hbar^2}{2m_j}\frac{\nabla_j^2 R}{R}) + V, \qquad (30)$$

$$\frac{\partial P}{\partial t} = -\sum_{j=1}^{N}\nabla_j \cdot (P\mathbf{v}_j). \qquad (31)$$

The probability density P and the particle velocities $\mathbf{v}_j$ are defined the same way as in the single particle case:

$$P(\mathbf{x}_1, ..., \mathbf{x}_N, t) = [R(\mathbf{x}_1, ..., \mathbf{x}_N, t)]^2, \qquad (32)$$

$$\mathbf{v}_j(\mathbf{x}_1, ..., \mathbf{x}_N, t) = \nabla_j S(\mathbf{x}_1, ..., \mathbf{x}_N, t)/m_j. \qquad (33)$$

(Here it must be remembered that the index j denotes the jth *particle* and that each vector $\mathbf{v}_j$ has three components.) The definition of the quantum potential Q for N particles is:

$$Q(\mathbf{x}_1, ..., \mathbf{x}_N, t) = -\sum_{j=1}^{N}\frac{\hbar^2}{2m_j}\frac{\nabla_j^2 R}{R}. \qquad (34)$$

In the single particle case the quantum potential is responsible for the interference effects of quantum mechanics, but in the N particle case Q is also responsible for quantum non-locality ([5], p. 57).

The classical counterpart of the Schrödinger equation can be deduced the same way as in the single particle case, and it is simply:

$$i\hbar\partial\Psi/\partial t = -[\sum_{j=1}^{N}\frac{\hbar^2}{2m_j}\nabla_j^2\Psi] - [\sum_{j=1}^{N}\exp(iS/\hbar)\nabla_j^2 R] + V\Psi, \qquad (35)$$

where R and S are defined as in (29). Using (34) we can write the equation (35) as:

$$i\hbar\partial\Psi/\partial t = -\sum_{j=1}^{N}\frac{\hbar^2}{2m_j}\nabla_j^2\Psi + (V - Q)\Psi. \qquad (36)$$

Now the particle velocities are given by (33) as in the quantum case.

8 A generalized family of equations

In the previous sections we have presented two fundamentally different methods for retrieving classical mechanics from quantum mechanics:

1. replacing $\hbar$ by $\alpha\hbar$ and letting $\alpha \to 0$, (section 4)
2. replacing V by $V - Q$, (section 5)

We will combine these two approaches to get a more general result. We shall consider the one particle case, since the generalization to many particles is trivial.

First, we can replace $\hbar$ by $\alpha\hbar$ in the equation (5) to get the equation

$$\frac{\partial S}{\partial t} = -[\frac{(\nabla S)^2}{2m} + V - \frac{(\alpha\hbar)^2}{2m}\frac{\nabla^2 R}{R}]. \tag{37}$$

This is equivalent to replacing $\hbar$ by $\alpha\hbar$ in the Schrödinger equation (1) and using the relation

$$\Psi = R\exp[iS/(\alpha\hbar)] \tag{38}$$

instead of (2).

Second, we can add the quantum potential Q multiplied by a constant $\pm\beta^2$ to the right side of the equation (37)

$$\frac{\partial S}{\partial t} = -[\frac{(\nabla S)^2}{2m} + V - \frac{(\alpha^2 \mp \beta^2)\hbar^2}{2m}\frac{\nabla^2 R}{R}]. \tag{39}$$

This equation coupled with the equation (6) is equivalent to the following modified Schrödinger equation:

$$i(\alpha\hbar)\partial\Psi/\partial t = -\frac{(\alpha\hbar)^2}{2m}\nabla^2\Psi + (V \mp \beta^2 Q)\Psi. \tag{40}$$

Now, it can be seen from the form of the equation (39) that if $\alpha^2 \mp \beta^2 = 0$ it becomes the Hamilton-Jacobi equation and thus (40) gives the classical mechanics and, similarly, if $\alpha^2 \mp \beta^2 = 1$ the equation (40) represents quantum mechanics. However, care must be taken to use the correct definition of S (38) and velocity (8), which using Ψ takes the form

$$\mathbf{v}(\mathbf{x}, t) = \frac{\alpha\hbar}{m} Im \frac{\nabla\Psi}{\Psi}. \tag{41}$$

9 Tables

We have collected the main results of our analysis of the classical and quantum equations into a table. For simplicity, only the single particle case is presented.

Ψ	Quantum Mechanics	Classical Mechanics
Wave eq.	$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$	$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + (V-Q)\Psi$
Quantum pot.	$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\lvert\psi\rvert}{\lvert\psi\rvert}$	$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\lvert\psi\rvert}{\lvert\psi\rvert}$
Prob. dens.	$P(\mathbf{x},t) = \lvert\Psi(\mathbf{x},t)\rvert^2$	$P(\mathbf{x},t) = \lvert\Psi(\mathbf{x},t)\rvert^2$
Velocity	$\mathbf{v}(\mathbf{x},t) = \frac{\hbar}{m} Im\frac{\nabla\Psi}{\Psi}$	$\mathbf{v}(\mathbf{x},t) = \frac{\hbar}{m} Im\frac{\nabla\Psi}{\Psi}$
Eq. of motion	$m\frac{d^2\mathbf{x}}{dt^2} = -\nabla(V+Q)$	$m\frac{d^2\mathbf{x}}{dt^2} = -\nabla V$
H.-Jacobi	$\frac{\partial S}{\partial t} = -[\frac{(\nabla S)^2}{2m} + V + Q]$	$\frac{\partial S}{\partial t} = -[\frac{(\nabla S)^2}{2m} + V]$
Cons. prob.	$\frac{\partial P}{\partial t} = -\nabla\cdot(P\mathbf{v})$	$\frac{\partial P}{\partial t} = -\nabla\cdot(P\mathbf{v})$

Generalized Schrödinger equation	$i(\alpha\hbar)\frac{\partial\Psi}{\partial t} = -\frac{(\alpha\hbar)^2}{2m}\nabla^2\Psi + (V \mp \beta^2 Q)\Psi$

10 Discussion

We could also keep to the standard interpretation of quantum mechanics in interpreting the equations (23) and (35). It would be interesting from a pedagogical point of view, since many features of quantum physics now have classical counterparts. Such concepts as the preparation of the system, unitary evolution, and collapse of the wave function could be studied and compared with their classical counterparts.

One crucial difference between classical particle mechanics and quantum mechanics is the existence of interference in quantum mechanics. In the causal interpretation it is precisely the quantum potential Q that is responsible for the interference effects. Since there is no interference in classical particle physics, it must be that adding the term $-Q\Psi$ to the right side of the Schrödinger equation somehow wipes out all the interference effects of quantum mechanics. However, this can be understood as a result of the vanishing of $\hbar$ from the equation (15) when compared with (5).

The approach presented in this paper gives the possibility of treating some particles of a problem as "classical" and the others as "quantum" particles. This can be achieved by omitting one or more terms from the second sum of the equation (35). One could, for example, study the behaviour of one "classical" particle in an environment of purely "quantum" particles, or vice versa. This could help in understanding how the combined effect of many randomly moving particles creates a (more or less) constant and uniform potential.

Since this paper has so far only considered non-relativistic particle quantum mechanics, it would be interesting to see what form the relativistic particle equations, or even field equations, would take if similar procedures to the one presented in this paper would be applied to them. The consequences to the Lorentz invariance of the equations would be one point of interest since the original classical relativistic equations are (by definition) Lorentz invariant and the corresponding quantum equations also have a covariant form.

References

[1] L. de Broglie, *La Nouvelle Dynamique des Quanta*, in [2], p. 105

[2] *Électrons et Photons: Rapports et Discussions du Cinquième Conseil de Physique tenu a Bruxelles du 24 au 29 Octobre 1927 sous les Auspices de l'Institut International de Physique Solvay*, Gauthiers-Villars, Paris (1928)

[3] D. Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables I*, Phys. Rev. **85** (1952), p. 166, also in [4], p. 369

[4] J. A. Wheeler and W. H. Zurek (editors), *Quantum Theory and Measurement*, Princeton University Press, Princeton (1983)

[5] D. Bohm and B. J. Hiley, *The Undivided Universe*, Routledge, London and New York (1993)