# Entropy and the individual iterations of the logistic map

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**Abstract**

It is presumed *a priori* that there is an entropy-area relationship inherent to the iterations of the logistic map. Several interesting results are produced.

## 1 The first iteration of the logistic map

The paper "Fibonacci order in the period-doubling cascade to chaos" by Linage *et al* [1] describes how the golden ratio

$$\phi = \frac{\sqrt{5}+1}{2} \simeq 1.61803. \tag{1}$$

can be found by analyzing the bifurcation diagram that is generated by compositing tens (or hundreds, *etc*) of iterations of the logistic map

$$x' = rx(1-x) = -rx^2 + rx. \tag{2}$$

This paper will instead describe how the golden ratio and many other interesting values can be found by analyzing the individual iterations of the logistic map. This paper presumes *a priori* that there is an entropy-area relationship inherent to the iterations of the logistic map, and that this relationship is simply one unit of entropy per unit area. This paper presumes *a priori* that entropy is measured in natural units (ie. base $e \simeq 2.71828$).

Figure 1 is a plot of the solutions $x'$ to the polynomial given in Eq. (2) for the interval $r = [0,4]$, $x = [0,1]$, where the magnitudes of the solutions are represented by brightness. Essentially, the plot shows the emergence at $r = 2$ of a continually widening trunk that effectively ends at $r = 4$. Note that the trunk is not quite fully mirror symmetric with regard to the line $x = 0.5$, insomuch that the trunk is slightly droopy. Figure 2 is an alternative version of Figure 1 with extremely high contrast. The pure white region in this second figure is the superlevel set of this polynomial where brightness $\geq 0.5$. This high contrast plot makes the emergence of the trunk at $r = 2$ all the more obvious.

Setting $x = 0.5$ as a constant, we get a horizontal cross-section along the middle of the trunk in the form of the expression

$$Brightness = \frac{r}{4}. \tag{3}$$

The indefinite integral of this expression, without the constant of integration, is

$$\int \left(\frac{r}{4}\right) dr = \frac{r^2}{8}. \tag{4}$$

Note that the area under the expression's solutions between $r = 2$, $r = 4$ is precisely

$$A_0 = \frac{4^2}{8} - \frac{2^2}{8} = 1.5 = \ln(e^{1.5}). \tag{5}$$

Also, the area under the expression's solutions between $r = 0$, $r = 4$ is precisely

$$A_1 = \frac{4^2}{8} - \frac{0^2}{8} = 2 = \ln(e^2). \tag{6}$$

Finally, the area under the line $x = 0.5$ between $r = 2$, $r = 4$ is precisely

$$A_2 = 0.5 \times (4-2) = 1 = \ln(e). \tag{7}$$

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## 2 The second iteration of the logistic map

The golden ratio can be found directly by analyzing the second iteration of the logistic map

$$x'' = rx'(1-x') = -r^3x^4 + 2r^3x^3 - r^3x^2 - r^2x^2 + r^2x. \tag{8}$$

Figure 3 is a plot of the solutions $x''$ to the polynomial given in Eq. (8) for the interval $r = [0,4]$, $x = [0,1]$, where the magnitudes of the solutions are represented by brightness. Essentially, the plot shows the emergence of a continually widening trunk that splits at $r = \sqrt{5}+1$ into two branches that effectively end at $r = 4$. Figure 4 is an alternative version of Figure 3 with extremely high contrast. This high contrast plot goes to show how the trunk emerges at $r = 2$, and then splits at $r = \sqrt{5}+1$. Of course, $2/(\sqrt{5}+1) = 1/\phi$ and $(\sqrt{5}+1)/2 = \phi$. Note that the trunk and branches are not quite fully mirror symmetric with regard to the line $x = 0.5$, insomuch that they are slightly droopy.

Setting $x = 0.5$ as a constant, we get a horizontal cross-section along the middle of the trunk in the form of the expression

$$Brightness = -\frac{r^3}{16} + \frac{r^2}{4}. \tag{9}$$

Figure 5 is a plot of this cross-section. The indefinite integral of the expression given in Eq. (9) is

$$\int \left(-\frac{r^3}{16} + \frac{r^2}{4}\right) dr = -\frac{r^4}{64} + \frac{r^3}{12}. \tag{10}$$

The area under the curve between $r = 2$, $r = \sqrt{5}+1$ is

$$A_0 \simeq 0.693853 = \ln(2.00141). \tag{11}$$

The area under the curve between $r = 0$, $r = \sqrt{5}+1$ is

$$A_1 \simeq 1.11052 = \ln(3.03594). \tag{12}$$

These two areas are shown in Figures 6, 7 respectively. Note that the area under the line $x = 0.5$ between $r = 2$, $r = \sqrt{5}+1$ is precisely

$$A_2 = 0.5 \times \left((\sqrt{5}+1) - 2\right) = 1/\phi = \phi - 1 = 0.618034 = \ln(1.85528). \tag{13}$$

## 3 The third iteration of the logistic map

The third iteration of the logistic map is

$$\begin{aligned} x''' = rx''(1-x'') = &-r^7x^8 + 4r^7x^7 - 6r^7x^6 + 4r^7x^5 - r^7x^4 \\ &-2r^6x^6 + 6r^6x^5 - 6r^6x^4 + 2r^6x^3 \\ &-r^5x^4 + 2r^5x^3 - r^5x^2 \\ &-r^4x^4 + 2r^4x^3 - r^4x^2 \\ &-r^3x^2 + r^3x. \end{aligned} \tag{14}$$

Figure 8 is a plot of the solutions $x'''$ to the polynomial given in Eq. (14) for the interval $r = [0,4]$, $x = [0,1]$. Essentially, the plot shows the emergence of a continually widening trunk that splits at $r = \sqrt{5}+1$ into three branches, and then splits one last time at roughly $r \simeq 3.83187$ into four branches that effectively end at $r = 4$. Figure 9 is an alternative version of Figure 8 with extremely high contrast. Note that the trunk and branches are not quite fully mirror symmetric with regard to the line $x = 0.5$, insomuch that they are slightly droopy.

Setting $x = 0.5$ as a constant, the indefinite integral of the horizontal cross-section along the middle of the trunk is

$$\int \left(-\frac{r^7}{256} + \frac{r^6}{32} - \frac{r^5}{16} - \frac{r^4}{16} + \frac{r^3}{4}\right) dr = -\frac{r^8}{2048} + \frac{r^7}{224} - \frac{r^6}{96} - \frac{r^5}{80} + \frac{r^4}{16}. \tag{15}$$

The area under the curve between $r = 2$, $r \simeq 3.83187$ is

$$A_0 \simeq 1.2499 \simeq \ln(3.49). \tag{16}$$

The area under the curve between $r = 0$, $r \simeq 3.83187$ is

$$A_1 \simeq 1.62966 = \ln(5.10216). \tag{17}$$

Note that the area under the line $x = 0.5$ between $r = 2$, $r \simeq 3.83187$ is

$$A_2 = 0.5 \times (3.83187 - 2) \simeq 0.915937 = \ln(2.49912), \tag{18}$$

which is a rough approximation of Catalan's constant

$$K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \simeq 0.915966. \tag{19}$$

# 4 Discussion

Although the argument given here for an entropy-area relationship is at best heuristic (and at worst plain numerology), the large number of interesting values produced by just the first three iterations of the logistic map alone seems to imply that further investigation is warranted.

It is interesting to note that the act of making a which-way choice is bound to occur during the act of moving along a branching object. Of course, frequently picked choices produce the most entropy. It is also interesting to note that the area value $A_0 \simeq \ln(2.00141)$ for the second iteration of the logistic map is very close to the well-known Kolmogorov-Sinai entropy of $\ln(2)$.

Note that finding the area $A_2$ for iterations four and above is not as simple as treating the region as a rectangle. This is because the solutions to the polynomial rise above and dip below the threshold value of $x = 0.5$ many times before the last dip (ie. the last branching) occurs, each taking a chunk out of the otherwise ideal rectangle. For instance, see Figures 10 and 11 for plots of the solutions $x''''''$ to the polynomial corresponding to the sixth iteration of the logistic map.

The following is a table of the area values for the first seven iterations of the logistic map:

| Iteration | Emergence $r$ | Last branching $r$ | $A_0$ | $\exp(A_0)$ | $A_1$ | $\exp(A_1)$ | $A_2$ | $\exp(A_2)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 1.5 | $e^{1.5}$ | 2 | $e^2$ | 1 | $e$ |
| 2 | 2 | $\sqrt{5}+1$ | 0.693853 | 2.00141 | 1.11052 | 3.03594 | 0.618034 | 1.85528 |
| 3 | 2 | 3.83187 | 1.2499 | 3.49 | 1.62966 | 5.10216 | 0.915937 | 2.49912 |
| 4 | 2 | 3.96027 | 1.17844 | 3.24929 | 1.53789 | 4.65477 | 0.974653 | 2.65025 |
| 5 | 2 | 3.99026 | 1.31363 | 3.71964 | 1.66051 | 5.26197 | 0.958451 | 2.60765 |
| 6 | 2 | 3.99758 | 1.13624 | 3.11504 | 1.47471 | 4.36978 | 0.960769 | 2.61371 |
| 7 | 2 | 3.99939 | 1.3432 | 3.83128 | 1.67573 | 5.34271 | 0.970574 | 2.63946 |

# References

[1] Linage G, Montoya F, Sarmiento A, Showalter K, Parmananda P. Fibonacci order in the period-doubling cascade to chaos. (2006) Physics Letters A 359

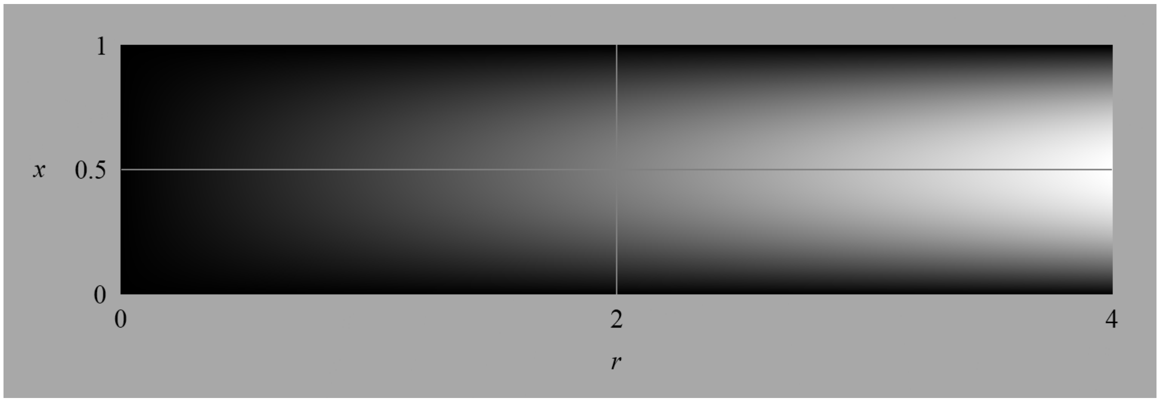


Figure 1: The solutions $x'$ to the polynomial given in Eq. (2) for the interval $r = [0,4]$, $x = [0,1]$.

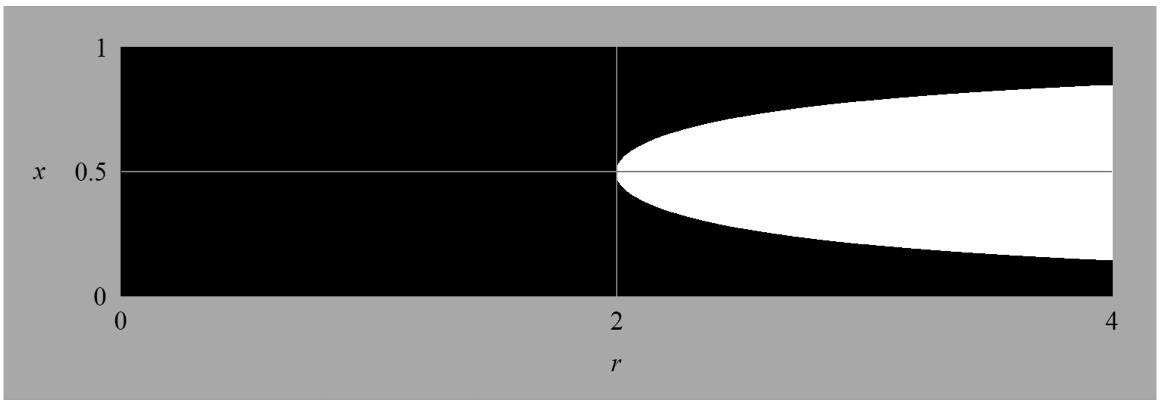


Figure 2: A high contrast version of Figure 1. The pure white region is the superlevel set of the polynomial where brightness $\geq 0.5$. This goes to show how the trunk emerges at $r = 2$.

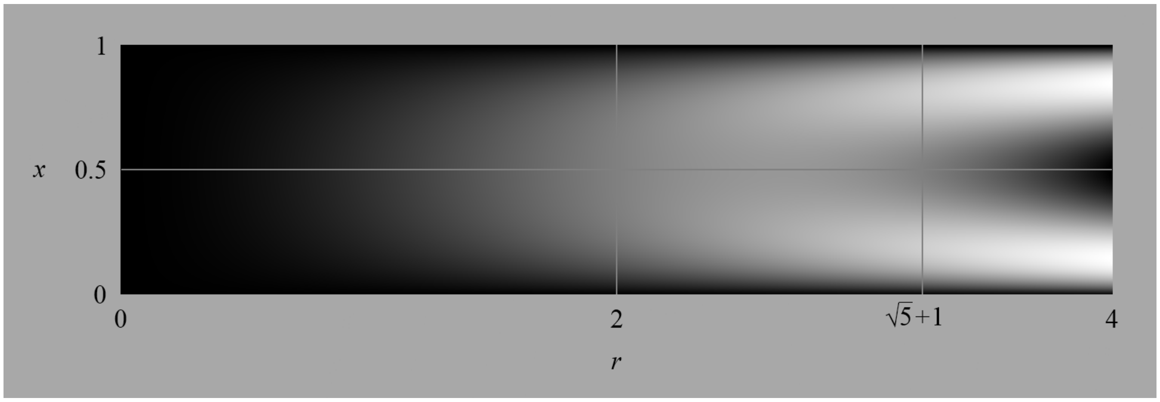


Figure 3: The solutions $x''$ to the polynomial given in Eq. (8) for the interval $r = [0,4]$, $x = [0,1]$.

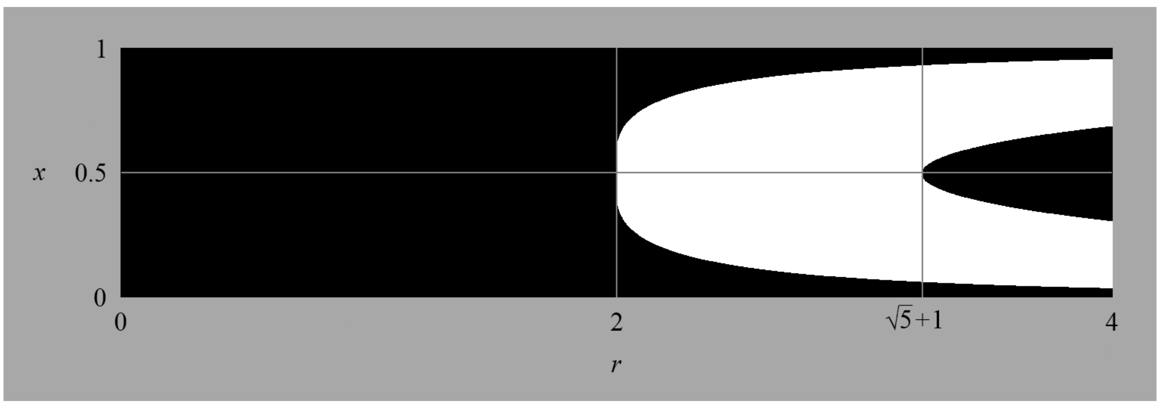


Figure 4: A high contrast version of Figure 3. The pure white region is the superlevel set of the polynomial where brightness $\geq 0.5$. This goes to show how the trunk emerges at $r = 2$, and then splits at $r = \sqrt{5}+1$ into two branches.

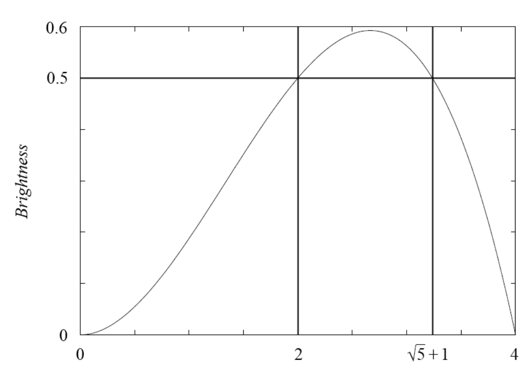


Figure 5: Cross-section of constant $x = 0.5$.

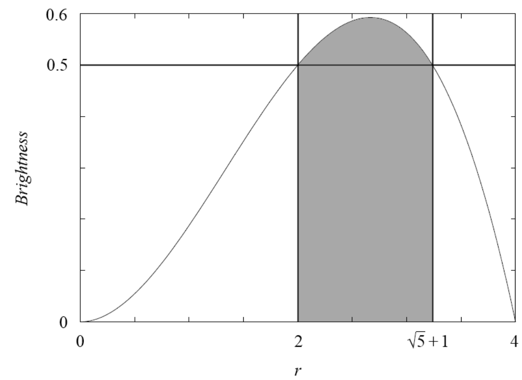


Figure 6: Cross-section of constant $x = 0.5$. The gray region under the curve has an area $A_0$ of $0.693853 = \ln(2.00141)$. Also, the gray subregion under the line $x = 0.5$ has an area $A_2$ of $1/\phi = \phi - 1 \simeq 0.618034 = \ln(1.85528)$.

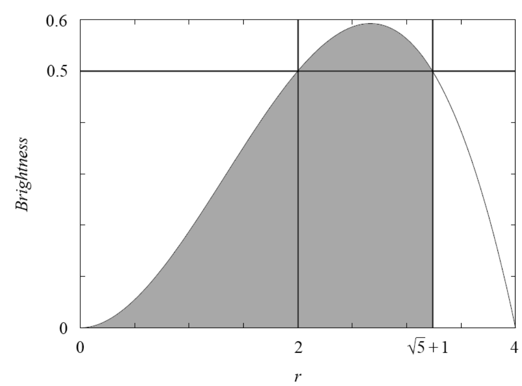


Figure 7: Cross-section of constant $x = 0.5$. The gray region under the curve has an area $A_1$ of $1.11052 = \ln(3.03594)$.

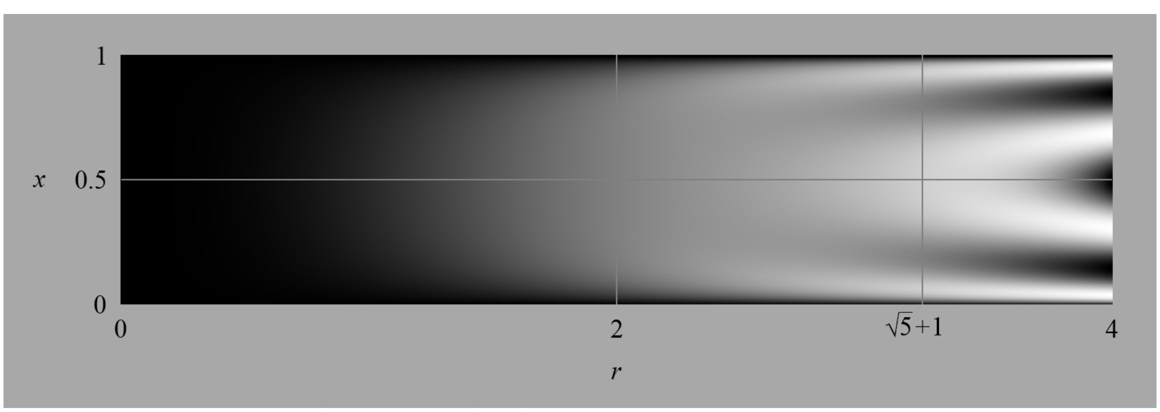


Figure 8: The solutions $x'''$ to the polynomial given in Eq. (14) for the interval $r = [0, 4]$, $x = [0, 1]$.

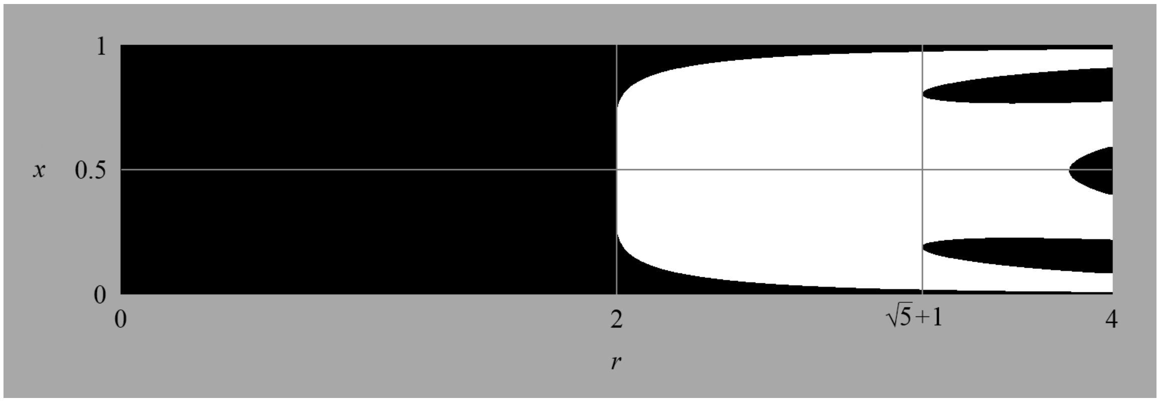


Figure 9: A high contrast version of Figure 8. The pure white region is the superlevel set of the polynomial where brightness $\geq 0.5$. This goes to show how the trunk emerges at $r = 2$, splits at $r = \sqrt{5}+1$ into three branches, and then splits one last time into four branches at roughly $r \simeq 3.83$.

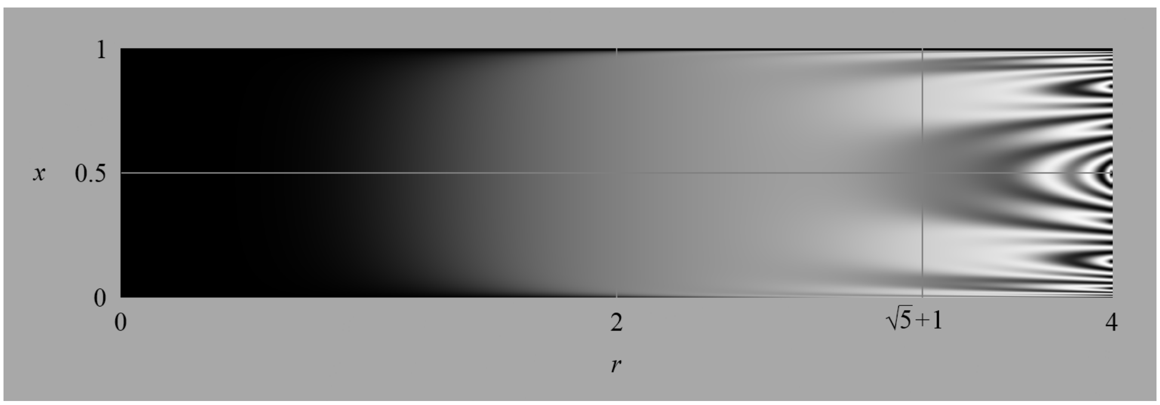


Figure 10: The solutions $x''''''$ to the polynomial corresponding to the sixth iteration of the logistic map for the interval $r = [0,4]$, $x = [0,1]$.

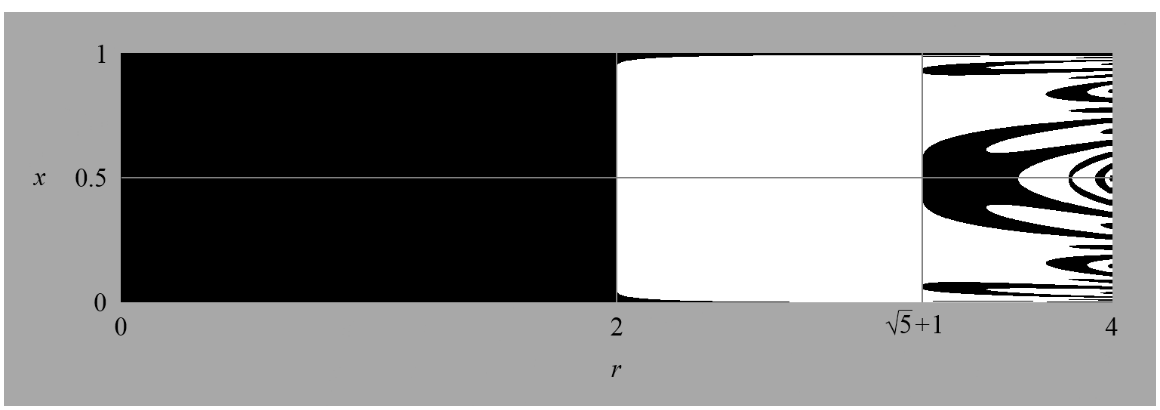


Figure 11: A high contrast version of Figure 10. The pure white region is the superlevel set of the polynomial where brightness $\geq 0.5$.