# Local fractional Mellin transform in fractal space

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**Abstract:** This paper deals with the theory and applications of the local fractional Mellin transform of the real order $\alpha$. We define the local fractional Mellin transform and its inverse transform. This is followed by several examples and the basic operational properties of local fractional Mellin transform. We discuss applications of local fractional Mellin transforms to local fractional boundary value problems.
**Keywords:** local fractional Mellin transform, local fractional boundary value problems, local fractional derivative；local fractional interval, zeta function.

## 1 Introduction

Local fractional calculus [3-5,7,8,10-19 ]is a generalization of differentiation and integration of the functions defined on fractal sets. In the last years has found use in studies of viscoelastic materials, as well as in many fields of science and engineering including electerical networks, probability, electromagnetic theory, diffusive transport and fluid flow [1,2,6,9,20-34].There are many definitions of local fractional derivatives and local fractional integrals (also called fractal calculus) [1-19]. Hereby we write down Gao-Yang-Kang definitions as follows. Gao-Yang- Kang local fractional derivative is denoted by [10-19]

$$f^{(\alpha)}(x_0)=\frac{df(x)}{dx^{\alpha}}|_{x=x_0}=\lim_{\delta x\to 0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}}, \quad \text{for } 0<\alpha\le 1, \qquad (1.1)$$

Where

$$\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(1+\alpha)\Delta(f(x)-f(x_0)),$$

and local fractional integral of $f(x)$ denoted by [10-19]

$${}_aI_b^{(\alpha)}f(t)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}f(t_j)(\Delta t_j)^{\alpha}, \qquad (1.2)$$

With $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,\ldots,\Delta t_j,\ldots\}$, where for $j=1,2,\ldots,N-1$ , $[t_j,t_{j+1}]$

is a partition of the interval $[a,b]$ and $t_0=a$ , $t_N=b$.

The purpose of this paper is to establish the local fractional Mellin transform based on Local fractional calculus and consider its application to local fractional equations with local fractional derivative. The remainder of the paper is organized as follows. The second section is devoted to the definition of the fractional Mellin transform and investigation of its properties including the inversion formula. In the third section we deal with the operational relations for the fractional Mellin transform. The last section contains some examples of application of the fractional Mellin transforms to the model partial differential equations of fractional order.

## 2 Local fractional Mellin transform and its inverse transform

In the section, we define the local fractional Mellin transform and its inverse transform, some examples are considered also.

**Definition 2.1** If $f(x)\in L_{1,\alpha}(\mathbb{R})$, variable $p$ is a complex number, then the local fractional Mellin transform of $f(x)$ is defined as

$$M_{\alpha}\{f(x)\}=f_{p}^{M,\alpha}(p):=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\infty}x^{\alpha(p-1)}f(x)(dx)^{\alpha} \tag{2.1}$$

The inverse local fractional Mellin transform has the form

$$f(x)=M_{\alpha}^{-1}(f_{p}^{M,\alpha}(p)):=\frac{1}{(2\pi i)^{\alpha}}\int_{c-i\infty}^{c+i\infty}x^{-p\alpha}f_{p}^{M,\alpha}(p)(dp)^{\alpha} \tag{2.2}$$

Where $L_{1,\alpha}(\mathbb{R})=\{f(x)\in C_{\alpha}(\mathbb{R}):\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\infty}|f(x)|(dx)^{\alpha}\}$ .Obviously, $M_{\alpha}$ and $M_{\alpha}^{-1}$ are linear integral operators.

**Example 2.1**

(a) If $f(x)=E(-n^{\alpha}x^{\alpha})$, where $n>0$, by substituting $nx=t$, then

$$\begin{aligned}M_{\alpha}\{E_{\alpha}(-n^{\alpha}x^{\alpha})\}&=f_{p}^{M,\alpha}(p)=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\infty}x^{\alpha(p-1)}E_{\alpha}(-n^{\alpha}x^{\alpha})(dx)^{\alpha}\\&=\frac{1}{\Gamma(1+\alpha)n^{\alpha p}}\int_{0}^{\infty}t^{\alpha(p-1)}E_{\alpha}(-t^{\alpha})(dt)^{\alpha}=\frac{\Gamma_{\alpha}(p)}{n^{\alpha p}}\end{aligned} \tag{2.3}$$

(b) If $f(x)=\frac{1}{(1+x)^{\alpha}}$, by putting $x=\frac{t}{1-t}$ or $t=\frac{x}{1+x}$,then

$$\begin{aligned}M_{\alpha}\{\frac{1}{(1+x)^{\alpha}}\}&=f_{p}^{M,\alpha}(p)=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{\alpha(p-1)}(1-t)^{-\alpha p}(dt)^{\alpha}\\&=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{\alpha(p-1)}(1-t)^{\alpha(1-p-1)}(dt)^{\alpha}=B_{\alpha}(p,1-p)=\Gamma(p)\Gamma(1-p)\end{aligned} \tag{2.4}$$

which is, by a result for the local fractional gamma function,

(c) If $f(x)=\frac{1}{E_{\alpha}(x^{\alpha})-1}$, by using

Since $\sum_{n=0}^{\infty}E_{\alpha}(-n^{\alpha}x^{\alpha})=\frac{1}{1-E_{\alpha}(-x^{\alpha})}$ and $\sum_{n=1}^{\infty}E_{\alpha}(-n^{\alpha}x^{\alpha})=\frac{1}{E_{\alpha}(x^{\alpha})-1}$,then we have

$$\begin{aligned}M_{\alpha}\{\frac{1}{E_{\alpha}(x^{\alpha})-1}\}&=f_{p}^{M,\alpha}(p)=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\infty}x^{\alpha(p-1)}\frac{1}{E_{\alpha}(x^{\alpha})-1}(dx)^{\alpha}\\&=\sum_{n=1}^{\infty}\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\infty}x^{\alpha(p-1)}E_{\alpha}(-n^{\alpha}x^{\alpha})(dx)^{\alpha}=\sum_{n=1}^{\infty}\frac{\Gamma_{\alpha}(p)}{n^{\alpha p}}=\Gamma_{\alpha}(p)\zeta_{\alpha}(p)\end{aligned} \tag{2.5}$$

Where $\zeta_\alpha(p)=\sum_{n=1}^{\infty}\frac{1}{n^{\alpha p}}$, $\operatorname{Re} p>1$, is the local fractional Riemann zeta function.

(d) If $f(x)=\frac{2^\alpha}{E_\alpha(2^\alpha x^\alpha)-1}$,then

$$\begin{aligned}M_\alpha\{\frac{2^\alpha}{E_\alpha(2^\alpha x^\alpha)-1}\}&= f_p^{M,\alpha}(p)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}\frac{2^\alpha}{E_\alpha(2^\alpha x^\alpha)-1}(dx)^\alpha\\&=2^\alpha\sum_{n=1}^{\infty}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}E_\alpha(-2^\alpha n^\alpha x^\alpha)(dx)^\alpha=2^\alpha\sum_{n=1}^{\infty}\frac{\Gamma_\alpha(p)}{(2n)^{\alpha p}}=2^{\alpha(1-p)}\Gamma_\alpha(p)\zeta_\alpha(p)\end{aligned}\qquad(2.6)$$

(e) If $f(x)=\frac{1}{(1+x)^{n\alpha}}$ ,then

$$M_\alpha\{\frac{1}{(1+x)^{n\alpha}}\}= f_p^{M,\alpha}(p)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}\frac{1}{(1+x)^{n\alpha}}(dx)^\alpha$$

which is, by setting $x=\frac{t}{1-t}$ or $t=\frac{x}{1+x}$

$$\begin{aligned}M_\alpha\{\frac{1}{(1+x)^{n\alpha}}\}&= f_p^{M,\alpha}(p)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha(p-1)}(1-t)^{\alpha(n-p-1)}(dt)^\alpha\\&=B_\alpha(p,n-p)=\frac{\Gamma_\alpha(p)\Gamma_\alpha(n-p)}{\Gamma_\alpha(n)}\end{aligned}\qquad(2.7)$$

where $B_\alpha(p,q)$ is the local fractional beta function.

Hence, $M_\alpha^{-1}\{\Gamma_\alpha(p)\Gamma_\alpha(n-p)\}=\frac{\Gamma_\alpha(n)}{(1+x)^{n\alpha}}$

(f) Find the local fractional Mellin transform of $\cos_\alpha k^\alpha x^\alpha$ and $\sin_\alpha k^\alpha x^\alpha$. It follows from (2.3) that

If $f(x)=E_\alpha(-i^\alpha k^\alpha x^\alpha)$, where $n>0$, then

$$M_\alpha\{E_\alpha(-i^\alpha k^\alpha x^\alpha)\}= f_p^{M,\alpha}(p)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}E_\alpha(-i^\alpha k^\alpha x^\alpha)(dx)^\alpha$$

which is, by putting $ikx=t$,

$$\begin{aligned}M_\alpha\{E_\alpha(-i^\alpha k^\alpha x^\alpha)\}&= f_p^{M,\alpha}(p)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}E_\alpha(-i^\alpha k^\alpha x^\alpha)(dx)^\alpha\\&=\frac{1}{\Gamma(1+\alpha)(ki)^{\alpha p}}\int_0^\infty t^{\alpha(p-1)}E_\alpha(-t^\alpha)(dt)^\alpha=\frac{\Gamma_\alpha(p)}{(ik)^{\alpha p}}=\frac{\Gamma_\alpha(p)i^{-\alpha p}}{k^{\alpha p}}=\frac{\Gamma_\alpha(p)(i^\alpha)^{-p}}{k^{\alpha p}}\\&=\frac{\Gamma_\alpha(p)(\cos_\alpha(\frac{\pi}{2})^\alpha+\sin_\alpha(\frac{\pi}{2})^\alpha)^{-p}}{k^{\alpha p}}=\frac{\Gamma_\alpha(p)}{k^{\alpha p}}[\cos_\alpha(\frac{p\pi}{2})^\alpha-\sin_\alpha(\frac{p\pi}{2})^\alpha]\end{aligned}$$

Hence, we obtain

$$M_\alpha\{E_\alpha(-i^\alpha k^\alpha x^\alpha)\}=\frac{\Gamma_\alpha(p)}{k^{\alpha p}}[\cos_\alpha(\tfrac{p\pi}{2})^\alpha-i^\alpha\sin_\alpha(\tfrac{p\pi}{2})^\alpha] \quad (2.8)$$

Separating real and imaginary parts, we find

$$M_\alpha\{\cos_\alpha k^\alpha x^\alpha\}=\frac{\Gamma_\alpha(p)}{k^{\alpha p}}\cos_\alpha(\tfrac{p\pi}{2})^\alpha \quad (2.9)$$

$$M_\alpha\{\sin_\alpha k^\alpha x^\alpha\}=\frac{\Gamma_\alpha(p)}{k^{\alpha p}}\sin_\alpha(\tfrac{p\pi}{2})^\alpha] \quad (2.10)$$

These results can be used to calculate the local fractional Fourier cosine and Fourier sine transforms of $x^{(p-1)\alpha}$. Result (2.9) can be written as

$$F_{\alpha,c}\{\frac{x^{(p-1)\alpha}}{2}\}=\frac{\Gamma_\alpha(p)}{k^{\alpha p}}\cos_\alpha(\tfrac{p\pi}{2})^\alpha \quad (2.11)$$

Or

$$F_{\alpha,c}\{x^{(p-1)\alpha}\}=\frac{2\Gamma_\alpha(p)}{k^{\alpha p}}\cos_\alpha(\tfrac{p\pi}{2})^\alpha \quad (2.12)$$

Similarly,

$$F_{\alpha,s}\{x^{(p-1)\alpha}\}=\frac{2\Gamma_\alpha(p)}{k^{\alpha p}}\sin_\alpha(\tfrac{p\pi}{2})^\alpha] \quad (2.13)$$

**3 Basic Operational Properties of local fractional Mellin Transforms**

If $M_\alpha\{f(x)\}=f_p^{M,\alpha}(p)$, then we have the following operational properties:

(a) (Scaling Property).

$$M_\alpha\{f(ax)\}=a^{-\alpha p}f_p^{M,\alpha}(p),\quad a>0\,. \quad (3.1)$$

**Proof.** Based on definition, by substituting $ax=t$ ,we obtain

$$M_\alpha\{f(ax)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}f(ax)(dx)^\alpha$$

$$=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty(\tfrac{t}{a})^{\alpha(p-1)}f(t)(d\tfrac{t}{a})^\alpha=\frac{1}{a^{\alpha p}}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty t^{\alpha(p-1)}f(t)(dt)^\alpha$$

$$=\frac{1}{a^{\alpha p}}f_p^{M,\alpha}(p)=a^{-\alpha p}f_p^{M,\alpha}(p)$$

(b) (Shifting Property).

$$M_\alpha\{x^{a\alpha}f(x)\}=f_p^{M,\alpha}(p+a) \quad (3.2)$$

**Proof.** By definition,we obtain

$$M_\alpha\{x^{a\alpha}f(x)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}x^{a\alpha}f(x)(dx)^\alpha$$

$$=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p+a-1)}f(t)(dx)^\alpha=f_p^{M,\alpha}(p+a)$$

(c) $$M_\alpha\{f(x^a)\}=\frac{1}{a^\alpha}f_p^{M,\alpha}(\tfrac{p}{a}) \quad (3.3)$$

**Proof.** By definition, let $t=x^a$ , we obtain

$$M_\alpha\{f(x^a)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}f(x^a)(dx)^\alpha$$
$$=\frac{1}{a^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty t^{\frac{\alpha(p-1)}{a}+\alpha(\frac{1}{a}-1)}f(t)(dt)^\alpha$$
$$=\frac{1}{a^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty t^{\alpha(\frac{p}{a}-1)}f(t)(dt)^\alpha=\frac{1}{a^\alpha}f_p^{M,\alpha}(\tfrac{p}{a})$$

Similarly, we have

$$M_\alpha\{\frac{1}{x^\alpha}f(\tfrac{1}{x})\}=f_p^{M,\alpha}(1-p) \qquad (3.4)$$

(d) (local fractional Mellin Transforms of Derivatives).

$$M_\alpha\{f^{(\alpha)}(x)\}=-\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+(p-2)\alpha)}f_p^{M,\alpha}(p-1) \qquad (3.5)$$

provided $x^{\alpha(p-1)}f(x)$ vanishes as $x\to 0$ and as $x\to\infty$.

$$M_\alpha\{f^{(2\alpha)}(x)\}=\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+(p-3)\alpha)}f_p^{M,\alpha}(p-2) \qquad (3.6)$$

More generally,

$$M_\alpha\{f^{(n\alpha)}(x)\}=(-1)^n\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+(p-(n+1))\alpha)}f_p^{M,\alpha}(p-n) \qquad (3.7)$$

provided $x^{\alpha(p-r-1)}f^{(r\alpha)}(x)=0$ as $x\to 0$ and as $x\to\infty$ for $r=0,1,2\ldots,n-1$.

**Proof.** by definition, We have

$$M_\alpha\{f^{(\alpha)}(x)\}=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}f^{(\alpha)}(x)(dx)^\alpha$$
$$=x^{\alpha(p-1)}f(x)|_0^\infty-\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x)(x^{\alpha(p-1)})^\alpha(dx)^\alpha$$
$$=-\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+(p-2)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-2)}f(x)(dx)^\alpha$$
$$=-\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+(p-2)\alpha)}f_p^{M,\alpha}(p-1)$$

The proofs of (3.6) and (3.7) are similar, we omit it.

(e) If $M_\alpha\{f(x)\}=f_p^{M,\alpha}(p)$ ,then

$$M_\alpha\{x^\alpha f^{(\alpha)}(x)\}=-\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p-1)\alpha)}f_p^{M,\alpha}(p) \qquad (3.8)$$

provided $x^{\alpha p}f(x)$ vanishes at $x=0$ and as $x\to\infty$.

$$M_\alpha\{x^{2\alpha} f^{(2\alpha)}(x)\} = \frac{\Gamma(1+(p+1)\alpha)}{\Gamma(1+(p-1)\alpha)} f_p^{M,\alpha}(p) \quad (3.9)$$

More generally,

$$M_\alpha\{x^{n\alpha} f^{(n\alpha)}(x)\} = (-1)^n \frac{\Gamma(1+(p+n-1)\alpha)}{\Gamma(1+(p-1)\alpha)} f_p^{M,\alpha}(p) \quad (3.10)$$

**Proof.** Applying definition, We obtain

$$\begin{aligned}
M_\alpha\{x^\alpha f^{(\alpha)}(x)\} &= \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)} x^\alpha f^{(\alpha)}(x)(dx)^\alpha \\
&= x^{\alpha(p-1)} x^\alpha f(x)|_0^\infty - \frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x)(x^{\alpha(p-1)}x^\alpha)^{(\alpha)}(dx)^\alpha \\
&= -\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x)(x^{\alpha p})^{(\alpha)}(dx)^\alpha \\
&= -\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p-1)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x)x^{\alpha(p-1)}(dx)^\alpha \\
&= -\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p-1)\alpha)} f_p^{M,\alpha}(p)
\end{aligned}$$

Similar arguments can be used to prove results (3.9) and (3.10).

(f) (The local fractional Mellin Transforms of Differential Operators).

If $M_\alpha\{f(x)\} = f_p^{M,\alpha}(p)$ ,then

$$M_\alpha\{(x^\alpha \frac{d^\alpha}{dx^\alpha})^2 f(x)\} = \frac{\Gamma(1+(p+1)\alpha)}{\Gamma(1+(p-1)\alpha)} f_p^{M,\alpha}(p) - \frac{T(1+\alpha)\Gamma(1+p\alpha)}{\Gamma(1+(p-1)\alpha)} f_p^{M,\alpha}(p) \quad (3.11)$$

**Proof.** Applying definition, We obtain

$$\begin{aligned}
M_\alpha\{(x^\alpha \frac{d^\alpha}{dx^\alpha})^2 f(x)\} &= M_\alpha\{x^{2\alpha} f^{(2\alpha)}(x) + T(1+\alpha)x^\alpha f^{(\alpha)}(x)\} \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}[x^{2\alpha} f^{(2\alpha)}(x) + T(1+\alpha)x^\alpha f^{(\alpha)}(x)](dx)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)} x^{2\alpha} f^{(2\alpha)}(x)(dx)^\alpha \\
&+ \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)} T(1+\alpha)x^\alpha f^{(\alpha)}(x)](dx)^\alpha \\
&= \frac{\Gamma(1+(p+1)\alpha)}{\Gamma(1+(p-1)\alpha)} f_p^{M,\alpha}(p) - \frac{T(1+\alpha)\Gamma(1+p\alpha)}{\Gamma(1+(p-1)\alpha)} f_p^{M,\alpha}(p)
\end{aligned}$$

(g) (The local fractional Mellin Transforms of Integrals).

Let $F(x) = \frac{1}{\Gamma(1+\alpha)}\int_0^x f(t)(dt)^\alpha$ , then

$$M_\alpha\{\frac{1}{\Gamma(1+\alpha)}\int_0^x f(t)dt\} = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}[\frac{1}{\Gamma(1+\alpha)}\int_0^x f(t)dt](dx)^\alpha \quad (3.12)$$

**Proof.** Let $F(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x f(t)(dt)^\alpha$ , obviously $F^{(\alpha)}(x)=f(x)$.By Application of (3.5) with $F^{(\alpha)}(x)$, then we have

$$\begin{aligned} M_\alpha\{F^{(\alpha)}(x)\} &= \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}F^{(\alpha)}(x)(dx)^\alpha \\ &= x^{\alpha(p-1)}F(x)|_0^\infty - \frac{1}{\Gamma(1+\alpha)}\int_0^\infty F(x)(x^{\alpha(p-1)})^\alpha (dx)^\alpha \\ &= -\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+(p-2)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-2)}F(x)(dx)^\alpha \end{aligned} \tag{3.13}$$

By (3.13), we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-2)}F(x)(dx)^\alpha \\ &= -\frac{\Gamma(1+(p-2)\alpha)}{\Gamma(1+(p-1)\alpha)}M_\alpha\{F^{(\alpha)}(x)\} \\ &= -\frac{\Gamma(1+(p-2)\alpha)}{\Gamma(1+(p-1)\alpha)}M_\alpha\{f(x)\} \end{aligned}$$

which is, replacing $p$ by $p+1$, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}F(x)(dx)^\alpha \\ &= -\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+p\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha p}f(x)(dx)^\alpha \\ &= -\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+p\alpha)} f_p^{M,\alpha}(p+1) \end{aligned}$$

Hence, we obtain

$$M_\alpha\{F(x)\} = -\frac{\Gamma(1+(p-1)\alpha)}{\Gamma(1+p\alpha)} f_p^{M,\alpha}(p+1) \tag{3.14}$$

(h) (Convolution Type Theorems).

If $M_\alpha\{f(x)\}=f_p^{M,\alpha}(p)$ and $M_\alpha\{g(x)\}=g_p^{M,\alpha}(p)$ ,then

$$M_\alpha\{f(x)*g(x)\} = M_\alpha\{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(\xi)g(\frac{x}{\xi})\frac{1}{\xi^\alpha}(d\xi)^\alpha\} = f_p^{M,\alpha}(p)g_p^{M,\alpha}(p) \tag{3.15}$$

$$M_\alpha\{f(x)\circ g(x)\} = M_\alpha\{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x\xi)g(\xi)(d\xi)^\alpha\} = f_p^{M,\alpha}(p)g_p^{M,\alpha}(1-p) \tag{3.16}$$

**Proof.** By definition，set $\frac{x}{\xi}=\eta$ we have,

$$
\begin{aligned}
&M_\alpha\{f(x)*g(x)\}\\
&=M_\alpha\{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(\xi)g(\frac{x}{\xi})\frac{1}{\xi^\alpha}(d\xi)^\alpha\}\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(\xi)g(\frac{x}{\xi})\frac{1}{\xi^\alpha}(d\xi)^\alpha](dx)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(\xi)\frac{1}{\xi^\alpha}(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}g(\frac{x}{\xi})(dx)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(\xi)\frac{1}{\xi^\alpha}(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty (\xi\eta)^{\alpha(p-1)}g(\eta)(d\xi\eta)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(\xi)\frac{1}{\xi^\alpha}(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \xi^{\alpha(p-1)}\xi^\alpha\eta^{\alpha(p-1)}g(\eta)(d\eta)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(\xi)\frac{1}{\xi^\alpha}\xi^{\alpha(p-1)}\xi^\alpha(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \eta^{\alpha(p-1)}g(\eta)(d\eta)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \xi^{\alpha(p-1)}f(\xi)(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \eta^{\alpha(p-1)}g(\eta)(d\eta)^\alpha\\
&=f_p^{M,\alpha}(p)g_p^{M,\alpha}(p)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&M_\alpha\{f(x)\circ g(x)\}=M_\alpha\{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x\xi)g(\xi)(d\xi)^\alpha\}\\
&=M_\alpha\{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x\xi)g(\xi)(d\xi)^\alpha\}\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}[\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x\xi)g(\xi)(d\xi)^\alpha](dx)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty g(\xi)(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}f(x\xi)(dx)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty g(\xi)(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty (\frac{\eta}{\xi})^{\alpha(p-1)}f(\eta)(d\frac{\eta}{\xi})^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty g(\xi)(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \xi^{-\alpha(p-1)-\alpha}\eta^{\alpha(p-1)}f(\eta)(d\eta)^\alpha\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty g(\xi)\xi^{-\alpha p}(d\xi)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \eta^{\alpha(p-1)}f(\eta)(d\eta)^\alpha\\
&=f_p^{M,\alpha}(p)\frac{1}{\Gamma(1+\alpha)}\int_0^\infty g(\xi)\xi^{\alpha(1-p-1)}(d\xi)^\alpha\\
&=f_p^{M,\alpha}(p)g_p^{M,\alpha}(1-p)
\end{aligned}
$$

Note that, in this case, the operation $\circ$ is not commutative.

Clearly （obviously）, putting $x=s$,

$$
\begin{aligned}
&M_\alpha^{-1}\{f_p^{M,\alpha}(1-p)g_p^{M,\alpha}(p)\}\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty g(st)f(t)(dt)^\alpha
\end{aligned}
$$

Sttting $g(t)=E_\alpha(-t^\alpha)$ and $g_p^{M,\alpha}(p)=\Gamma_\alpha(p)$, we obtain the local fractional Laplace transform (yang-Laplace transform) of $f(t)$.

$$M_\alpha^{-1}\{f_p^{M,\alpha}(1-p)g_p^{M,\alpha}(p)\} = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha t^\alpha) f(t)(dt)^\alpha = L_\alpha\{f(t)\} = f_s^{L,\alpha}(s) \quad (3.17)$$

（i）(Parseval' s Type Property).

If $M_\alpha\{f(x)\} = f_p^{M,\alpha}(p)$ and $M_\alpha\{g(x)\} = g_p^{M,\alpha}(p)$ ,then

$$M_\alpha\{f(x)g(x)\} = \frac{1}{(2\pi i)^\alpha}\int_{c-i\infty}^{c+i\infty} f_p^{M,\alpha}(s) g_p^{M,\alpha}(p-s)(ds)^\alpha \quad (3.18)$$

Or, equivalently,

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)} f(x)g(x)(dx)^\alpha = \frac{1}{(2\pi i)^\alpha}\int_{c-i\infty}^{c+i\infty} f_p^{M,\alpha}(s) g_p^{M,\alpha}(p-s)(ds)^\alpha \quad (3.19)$$

In particular, when $p=1$, we obtain the Parseval formula for the local fractional Mellin transform,

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty f(x)g(x)(dx)^\alpha = \frac{1}{(2\pi i)^\alpha}\int_{c-i\infty}^{c+i\infty} f_p^{M,\alpha}(s) g_p^{M,\alpha}(1-s)(ds)^\alpha \quad (3.20)$$

**Proof.** By definition, we have

$$\begin{aligned} & M_\alpha\{f(x)g(x)\} = \\ & = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)} f(x)g(x)(dx)^\alpha \\ & = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)} g(x)[\frac{1}{(2\pi i)^\alpha}\int_{c-i\infty}^{c+i\infty} x^{-s\alpha} f_p^{M,\alpha}(s)(ds)^\alpha](dx)^\alpha \\ & = \frac{1}{(2\pi i)^\alpha}\int_{c-i\infty}^{c+i\infty} f_p^{M,\alpha}(s)(ds)^\alpha \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)} x^{-s\alpha} g(x)(dx)^\alpha \\ & = \frac{1}{(2\pi i)^\alpha}\int_{c-i\infty}^{c+i\infty} f_p^{M,\alpha}(s)(ds)^\alpha \frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-s-1)} g(x)(dx)^\alpha \\ & = \frac{1}{(2\pi i)^\alpha}\int_{c-i\infty}^{c+i\infty} f_p^{M,\alpha}(s) g_p^{M,\alpha}(p-s)(ds)^\alpha \end{aligned} \quad (3.21)$$

When $p=1$, the above result becomes (3.20).

## 4 Applications of the local fractional Mellin Transforms

We consider the following local fractional boundary value problem

$$x^{2\alpha}\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} + x^\alpha\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^{2\alpha}u}{\partial y^{2\alpha}} = 0, \quad 0<x<\infty, \quad 0<y<1, \quad (4.1)$$

$$u(x,0)=0\,,\ \ u(x,1)=\begin{cases}A, & 0\le x\le 1\\ 0 & x>1\end{cases}\,, \tag{4.2}$$

where $A$ is a constant.

We apply the local fractional Mellin transform of $u(x,y)$ with respect to $x$ defined by

$$\tilde{u}(p,y)=\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(p-1)}u(x,y)(dx)^\alpha$$

to reduce the given system into the form

$$\frac{d^{2\alpha}\tilde{u}}{dy^{2\alpha}}+\frac{\Gamma(1+(p+1)\alpha)-\Gamma(1+p\alpha)}{\Gamma(1+(p-1)\alpha)}\tilde{u}=0\,,\quad 0<y<1$$

$$\tilde{u}(p,0)=0\,,\quad \tilde{u}(p,1)=A\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha(p-1)}(dx)^\alpha=A\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}$$

The solution of the transformed problem is

$$\tilde{u}(p,y)=A\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\frac{\sin_\alpha p^\alpha y^\alpha}{\sin_\alpha p^\alpha}\,,\quad 0<\operatorname{Re}p<1$$

The inverse local fractional Mellin transform gives.

$$u(x,y)=\frac{A}{(2\pi i)^\alpha}\int_{c-i\infty}^{c+i\infty}x^{-p\alpha}\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\frac{\sin_\alpha p^\alpha y^\alpha}{\sin_\alpha p^\alpha}(dp)^\alpha \tag{4.3}$$

where $\tilde{u}(p,y)$ is analytic in the vertical strip $0<\operatorname{Re}p=c<\pi$. The integrand of (4.3) has simple poles at $p=n\pi$, $n=1,2,3,\ldots$ which lie inside a semicircular contour in the right half plane. Evaluating (4.3) by local fractional theory of residues gives the solution for $x>1$ as

$$u(x,y)=\frac{A^\alpha}{\pi^\alpha}\sum_{n=1}^{\infty}\frac{1}{n^\alpha}(-1)^n x^{-n\pi\alpha}\sin_\alpha n^\alpha\pi^\alpha y^\alpha \tag{4.4}$$

**References**


[1] J. Tenreiro Machado, Virginia Kiryakova,Francesco Mainardi, Recent History of Fractional Calculus,Communications in Nonlinear Science and Numerical Simulations,Elsevier, vol. 16, Issue 3, pp. 1140-1153, March 2011.

[2] J. A. Tenreiro Machado, Manuel F. Silva, Ramiro S. Barbosa, Isabel S. Jesus, Cecília M. Reis, Maria G. Marcos, Alexandra F. Galhano, Some Applications of Fractional Calculus in Engineering, Mathematical Problems in Engineering, Hindawi, vol. 2010, Article ID 639801, 2009. doi:10.1155/2010/639801

[3] S.G. Samko, A.A. Kilbas, O.I. Marichev: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, 1993.

[4] K.M.Kolwankar, and A.D.Gangal, Fractional differentiability of nowhere differentiable functions and dimensions, Chaos, 6 (4), 1996, 505–513.

[5] A.Carpinteri, P.Cornetti, A fractional calculus approach to the description of stress and strain localization in fractal media, Chaos, Solitons and Fractals, 13, 2002,85–94.

[6] A.Carpinteri , B.Chiaia, P.Cornetti, On the mechanics of quasi-brittle materials with a fractal microstructure, Engineering Fracture Mechanics, 70,2003,2321–2349.
[7] A. Babakhan, V.D.Gejji, On calculus of local fractional derivatives, J. Math. Anal. Appl.,270,2002,66–79.
[8] F.B.Adda, J.Cresson,About non-differentiable functions, J. Math. Anal. Appl.,263(2001),721–737 .
[9] F.B. Adda, J.Cresson. Fractional differential equations and the Schrödinger equation. Applied Mathematics and Computation 161 (2005) 323–345.
[10] F. Gao, X.Yang, Z. Kang. Local fractional Newton's method derived from modified local fractional calculus, In: Proc. of CSO 2009, 2009, 228–232.
[11] X.Yang, F. Gao. The fundamentals of local fractional derivative of the one-variable nondifferentiable functions, World Sci-Tech R&D, 31(5), 2009, 920-921.
[12] X.Yang, L.Li, R.Yang. Problems of Local fractional definite integral of the one-variable nondifferentiable function, World Sci-Tech R&D, 31(4), 2009, 722-724.
[13] X.Yang, F.Gao. Fundamentals of local fractional iteration of the continuously nondifferentiable functions derived from local fractional calculus. In: Proc. of CSIE2011, Springer, 2011.
[14] X.Yang, Research on fractal mathematics and some applications in mechanics. M.S.thesis, China University of Mining and Technology, 2009.6.
[15] J.H He. A new fractional derivation. Thermal Science.15,1, 2011, 145-147.
[16] W. Chen. Time-space fabric underlying anomalous disusion. Chaos, Solitons and Fractals 28 (2006) 923–929.
[17] Zhong, W. P., Gao, F.: Application of the Yang-Laplace transforms to solution to nonlinear fractional wave equation with fractional derivative. In: Proc. of the 2011 3rd International Conference on Computer Technology and Development, ASME, 2011.
[18] X. Yang, Local Fractional Integral Transforms, \textit{Progress in Nonlinear Science}, 4,2011, 1-225.
[19]X. Yang, Local Fractional Functional Analysis and Its Applications. Asian Academic publisher Limited, Hong Kong, China, 2011.
[20] K. Assaleh; W.M. Ahmad,Modeling of speech signals using fractional calculus 9th International Symposium on Signal Processing and Its Applications, 2007. ISSPA 2007. 12-15 Feb. 2007 Page(s):1 - 4
[21] J. F. Douglas, Some applications of fractional calculus to polymer science, Advances in chemical physics, Vol 102, John Wiley & Sons, Inc. Applications of fractional calculus 1031
[22] Z. E. A. Fellah, C.Depollier, Application of fractional calculus to the sound waves propagation in rigid porous materials: Validation via ultrasonic measurement, Acta Acustica vol.88(2002)34-39
[23] V. V. Kulish and Jos´ e L. LageApplication of Fractional Calculus to Fluid MechanicsJ. Fluids Eng. – September 2002 – Volume 124, Issue 3, 803 (4pages)
[24] K. B. Oldham and J. Spanier, The Fractional Calculus , Academic Press,INC. 1974
[25] R. Magin, Fractional Calculus in Bioengineering, Part1,Crit Rev Biomed Eng 2004; 32(1):1-104
[26] R. L. Magin, Modeling the Cardiac Tissue Electrode Interface Using Fractional CalculusJournal of Vibration and Control, Vol. 14, No. 9-10, 1431-1442 (2008)

[27] Margulies, Timothy Wave propagation in viscoelastic horns using a fractional calculus rheology modelAcoustical Society of America Journal, Volume 114, Issue 4, pp. 2442-2442 (2003).
[28] B. Mathieu, P. Melchior, A. Oustaloup, Ch. Ceyral, Fractional differentiation for edge detectionFractional Signal Processing and Applications Volume 83, Issue 11, Pages 2285-2480 (November 2003)
[29] J. Munkhammar, Riemann - Liouville Fractional Derivatives and the Taylor- Riemann Series, U. U. D. M. Project report 2004:7
[30] I. Podlubny, Geometric and physical interpretation of fractional integration andfractional differentiation, fractional calculus and applied analysis, Vol 5, Number4(2002)
[31] N. Sebaa, Z. E. A. Fellah, W. Lauriks, C. Depollier, Application of fractional calculus to ultrasonic wave propagation in human cancellous bone, Signal Processing archive Volume 86 , Issue 10 (2006)2668 - 2677
[32] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity Molecular and Quantum Acoustics Vol.23,397-404(2002)
[33] J.I. Su´arez , B.M. Vinagre , A.J. Calder´on , C.A. Monje and Y.Q. Chen Using Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles Lecture Notes in Computer Science, Springer Berlin / Heidelberg, Volume 2809/2004 1032 M. Dalir and M. Bashour
[34] B. M. Vinagre and YangQuan Chen, Fractional Calculus Applications In Automatic Control and Robotics, 41st IEEE Conference on decision and control Tutoral Workshop2, Las Vegas, Desember 2002