Vertex-only bifurcation diagrams are deceptively simple

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March 15, 2012

Abstract

By plotting the polynomials corresponding to several iterations of the logistic map, it is found that the entropy of a branching path can be larger than what is intuitively expected.

The bifurcation diagram is often used to analyze the cycles of various lengths that are inherent to the iterations of the logistic map. Most importantly, the diagram shows how the length of a cycle often doubles, which produces a cascading series of clearly-defined two-way branches, two-way sub-branches, *etc*. However, the diagram is often drawn using only the vertices x, x', *etc*, which effectively ignores the fact that each iteration is representative of a polynomial. By drawing the vertex-to-vertex line segments on top of the vertex-only diagram, it can be seen that the choices to be made when moving along a cascading series of two-way branches are not as simple as they first seem.

1 The bifurcation diagram of the logistic map

The first iteration of the logistic map is

$$x' = rx(1-x) = -rx^2 + rx, \tag{1}$$

where $x = [0,1]$, $r = [0,4]$, and $x' = [0,1]$. By feeding the solution x' back to the map, we obtain the second iteration

$$\begin{aligned} x'' = rx'(1-x') = & -r^3x^4 + 2r^3x^3 - r^3x^2 \\ & -r^2x^2 + r^2x. \end{aligned} \tag{2}$$

Likewise for the third iteration

$$\begin{aligned} x''' = rx''(1-x'') = & -r^7x^8 + 4r^7x^7 - 6r^7x^6 + 4r^7x^5 - r^7x^4 \\ & -2r^6x^6 + 6r^6x^5 - 6r^6x^4 + 2r^6x^3 \\ & -r^5x^4 + 2r^5x^3 - r^5x^2 \\ & -r^4x^4 + 2r^4x^3 - r^4x^2 \\ & -r^3x^2 + r^3x, \end{aligned} \tag{3}$$

and so on and so forth.

For instance, see Figures 1 and 2 for a plot of the solutions to the 755th iteration for $x = 0.5$, $r = [0,4]$. Note that in Figure 2 there are line segments connecting the vertices that are adjacent along the r axis. This signifies that the plot is of a polynomial, and not of a set of disconnected vertices like in Figure 1.

Also for instance, see Figures 3 and 4 for a plot of the solutions to the 755th iteration for $x = 0.7$, $r = [0,4]$. Note that there are now a small number of prominent quasi-discontinuities in the solutions to the polynomial. Note that these quasi-discontinuities seem to span just between directly neighbouring branches. It seems that these so-called quasi-discontinuities are likely not genuine discontinuities since the polynomial corresponding to the 775th iteration is differentiable.

Figures 5 and 6 are a plot of the solutions to the 755th to 770th iteration for $x = 0.7$, $r = [0,4]$. Note that with multiple iterations all plotted together like this, the quasi-discontinuities are detectable only in Figure 6, where there are line segments connecting the vertices that are adjacent along the r axis. Also see Figure 7 for a plot of the solutions to the 755th to 770th

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iteration for $x = 0.71$, $r = [0,4]$. Note that there are now quite a large number of prominent quasi-discontinuities in the solutions to the polynomial.

Finally, see Figure 8 for a plot of the solutions to the 548th iteration for $x = 0.91$, $r = [0,4]$. Note that there are prominent quasi-discontinuities. Note that these quasi-discontinuities no longer span just between directly neighbouring branches.

For more information on the logistic map, see Refs. [1,2].

2 The entropy of a branching path, with and without (quasi-)discontinuities

Consider the branching path shown in Figure 9. The branching path starts with the segment A, which branches into segments B and C. Additionally, segment B branches into segments D and E, and segment C branches into segments F and G. Since the branching path starts with one segment (ie. A) and there are $(b = 2)$ levels of $(c = 2)$-way branching, the total number of end segments (ie. D, E, F, G) is $(c^b = 4)$. The four paths corresponding to these start and end segments are A $\rightarrow$ B $\rightarrow$ D, A $\rightarrow$ B $\rightarrow$ E, A $\rightarrow$ C $\rightarrow$ F, A $\rightarrow$ C $\rightarrow$ G. For instance, see the path A $\rightarrow$ B $\rightarrow$ E in Figure 10. Presuming that these four paths are all picked with equal relative frequency (ie. each is picked $(c^{-b} = 1/4)$th of the time), the entropy of the branching path is simply

$$S = \ln(4) \simeq 1.38629. \tag{4}$$

Given the quasi-discontinuities inherent to the iterations of the logistic map, it seems that the path does not necessarily have to be as simple as A $\rightarrow$ B $\rightarrow$ E. For instance, see the not so simple path A $\rightarrow$ B $\rightarrow$ Y $\rightarrow$ C $\rightarrow$ Z $\rightarrow$ B $\rightarrow$ E in Figure 11, where the segment Y spans between B and C, and then the segment Z spans between C and B. Note that although this not so simple path shares the same start and end segments as the simple path A $\rightarrow$ B $\rightarrow$ E, the two paths are not entirely the same. Also note that although these paths share the same end segment, this is no guarantee that they will continue to share all subsequent segments. Presuming that the four simple paths and this one not so simple path are all chosen with equal relative frequency, the entropy of the branching path becomes simply

$$S = \ln(5) \simeq 1.60944. \tag{5}$$

Of course, the one not so simple path may not be chosen as frequently as the others, which would bring the entropy back down. For instance, where $p_5 = 0.001$ represents how often the one not so simple path is chosen, and $p_1 = p_2 = p_3 = p_4 = (1 - p_5)/4$ represent how often the four simple paths are chosen, the entropy is not so simple

$$S = -\sum_{i=1}^{i=5} p_i \ln(p_i) \simeq 1.39282 \simeq \ln(4.02617). \tag{6}$$

In effect, the path choice can be a little more than four-way.

Of course, the path may get a little repetitive like A $\rightarrow$ B $\rightarrow$ Y $\rightarrow$ C $\rightarrow$ Z $\rightarrow$ B $\rightarrow$ Y $\rightarrow$ C $\rightarrow$ Z $\rightarrow$ B $\rightarrow$ E. Note that this path is not the entirely the same as the path A $\rightarrow$ B $\rightarrow$ Y $\rightarrow$ C $\rightarrow$ Z $\rightarrow$ B $\rightarrow$ E. Also, note again that Figure 8 shows that the quasi-discontinuities do not necessarily have to span just between directly neighbouring segments. An analogous path would be something like A $\rightarrow$ B $\rightarrow$ E $\rightarrow$ X $\rightarrow$ G, where the segment X spans between E and G.

It seems that the act of spanning is equal to the acts of making a delayed choice, tunneling, and jumping.

For more information on entropy, see Ref. [3].

3 Conclusion

Given the vertex-only bifurcation diagram in Figure 5, it first seems that each of the simple branches (ie. the branch at $r = 3$) represents a simple two-way choice. However, this first impression is not correct. The line segment bifurcation diagram in Figure 6 instead goes to show that there are quasi-discontinuities in the iterations of the logistic map. In the end, these quasi-discontinuities can make for paths that are more complicated than first expected. For instance, see Figure 7.

References

[1] Devaney RL. An Introduction to Chaotic Dynamical Systems, 2E. (2003) ISBN-13: 978-0813340852

[2] Peitgen H-O, Jürgens H, Saupe D. Chaos and Fractals: New Frontiers of Science, 2E. (2004) ISBN: 978-0387202297

[3] Cover TM, Thomas JA. Elements of Information Theory, 2E. (2006) ISBN: 978-0471241959

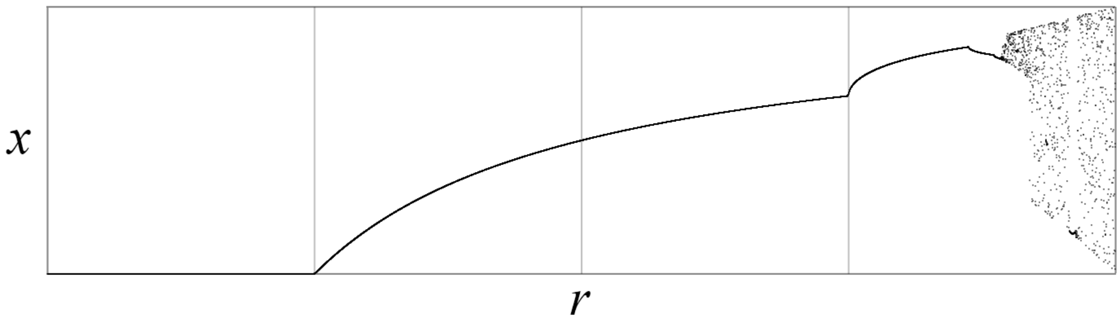

Figure 1: The solutions to the 755th iteration of the logistic map for $x = 0.5$, $r = [0,4]$. Only vertices are drawn.

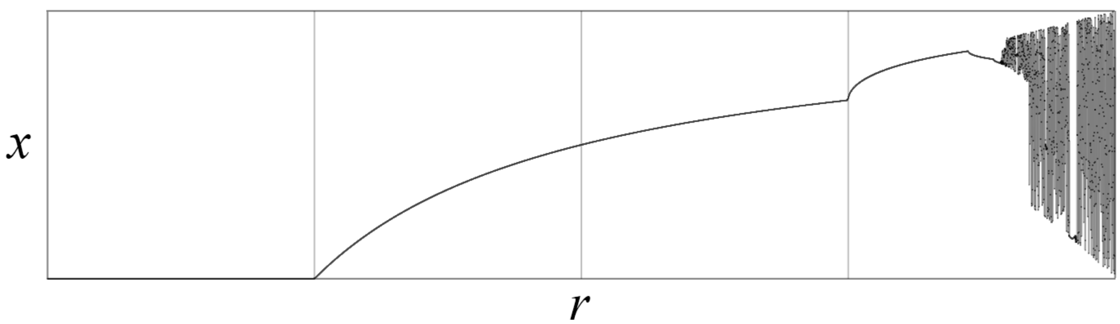

Figure 2: The solutions to the 755th iteration of the logistic map for $x = 0.5$, $r = [0,4]$. Both vertices and line segments are drawn.

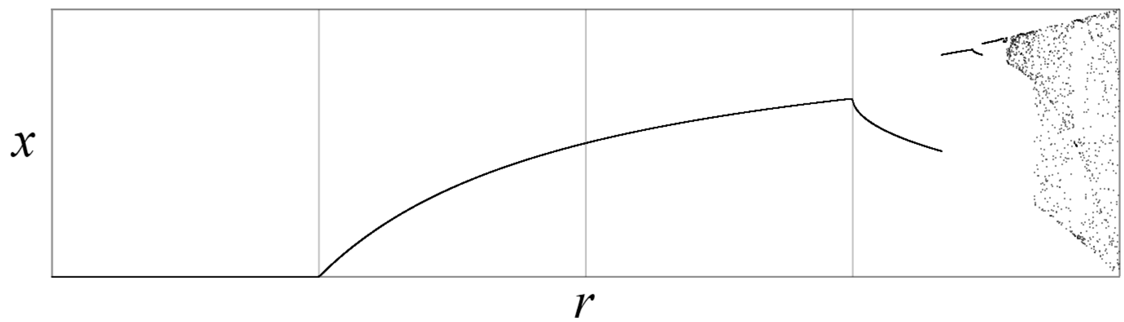

Figure 3: The solutions to the 755th iteration of the logistic map for $x = 0.7$, $r = [0,4]$. Only vertices are drawn.

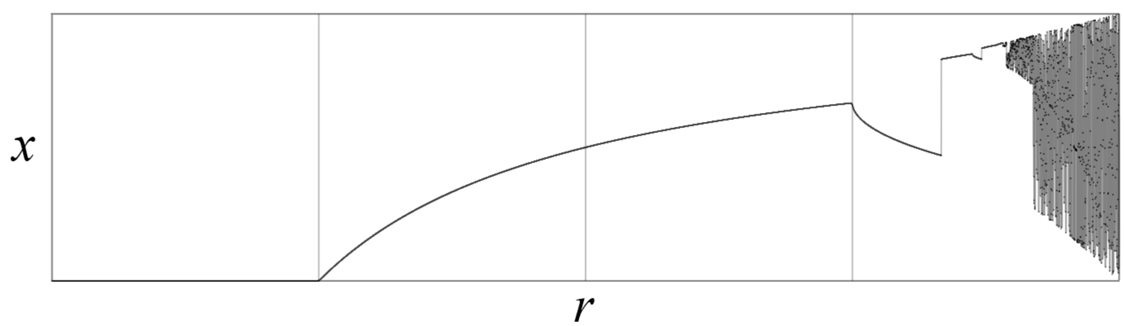

Figure 4: The solutions to the 755th iteration of the logistic map for $x = 0.7$, $r = [0,4]$. Both vertices and line segments are drawn.

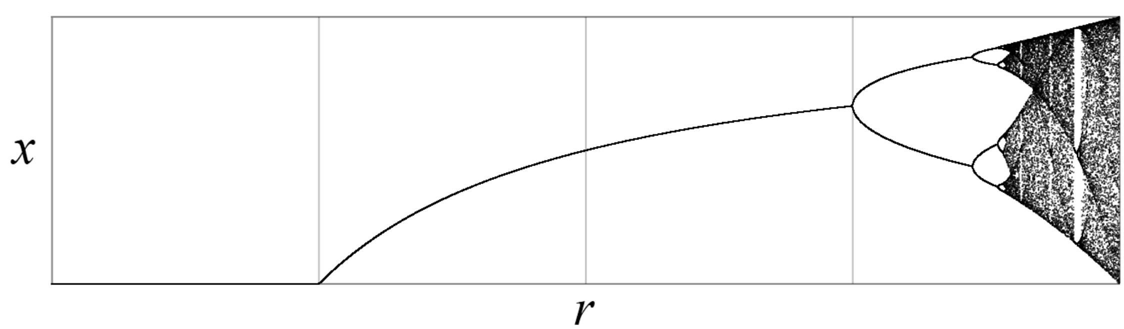

Figure 5: The solutions to the 755th to the 770th iteration of the logistic map for $x = 0.7$, $r = [0,4]$. Only vertices are drawn.

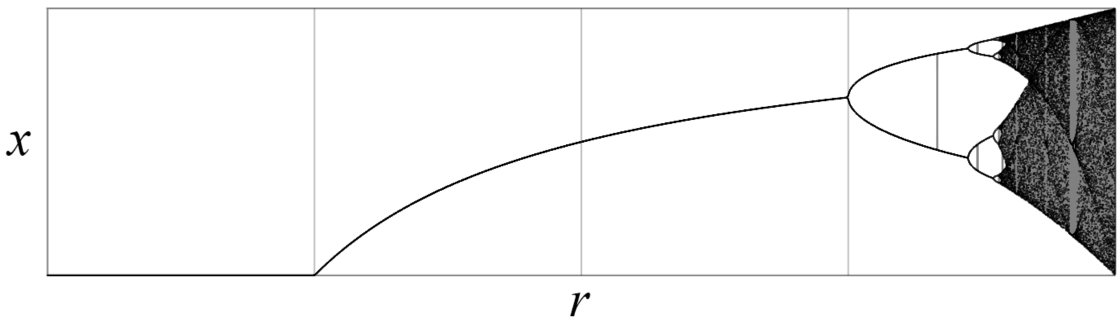

Figure 6: The solutions to the 755th to the 770th iteration of the logistic map for $x = 0.7$, $r = [0,4]$. Both vertices and line segments are drawn.

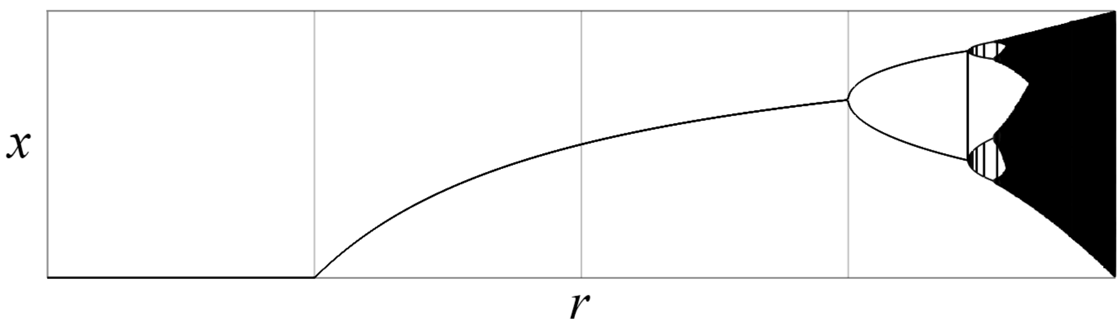

Figure 7: The solutions to the 755th to the 770th iteration of the logistic map for $x = 0.71$, $r = [0,4]$. Only line segments are drawn.

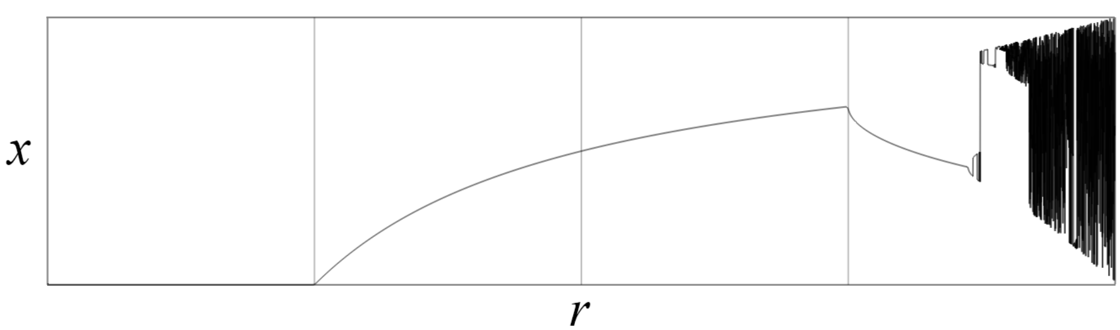

Figure 8: The solutions to the 548th iteration of the logistic map for $x = 0.91$, $r = [0,4]$. Only line segments are drawn.

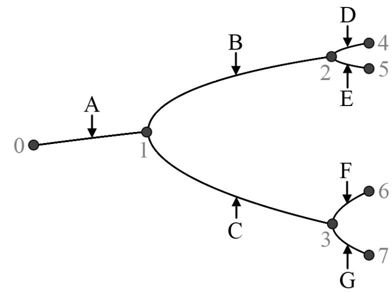

Figure 9: A branching path: A branches into B, C. B branches into D, E. C branches into F, G.

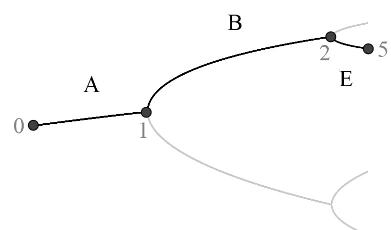

Figure 10: One simple path: A → B → E.

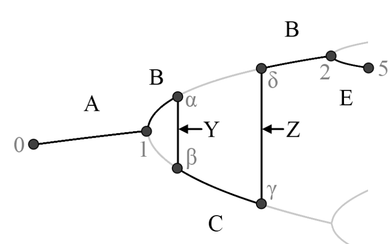

Figure 11: One not so simple path: A → B → Y → C → Z → B → E. Note that although this path shares the same start and end segments as the simple path A → B → E (ie. see Figure 10), the two paths are not entirely the same.