On A Theorem of C.L. Siegel Regarding Binary Quadratic Forms

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Abstract. We prove a special case of C.L. Siegel's theorem regarding the class number h(Δ) of binary quadratic forms with fundamental discriminant Δ.

1. INTRODUCTION. C. L. Siegel [1] proved the following theorem. If h(Δ) denotes the number of classes of binary quadratic forms with fundamental discriminant Δ then if Δ= -D<0 then Siegel's theorem asserts

$$\lim_{D\to\infty}\frac{\ln h(-D)}{\ln D}=\frac{1}{2}$$

Siegel proved this theorem using analytic methods. We prove a special case using comparatively elementary methods, namely,

Theorem 1. $$\lim_{p\to\infty}\frac{\ln h(-p)}{\ln p}=\frac{1}{2} \quad \text{where} \quad p \text{ is a prime} \equiv 3\ (mod\ 4)$$

Let R and N denote the sum of the quadratic residues and non-residues respectively of p.
Then by a theorem of Dirichlet [2]

$$h(-p)=\frac{N-R}{p}=\frac{p\frac{(p-1)}{2}-2R}{p} \qquad p\equiv 3\ (mod\ 4) \tag{1}$$

Since the quadratic residues are generated by $i^2 \;\; i=1,2,\ldots\cdot\left(p-\frac{1}{2}\right)$ and since each square lies in one of the intervals ($kp,(k+1)p$) $k=0,1,2\ldots\frac{p-3}{4}$ then $R=\sum_{n=1}^{\frac{p-1}{2}}n^2-\sum_{n=1}^{\frac{p+1}{4}}\left(\lfloor\sqrt{np}\rfloor-\left\lfloor\sqrt{(n-1)p}\right\rfloor\right)(n-1)p$

where we have made use of the floor function. Simplifying the algebra, we obtain

$$h(-p) = \frac{p^2 - 3p + 2}{6} - 2\sum_{n=1}^{\frac{p-3}{4}} \sqrt{np} + 2\sum_{n=1}^{\frac{p-3}{4}} \left\{\sqrt{np}\right\} \tag{2}$$

Lemma 1. $\lim_{p \to \infty} 2p^{\frac{1}{2}} \sum_{n=1}^{\frac{p-3}{4}} \left\{\sqrt{n}\right\} = \frac{p-3}{4}$

What this means is that the numbers $x_n = \sqrt{np} - \left\lfloor \sqrt{np} \right\rfloor$ n=1,2,3.... are uniformly distributed on [0,1]. This follows from the fact that $f_p(n) = \sqrt{np}$ satisfies the following four conditions as shown in [3].

(1) $f_p(n)$ is continuously differentiable.

(2) $f_p(n)$ is monotone increasing to ∞ $as\ n \to \infty$.

(3) $f'_p(n)$ is monotone decreasing to 0 as $n \to \infty$

(4) $nf'_p(n)\ tends\ to\ \infty\ as\ n \to \infty$

Lemma 2. $\lim_{p \to \infty} -2p^{\frac{1}{2}} \sum_{1}^{\frac{p-3}{4}} \sqrt{n} = -\frac{p^2}{6} + \frac{p}{4} + C_1 + C_2\, p^{\frac{1}{2}}$ where $C_1\ and\ C_2$ are constants.

Using Euler Summation as defined in [4]

$\sum_{a \le k < b} f(k) = \int_a^b f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{k-1}(x)\Big|_a^b + R_m$ where the B_k are the Bernoulli numbers. Let $f(k) = \sqrt{k}$ then as shown in [5]

$$\sum_{k=1}^{n} \sqrt{k} = \left(\frac{2}{3}\right)\left(n^{\frac{3}{2}} - 1\right) + \frac{1}{2}\sqrt{n} + C + O\left(n^{-\frac{1}{2}}\right) \quad \text{where C is a constant.} \tag{3}$$

Letting $n = \frac{p-3}{4}\ and\ multiplying\ (3)\ by\ -2p^{\frac{1}{2}}$ we obtain

$$\sum_{1}^{\frac{p-3}{4}} \sqrt{k} = -\frac{1}{6}p^2\left(1 - \frac{3}{p}\right)^{\frac{3}{2}} + \frac{4}{3}p^{\frac{1}{2}} - \frac{1}{2}p\left(1 - \frac{3}{p}\right)^{\frac{1}{2}} + C_3\, p^{\frac{1}{2}} + C_4 \tag{4}$$

Making use of the expansion $\left(1-\frac{3}{p}\right)^{\frac{3}{2}}=1+\frac{3}{2}\left(-\frac{3}{p}\right)+\frac{3}{8}\left(-\frac{3}{p}\right)^{2}+\ldots\ldots\ldots$ (5)

and multiplying the right side of (5) by $-\frac{1}{6}p^{2}$ and taking $\lim_{p\to\infty}$ we obtain

$$\left(-\frac{1}{6}p^{2}+\frac{3}{4}p+\frac{9}{48}\right)\sim\left(-\frac{1}{6}p^{2}\left(1-\frac{3}{p}\right)^{\frac{3}{2}}\right) \tag{6}$$

Similarly, expanding $\left(1-\frac{3}{2}\right)^{\frac{1}{2}}=1+\frac{1}{2}\left(-\frac{3}{p}\right)-\frac{1}{8}\left(-\frac{3}{p}\right)^{2}+\ldots\ldots\ldots$ (7)

multiplying by $-\frac{1}{2}p$ and taking $\lim_{p\to\infty}$ we obtain $\left(-\frac{1}{2}p+\frac{3}{4}\right)\sim\left(-\frac{1}{2}p\left(1-\frac{3}{2}\right)^{\frac{1}{2}}\right)$ (8)

Combining the results of (6) and (8) with the remaining terms in (4) we get

$$\lim_{p\to\infty}\sum_{1}^{\frac{p-3}{4}}\sqrt{k}=-\frac{1}{6}p^{2}+\frac{p}{4}+Cp^{\frac{1}{2}}+C \tag{9}$$

which establishes Lemma 2

Substituting the results of Lemma's 1 & 2 into (2) yields

$$\lim_{p\to\infty}h(-p)=C_{1}p^{\frac{1}{2}}+C_{2} \tag{10}$$

Dividing (9) by $\ln p$ and taking limits we get $\lim_{p\to\infty}\frac{h(-p)}{\ln(p)}=\frac{1}{2}$ which is the intended result.

REFERENCES

1. A.O. Gelfond & Y.V. Linnik, *Elementary Methods In Analytic Number Theory*, Rand McNally Mathematics Series, Rand McNally & Co. 1965, 217

2. H. Davenport, *The Higher Arithmetic*, Hutchinson's UniversityLibrary, 1952, p 148.

3. G. Polya & G. Szego, *Problems & Theorems In Analysis, Volume 1*, Springer 1998, p 90, Problem 174.

4. R.L. Graham, D. Knuth, & O. Patashnik, *Concrete Mathematics,1989, Addison Wesley, p 455*

5. M. Wildon, *Notes on Bernoulli Numbers & Euler Summation, (2006*), available at http://www..ma.rhul.ac.uk.