The finite Yang-Laplace Transform in fractal space

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Abstract: In this paper, we establish finte Yang-Laplace Transform on fractal space, considered some properties of finte Yang-Laplace Transform.
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1 Introduction

Local fractional calculus has played an important role in areas ranging from fundamental science to engineering in the past ten years [1-18]. It is significant to deal with the continuous functions (fractal functions), which are irregular in the real world. Recently, Yang-Laplace transform based on the local fractional calculus was introduced [9] and Yang continued to study this subject [10]. The Yang-Laplace transform of $f(x)$ is given by [9,10]

$$L_{\alpha}\{f(x)\}=f_s^{L,\alpha}(s):=\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}E_{\alpha}(-s^{\alpha}x^{\alpha})f(x)(dx)^{\alpha},\quad 0<\alpha\le 1, \tag{1.1}$$

And its Inverse formula of Yang- Laplace's transforms as follows

$$f(x)=L_{\alpha}^{-1}(f_s^{L,\alpha}(s)):=\frac{1}{(2\pi)^{\alpha}}\int_{\beta-i\infty}^{\beta+i\infty}E_{\alpha}(s^{\alpha}x^{\alpha})f_s^{L,\alpha}(s)(ds)^{\alpha} \tag{1.2}$$

The purpose of this paper is to establish the finte Yang-Laplace Transforms based on the Yang-Laplace transforms and consider its some properties.

2 The Finite Yang-Laplace Transform and its properties

In the section, both finite Yang-Laplace transform and its inverse are defined from the corresponding Yang-Laplace transform and its inverse .

Definition 2.1 (The Finite Yang-Laplace Transform). If $f(x)$ is a continuous or piecewise continuous function on a finite interval $0<x<T$, the finite local fractional Laplace transform of $f(x)$ is defined by

$$L_{\alpha,T}\{f(x)\}=\tilde{f}_s^{L,\alpha}(s,T):=\frac{1}{\Gamma(1+\alpha)}\int_0^{T}E_{\alpha}(-s^{\alpha}x^{\alpha})f(x)(dx)^{\alpha} \tag{2.1}$$

where s is a real or complex number and T is a finite number that may be positive or negative so that (2.1) can be defined in any interval $(-T_1,T_2)$. Clearly, $L_{\alpha,T}$ is a linear integral transformation.

The inverse finite Yang-Laplace transform is defined by the complex integral

$$f(x)=L_{\alpha,T}^{-1}(\tilde{f}_s^{L,\alpha}(s,T)):=\frac{1}{(2\pi i)^{\alpha}}\int_{\beta-i\infty}^{\beta+i\infty}E_{\alpha}(s^{\alpha}x^{\alpha})\tilde{f}_s^{L,\alpha}(s,T)(ds)^{\alpha} \tag{2.2}$$

where the integral is taken over any open contour Γ joining any two points $\beta - iR$ and $\beta + iR$ in the finite complex s plane as $R \to \infty$.

If $f(x)$ is almost piecewise continuous, that is, it has at most a finite number of simple discontinuities in $0 \le x \le T$. Moreover, in the intervals where $f(x)$ is continuous, it satisfies a Lipschitz condition of order $\gamma > 0$. Under these conditions, it can be shown that the inversion integral (2.2) is equal to.

$$\frac{1}{(2\pi i)^\alpha}\int_\Gamma E_\alpha(s^\alpha x^\alpha)\tilde{f}_s^{L,\alpha}(s,T)(ds)^\alpha = \frac{1}{2}[f(x-0)+f(x+0)] \quad , \tag{2.3}$$

where Γ is an arbitrary open contour that terminates with finite constant β as $R \to \infty$. This is due to the fact that $\tilde{f}_s^{L,\alpha}(s,T)$ is an entire function of s.

Example 2.1 if $f(x) = 1$, then

$$L_{\alpha,T}\{1\} = -\frac{1}{s^\alpha}E_\alpha(-s^\alpha x^\alpha)\big|_0^T = -[E_\alpha(-s^\alpha T^\alpha)-1] = [1-E_\alpha(-s^\alpha T^\alpha)] \tag{2.4}$$

Example 2.2 if $f(x) = E_\alpha(a^\alpha x^\alpha)$,

$$\begin{aligned} L_{\alpha,T}\{E_\alpha(a^\alpha x^\alpha)\} &= -\frac{1}{s^\alpha - a^\alpha}[E_\alpha(-(s^\alpha - a^\alpha)T^\alpha)-1] \\ &= \frac{1}{s^\alpha - a^\alpha}[1-E_\alpha(-(s^\alpha - a^\alpha)T^\alpha)] \end{aligned} \tag{2.5}$$

Theorem 2.1 if $L_{\alpha,T}\{f(x)\} = \tilde{f}_s^{L,\alpha}(s,T)$,then

$$L_{\alpha,T}\{E_\alpha(-a^\alpha x^\alpha)f(x)\} = \tilde{f}_s^{L,\alpha}(s+a,T) \qquad \text{(Shifting)} \tag{2.6}$$

$$L_{\alpha,T}\{f(ax)\} = \frac{1}{a^\alpha}\tilde{f}_s^{L,\alpha}\left(\frac{s}{a},aT\right) \qquad \text{(Scaling)} \tag{2.7}$$

Proof

$$\begin{aligned} L_{\alpha,T}\{E_\alpha(-a^\alpha x^\alpha)f(x)\} &= \frac{1}{\Gamma(1+\alpha)}\int_0^T E_\alpha(-a^\alpha x^\alpha)E_\alpha(-s^\alpha x^\alpha)f(x)(dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^T E_\alpha(-(s+a)^\alpha x^\alpha)f(x)(dx)^\alpha = \tilde{f}_s^{L,\alpha}(s+a,T) \end{aligned}$$

Let $y = ax$, we have

$$\begin{aligned} L_{\alpha,T}\{f(ax)\} &= \frac{1}{\Gamma(1+\alpha)}\int_0^T E_\alpha(-s^\alpha x^\alpha)f(ax)(dx)^\alpha \\ &= \frac{1}{a^\alpha\Gamma(1+\alpha)}\int_0^{aT}E_\alpha\left(-\frac{s^\alpha}{a^\alpha}x^\alpha\right)f\left(\tfrac{y}{a}\right)(dy)^\alpha = \tilde{f}_s^{L,\alpha}\left(\frac{s}{a},aT\right) \end{aligned}$$

Theorem 2.2 (Finite local fractional Laplace Transforms of Derivatives).

if $L_{\alpha,T}\{f(x)\} = \tilde{f}_s^{L,\alpha}(s,T)$,then

$$L_{\alpha,T}\{f^{(\alpha)}(x)\} = s^\alpha\tilde{f}_s^{L,\alpha}(s,T) - f(0) + E_\alpha(-s^\alpha T^\alpha)f(T) \tag{2.8}$$

$$L_{\alpha,T}\{f^{(2\alpha)}(x)\}=s^{2\alpha}\tilde{f}_s^{L,\alpha}(s,T)-s^{\alpha}f(0)$$
$$-f^{(\alpha)}(0)+s^{\alpha}f(T)E_{\alpha}(-s^{\alpha}T^{\alpha})+f^{(\alpha)}(T)E_{\alpha}(-s^{\alpha}T^{\alpha}) \qquad (2.9)$$

More generally,

$$L_{\alpha,T}\{f^{(n)}(x)\}=s^{n\alpha}\tilde{f}_s^{L,\alpha}(s,T)-\sum_{k=1}^{n}s^{(n-k)\alpha}f^{((k-1)\alpha)}(0)+E_{\alpha}(-s^{\alpha}T^{\alpha})\sum_{k=1}^{n}s^{(n-k)\alpha}f^{((k-1)\alpha)}(T) \qquad (2.10)$$

Proof. Integrating by parts, we have

$$L_{\alpha,T}\{f^{(\alpha)}(x)\}=s^{\alpha}\tilde{f}_s^{L,\alpha}(s,T)+E_{\alpha}(-s^{\alpha}T^{\alpha})f(T)-f(0)$$

Repeating this process gives (2.8). By induction, we can prove (2.9).

Theorem 2.3 (Finite local fractional Laplace Transform of Integrals). If

$$F(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x f(t)(dt)^{\alpha} \qquad (2.11)$$

so that $F^{(\alpha)}(x)=f(x)$ for all x, then

$$L_{\alpha,T}\{F(x)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^T E_{\alpha}(-s^{\alpha}x^{\alpha})F(x)(dx)^{\alpha}$$

Proof . We have from (2.10)

$$L_{\alpha,T}\{F^{(\alpha)}(x)\}=s^{\alpha}L_{\alpha,T}\{F(x)\}-F(0)+E_{\alpha}(-s^{\alpha}T^{\alpha})F(T)$$

Or

$$L_{\alpha,T}\{f(x)\}=s^{\alpha}L_{\alpha,T}\{\frac{1}{\Gamma(1+\alpha)}\int_0^x f(t)(dt)^{\alpha}\}+E_{\alpha}(-s^{\alpha}T^{\alpha})F(T)$$

Hence

$$L_{\alpha,T}\{\frac{1}{\Gamma(1+\alpha)}\int_0^x f(t)(dt)^{\alpha}\}=\frac{1}{s^{\alpha}}[L_{\alpha,T}\{f(x)\}-E_{\alpha}(-s^{\alpha}T^{\alpha})F(T)]$$

$$L_{\alpha,T}\{F(x)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^T E_{\alpha}(-s^{\alpha}x^{\alpha})F(x)(dx)^{\alpha}$$

Theorem 2.4 if $L_{\alpha,T}\{f(x)\}=\tilde{f}_s^{L,\alpha}(s,T)$,then

$$\frac{d^{\alpha}\tilde{f}_s^{L,\alpha}(s,T)}{ds^{\alpha}}=L_{\alpha,T}\{(-x)^{\alpha}f(x)\} \qquad (2.12)$$

$$\frac{d^{2\alpha}\tilde{f}_s^{L,\alpha}(s,T)}{ds^{2\alpha}}=L_{\alpha,T}\{(-x)^{2\alpha}f(x)\} \qquad (2.13)$$

More generally,

$$\frac{d^{n\alpha}\tilde{f}_s^{L,\alpha}(s,T)}{ds^{n\alpha}}=L_{\alpha,T}\{(-x)^{n\alpha}f(x)\} \qquad (2.14)$$

Proof.

$$\frac{d^{\alpha}\tilde{f}_s^{L,\alpha}(s,T)}{ds^{\alpha}}=\frac{1}{\Gamma(1+\alpha)}\int_0^T E_{\alpha}(-s^{\alpha}x^{\alpha})(-x)^{\alpha}f(x)(dx)^{\alpha}=L_{\alpha,T}\{(-x)^{\alpha}f(x)\}$$

Similarly, we obtain （2.13）and (2.14).

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