The local fractional Stieltjes Transform in fractal space

Guang-Sheng Chen
Department of Computer Engineering, Guangxi Modern Vocational Technology College,
Hechi,Guangxi, 547000, P.R. China
E-mail address: cgswavelets@126.com(Chen)

Abstract: This paper deals with the theory of the local fractional Stieltjes transform. We derive the Stieltjes transform. This is followed by several examples and the basic operational properties of Stieltjes transforms.

Keywords: fractal space, local fractional Stieltjes Transform, local fractional integrals, local fractional derivative

1 Introduction

Local fractional calculus has played an important role in areas ranging from fundamental science to engineering in the past ten years [1-18]. It is significant to deal with the continuous functions (fractal functions), which are irregular in the real world. Recently, Yang-Laplace transform based on the local fractional calculus was introduced [9] and Yang continued to study this subject [10]. The Yang-Laplace transform of $f(x)$ is given by [9,10]

$$L_\alpha\{f(x)\} = f_s^{L,\alpha}(s) := \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha\ ,\quad 0<\alpha\le 1, \tag{1.1}$$

And its Inverse formula of Yang- Laplace's transforms as follows

$$f(x) = L_\alpha^{-1}(f_s^{L,\alpha}(s)) := \frac{1}{(2\pi)^\alpha}\int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(s)(ds)^\alpha \tag{1.2}$$

The purpose of this paper is to establish local fractional Stieltjes Transforms based on the Yang-Laplace transforms and consider its application to local fractional equations with local fractional derivative. This paper is organized as follows. In section 2, local fractional Stieltjes Transforms is derived; Section 3 presents Properties of local fractional Stieltjes Transforms Transforms.

2 Definition of the Stieltjes Transform and Examples

In the section, we define Stieltjes transform and show some examples of Stieltjes transform.

We use the local fractional Laplace transform (Yang-Laplace transform) of $L_\alpha\{f(t)\} = f_s^{L,\alpha}(s)$ with respect to s to define the Stieltjes transform of $f(t)$. Clearly,

$$\begin{aligned} L_\alpha\{f_s^{L,\alpha}(s)\} = \tilde{f}_\alpha(z) &= \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha z^\alpha) f_s^{L,\alpha}(s)(ds)^\alpha \\ &= \int_0^\infty E_\alpha(-s^\alpha z^\alpha)(ds)^\alpha \int_0^\infty E_\alpha(-s^\alpha t^\alpha) f(t)(dt)^\alpha \end{aligned} \tag{2.1}$$

Interchanging the order of integration and evaluating the inner integral, we obtain

$$\tilde{f}_\alpha(z) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{f(t)}{(t+z)^\alpha}(dt)^\alpha \tag{2.2}$$

The Stieltjes transform of a function $f(t)$ on $0 \le t < \infty$ is denoted by $\tilde{f}_\alpha(z)$ and defined by

$$S_\alpha\{f(t)\} = \tilde{f}_\alpha(z) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{f(t)}{(t+z)^\alpha}(dt)^\alpha \qquad (2.3)$$

where z is a complex variable in the cut plane $|\arg z| < \pi$.

If $z = x$ is real and positive, then

$$S_\alpha\{f(t)\} = \tilde{f}_\alpha(x) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{f(t)}{(t+x)^\alpha}(dt)^\alpha \qquad (2.4)$$

Differentiating (2.4) with respect to x, we obtain

$$\frac{d^{n\alpha}\tilde{f}_\alpha(x)}{dx^{n\alpha}} = \frac{\Gamma(1-\alpha)}{\Gamma(1-(n+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{f(t)}{(t+x)^{(n+1)\alpha}}(dt)^\alpha \quad , \quad n = 1, 2, \ldots \qquad (2.5)$$

Example 2.1 Find the Stieltjes transform of each of the following functions

(a) $f(t) = \frac{1}{(t+a)^\alpha}$, (b) $f(t) = t^{(\beta-1)\alpha}$.

(a) We have, by definition,

$$\tilde{f}_\alpha(z) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{f(t)}{(t+z)^\alpha}(dt)^\alpha = \frac{1}{(a-z)^\alpha}\ln_\alpha\left|\frac{a^\alpha}{z^\alpha}\right| \qquad (2.6)$$

(b) set $x = \frac{t}{z}$,we have

$$\tilde{f}_\alpha(z) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{f(t)}{(t+z)^\alpha}(dt)^\alpha = z^{(\beta-1)\alpha}\Gamma(\beta)\Gamma(1-\beta) \qquad (2.7)$$

Example 2.2 Show that

$$S_\alpha\{\sin_\alpha k^\alpha\sqrt{t^\alpha}\} = 2^\alpha E_\alpha(-k^\alpha\sqrt{z^\alpha}) \quad , k > 0 \qquad (2.8)$$

We have, by definition $u = \sqrt{t}$, $t = u^2$

$$S_\alpha\{\sin_\alpha k^\alpha\sqrt{t^\alpha}\} = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{2^\alpha u^\alpha \sin_\alpha k^\alpha u^\alpha}{u^{2\alpha}+z^\alpha}(du)^\alpha = 2^\alpha E_\alpha(-k^\alpha\sqrt{z^\alpha})$$

3 Basic Operational Properties of Stieltjes Transforms

The following properties hold for the Stieltjes transform:

(a)

$$S_\alpha\{f(t+a)\} = \tilde{f}_\alpha(z-a) \qquad (3.1)$$

(b) $$S_\alpha\{f(at)\} = \tilde{f}_\alpha(az), a > 0 \qquad (3.2)$$

(c) $$S_\alpha\{t^\alpha f(t)\} = -z^\alpha\tilde{f}_\alpha(z) + \frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(t)(dt)^\alpha \qquad (3.3)$$

provided the integral on the right hand side exists.

(d) $$S_\alpha\{\frac{f(t)}{(t+a)^\alpha}\}=\frac{1}{(a-z)^\alpha}[\tilde{f}_\alpha(z)-\tilde{f}_\alpha(a)] \quad (3.4)$$

(e) $$S_\alpha\{\frac{1}{t^\alpha}f(\frac{a}{t})\}=\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\frac{1}{z^\alpha}\tilde{f}_\alpha(\frac{a}{z}),\quad a>0 \quad (3.5)$$

Pfoof . (a) We have, by definition, $t+a=\eta$

$$S_\alpha\{f(t+a)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty\frac{f(\eta)}{(\eta+z-a)^\alpha}(d\eta)^\alpha=\tilde{f}_\alpha(z-a)$$

(b) We have, by definition, $u=at$

$$S_\alpha\{f(at)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty\frac{f(u)}{(u+az)^\alpha}(du)^\alpha=\tilde{f}_\alpha(az)$$

(c) We have from the definition

$$S_\alpha\{t^\alpha f(t)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty\frac{t^\alpha f(t)}{(t+z)^\alpha}(dt)^\alpha$$
$$=-z^\alpha\tilde{f}_\alpha(z)+\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(t)(dt)^\alpha$$

This gives the desired result

(d) We have, by definition,

$$S_\alpha\{\frac{f(t)}{(t+a)^\alpha}\}=\frac{1}{(a-z)^\alpha}[\tilde{f}_\alpha(z)-\tilde{f}_\alpha(a)]$$

(e) We have, by definition, $\eta=\frac{a}{t}$

$$S_\alpha\{\frac{1}{t^\alpha}f(\frac{a}{t})\}=\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\frac{1}{z^\alpha}\tilde{f}_\alpha(\frac{a}{z})$$

Theorem 3.1 (Stieltjes Transforms of Derivatives). If $S_\alpha\{f(t)\}=\tilde{f}_\alpha(z)$,then

$$S_\alpha\{f^{(\alpha)}(t)\}=-\frac{f(0)}{z^\alpha}-\frac{d^\alpha\tilde{f}_\alpha(z)}{dz^\alpha} \quad (3.6)$$

$$S_\alpha\{f^{(2\alpha)}(t)\}=-\frac{f^{(\alpha)}(0)}{z^\alpha}+\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\frac{f(0)}{z^{2\alpha}}+\frac{\Gamma(1-\alpha)}{\Gamma(1-3\alpha)}\frac{d^{2\alpha}\tilde{f}_\alpha(z)}{dz^{2\alpha}} \quad (3.7)$$

Proof. Appling the definition and integrating by parts, we have

$$S_\alpha\{f^{(\alpha)}(t)\}=-\frac{f(0)}{z^\alpha}-\frac{d^\alpha\tilde{f}_\alpha(z)}{dz^\alpha}$$

This proves result (3.6).

Similarly, other results can readily be proved.

$$S_{\alpha}\{f^{(2\alpha)}(t)\}=-\frac{f^{(\alpha)}(0)}{z^{\alpha}}+\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\frac{f(0)}{z^{2\alpha}}+\frac{\Gamma(1-\alpha)}{\Gamma(1-3\alpha)}\frac{d^{2\alpha}\tilde{f}_{\alpha}(z)}{dz^{2\alpha}}$$

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