Local fractional Improper integral in fractal space

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Abstract: In this paper we study Local fractional improper integrals on fractal space. By some mean value theorems for Local fractional integrals, we prove an analogue of the classical Dirichlet-Abel test for Local fractional improper integrals.

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1 Introduction

Local fractional calculus [1-8] played an important role in fractal mathematics and engineering, especially in nonlinear phenomena.Very recently, Yang [9-11] gave Local fractional integral of $f(x)$ as follows:

$$_aI_b^{(\alpha)}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,\ldots,\Delta t_j,\ldots\}$, where for $j=1,2,\ldots,N-1$, $t_0=a$ and $t_N=b$, $[t_j,t_{j+1}]$ is a partition of the interval $[a,b]$.

The set up of this paper is as follows. In Section 2 we study Local fractional improper integrals of first kind. Finally, in Section 3, we deal with Local fractional improper integrals of second kind on fractal space.

2 Local fractional improper integrals of first kind

Definition 2.1 *(Local fractional improper integral of type 1) Local fractional improper integrals of type 1 are evaluated as follows:*

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If ${}_aI_t^{(\alpha)}f(x) = F(t) = \frac{1}{\Gamma(1+\alpha)}\int_a^t f(x)(dx)^\alpha$ *exists for all* $t \geq a$*, then we define*

$$\begin{aligned} {}_aI_\infty^{(\alpha)}f(x) &= \frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha \\ &= \lim_{t\to\infty}\frac{1}{\Gamma(1+\alpha)}\int_a^t f(x)(dx)^\alpha = \lim_{t\to\infty} {}_aI_t^{(\alpha)}f(x) \end{aligned} \tag{2.1}$$

provided the limit exists as a finite number. In this case, $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$ *is said to be convergent (or to converge). Otherwise,* $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$ *is said to be divergent (or to diverge).*

Proposition 2.1 *suppose* $f(x) \geq 0$ *for all* $x \in [a,\infty)$*,Then the local fractional integral* $F(t)$ *is increasing on* $[a,\infty)$*.*

Proof. Take t_1,$t_2 \in [a,\infty)$ with $t_1 < t_2$. By the property of additivity of the domain of integration.

$$\begin{aligned} F(t_2) &= \frac{1}{\Gamma(1+\alpha)}\int_a^{t_2} f(x)(dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_a^{t_1} f(x)(dx)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_{t_1}^{t_2} f(x)(dx)^\alpha \\ &= F(t_1) + \frac{1}{\Gamma(1+\alpha)}\int_{t_1}^{t_2} f(x)(dx)^\alpha. \end{aligned}$$

The last integral is ≥ 0 by Property 2.18.[9] ,Therefore $F(t_2) \geq F(t_1)$.

Proposition 2.2 *if both* $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$ *and* $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty g(x)(dx)^\alpha$ *are convergent ,* k_1 k_2 *are constant , then*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^\infty [k_1f(x) + k_2g(x)](dx)^\alpha,$$

is convergent,and

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_a^\infty [k_1f(x) + k_2g(x)](dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}k_1\int_a^\infty f(x)dx)^\alpha + \frac{1}{\Gamma(1+\alpha)}k_2\int_a^\infty g(x)(dx)^\alpha \end{aligned}$$

Proposition 2.3 *if* $f(x)$ *is local fractional integrable for any infite interval* $[a,t]$ *,* $a < b$*, then*

(1) $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$ *and* $\frac{1}{\Gamma(1+\alpha)}\int_b^\infty f(x)(dx)^\alpha$ *both converge or both diverge.*

(2) $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_b^\infty f(x)(dx)^\alpha$*,*
for $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$ *and* $\frac{1}{\Gamma(1+\alpha)}\int_b^\infty f(x)(dx)^\alpha$ *both converge.*

The existence of the limit $\lim_{t\to\infty} F(t)$ is equivalent to the conditions of Cauchy's criterion for the existence of the limit of a function, which reads: For any $\varepsilon > 0$ there exists $t_0 > a$ such that for all t_1,t_2 with $t_1 > t_0$ and $t_2 > t_0$ the inequality $|F(t_1) - F(t_2)| < \varepsilon^\alpha$ holds. So we can express the following Cauchy criterion for existence of an local fractional improper integral of first kind.

Theorem 2.1 . *For the existence of the local fractional integral (2.1) it is necessary and sufficient that for any given ε there exists $t_0 > a$ such that*

$$\left|\frac{1}{\Gamma(1+\alpha)}\int_{t_1}^{t_2} f(x)(dx)^\alpha\right| < \varepsilon^\alpha,$$

for any t_1,t_2 satisfying the inequalities $t_1 > t_0$ and $t_2 > t_0$.

An local fractional integral of type (2.1) is said to be absolutely convergent provided the integral

$$\frac{1}{\Gamma(1+\alpha)}\int_{t_1}^{t_2} |f(x)|(dx)^\alpha,$$

of the modulus of the function $f(x)$ is convergent. If an local fractional integral is convergent, but not absolutely convergent, it is called conditionally convergent.

Theorem 2.2 *if $f(x)$ is integrable for any infite interval and $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty |f(x)|(dx)^\alpha$ is convergent ,then*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha,$$

is convergent. And

$$\left|\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha\right| \leq \frac{1}{\Gamma(1+\alpha)}\int_a^\infty |f(x)|(dx)^\alpha.$$

Proof. This follows from the inequality (see[9])

$$\left|\frac{1}{\Gamma(1+\alpha)}\int_{t_1}^{t_2} f(x)(dx)^\alpha\right| \leq \frac{1}{\Gamma(1+\alpha)}\int_{t_1}^{t_2} |f(x)|(dx)^\alpha,$$

and Theorem 1.

A convergent local fractional improper integral may not be absolutely convergent. But, of course, a convergent local fractional improper integral of a nonnegative function is always absolutely convergent.

Let us now consider the local fractional integral (2.1) with a nonnegative function $f(x)$. In this case the function $F(t)$ defined by definition 1 is obviously nondecreasing. Therefore, if it is bounded, i.e., if $F(t) \leq M(t > a)$ for some $M > 0$, then local fractional integral (2.1) is convergent:

$${}_aI_\infty^{(\alpha)} f(x) = \lim_{t\to\infty} {}_aI_t^{(\alpha)} f(x) = \lim_{t\to\infty} F(t) \leq M.$$

If $F(t)$ is unbounded, then integral (2.1) is divergent:

$${}_aI_\infty^{(\alpha)} f(x) = \lim_{t\to\infty} {}_aI_t^{(\alpha)} f(x) = \lim_{t\to\infty} F(t) = \infty.$$

Hence we have the following result.

Theorem 2.3 *An local fractional integral (2.1) with $f(x) \geq 0$ for all $x \geq a$ is convergent if and only if there exists a constant $M > 0$ such that*

$$F(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^t f(x)(dx)^\alpha \leq M, \quad \textit{for} \quad t \geq a.$$

The value of the local fractional improper integral is then not greater than M.

Proof. By Proposition 1, $F(t)$ is increasing on $[a, \infty)$. Then

$$\lim_{t \to +\infty} F(t) = \sup\{F(t) | t \geq a\} = M > 0,$$

and the theorem follows

$$\frac{1}{\Gamma(1+\alpha)} \int_a^t f(x)(dx)^\alpha \leq M,$$

for every $t \geq a$ whenever the integral converges.

Now we present the following comparison test.

Theorem 2.4 *Let the inequalities $0 \leq f(x) \leq g(x)$ be satisfied for all $x \in [a, \infty)$. Then the convergence of the local fractional improper integral*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^\infty g(x)(dx)^\alpha, \tag{2.2}$$

implies the convergence of the local fractional improper integral

$$\frac{1}{\Gamma(1+\alpha)} \int_a^\infty f(x)(dx)^\alpha \quad , \tag{2.3}$$

and the inequality

$$\frac{1}{\Gamma(1+\alpha)} \int_a^\infty f(x)(dx)^\alpha \leq \frac{1}{\Gamma(1+\alpha)} \int_a^\infty g(x)(dx)^\alpha.$$

while the divergence of integral (2.3) implies the divergence of local fractional integral (2.2).

proof. Let $F_1(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^t f(x)(dx)^\alpha$ and $F_2(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^t g(x)(dx)^\alpha$ for $t \geq a$, since $0 \leq f(x) \leq g(x)$ for every $x \geq a$,then

$$F_1(t) \leq F_2(t), \tag{2.4}$$

And (2.2) converges, then there exists a $M > 0$ such that

$$F_2(t) \leq M, \quad \text{for} \quad t \geq a. \tag{2.5}$$

From (2.4) and (2.5) ,we have

$$F_1(t) \leq M, \quad \text{for} \quad t \geq a. \tag{2.6}$$

By (2.6),then $\lim\limits_{t \to +\infty} F_1(t)$ exists and is finite. Hence (2.3) converges, also

$$\lim_{t \to +\infty} F_1(t) \leq \lim_{t \to +\infty} F_2(t) \leq M. \tag{2.7}$$

And we obtain
$$\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha \leq \frac{1}{\Gamma(1+\alpha)}\int_a^\infty g(x)(dx)^\alpha.$$

To avoid troublesome details of working with inequalities in practice, it is often convenient to use the following theorem rather than to use the comparison test directly.

Theorem 2.5 *(Limit Comparison Test) Suppose* $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$ *and*
$$\frac{1}{\Gamma(1+\alpha)}\int_a^\infty g(x)(dx)^\alpha,$$
are local fractional improper integrals of the first kind with positive integrands, and suppose that the limit
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L^\alpha, \tag{2.8}$$
exists (finite) and is not zero. Then the local fractional integrals are simultaneously convergent or divergent.

Proof. It follows from (2.8) that for any $\varepsilon \in (0, L)$ there exists $t_0 \in (a, \infty)$, such that
$$L^\alpha - \varepsilon^\alpha < \frac{f(x)}{g(x)} < L^\alpha + \varepsilon^\alpha, \text{for all} \quad t_0 > a.$$
and, since $g(x) > 0$, we have
$$(L^\alpha - \varepsilon^\alpha)g(x) < f(x) < (L^\alpha + \varepsilon^\alpha)g(x), \text{for all} \quad t_0 > a. \tag{2.9}$$
The convergence of the local fractional integral $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty g(x)(dx)^\alpha$ implies the convergence of the local fractional integral $\frac{1}{\Gamma(1+\alpha)}\int_{t_0}^\infty g(x)(dx)^\alpha$ and hence also the convergence of the local fractional integral $\frac{1}{\Gamma(1+\alpha)}\int_{t_0}^\infty (L^\alpha + \varepsilon^\alpha)g(x)(dx)^\alpha$. Therefore, by virtue of Theorem 4, the local fractional integral $\frac{1}{\Gamma(1+\alpha)}\int_{t_0}^\infty g(x)(dx)^\alpha$ also converges, and, together with it, so does the local fractional integral $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$. Conversely, if $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$is convergent, then $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty g(x)(dx)^\alpha$ is also convergent, which can be proved similarly using the left part of the inequalities (2.9).

Remark 2.1. For the local fractional integrals described in Theorem 5, suppose that $L = 0$. Then, if $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty g(x)(dx)^\alpha$ is convergent, so is $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$. Or, alternatively, suppose that $L = \infty$. Then, if $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty g(x)(dx)^\alpha$ is divergent, so is $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha$.

From Theorem 2.2 and Theorem 2.5, the following result is straightforward

Corollary 2.1 *Let* $|f(x)| \leq g(x)$ *for all* $x \in [a, \infty)$*. Then the convergence of the local fractional integral* $\frac{1}{\Gamma(1+\alpha)}\int_a^\infty g(x)(dx)^\alpha$ *implies the convergence of the integral*
$$\frac{1}{\Gamma(1+\alpha)}\int_a^\infty f(x)(dx)^\alpha.$$

If an local fractional integral is conditionally convergent, the demonstration of its convergence is usually a more delicate matter. Many of the instances of practical importance can be handled by the following theorem, which is an analogue of the well-known Dirichlet-Abel test for local fractional improper integrals on R^{α}.

Theorem 2.6 *Let the following conditions be satisfied.*

(1) $f(x)$ is local fractional integrable from a to any point $t \in [a, \infty)$, and the local fractional integral

$$F(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^t f(x)(dx)^{\alpha},$$

is bounded for all $t \geq a$.

(2) $g(x)$ is monotone on $[a, \infty)$ and $\lim_{x\to\infty} g(x) = 0$,

Then the local fractional improper integral of first kind of the form

$$\frac{1}{\Gamma(1+\alpha)} \int_a^{\infty} f(x)g(x)(dx)^{\alpha}, \tag{2.10}$$

is convergent.

Proof. Applying the mean value theorem for Local fractional integrals [12], we can write, for any $t_1, t_2 \in [a, \infty)$ with $t_2 > t_1 \geq a$

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)} \int_{t_1}^{t_2} f(x)g(x)(dx)^{\alpha} \\ &= [g(t_1) - g(t_2)]F(\xi) + g(t_2)\frac{1}{\Gamma(1+\alpha)} \int_{t_1}^{t_2} f(x)(dx)^{\alpha},\end{aligned}$$

where $F(\xi)$ is between $\inf\limits_{t\in[t_1,t_2]} F(t)$ and $\sup\limits_{t\in[t_1,t_2]} F(t)$. Let us suppose that M is a bound for $|F(t)|$ on $[a, \infty)$, that is, $|F(t)| \leq M$. Then we have from (2.11), taking into account that the local fractional integral on the right hand side of (2.11) is equal to $F(t_2) - F(t_1)$,

$$\left|\frac{1}{\Gamma(1+\alpha)} \int_{t_1}^{t_2} f(x)g(x)(dx)^{\alpha}\right| \leq M[|g(t_1)| + 3|g(t_2)|]. \tag{2.12}$$

Using this inequality, it is not difficult to complete the proof.

Let $\varepsilon > 0$ be arbitrary. Since $g(x) \to 0$ as $x \to 0$, we can choose a number $t_0 > a$ such that $|g(x)| < \frac{\varepsilon^{\alpha}}{4K}$ for all $x > t_0$.Hence, and from (2.12), it follows that for all t_1 and t_2 with $t_1 > t_0$and $t_2 > t_0$ the inequality

$$\left|\frac{1}{\Gamma(1+\alpha)} \int_{t_1}^{t_2} f(x)g(x)(dx)^{\alpha}\right| < \varepsilon^{\alpha},$$

holds. Consequently, by the Cauchy criterion (Theorem 2.1) the integral (2.10) is convergent.

Remark2.2. Integrals with $-\infty$ as a limit of integration may be treated by methods parallel to those given above.

3 Local fractional improper integrals of second kind

Definition 3.1 *(Local fractional improper integral of type 2) local fractional improper integrals of type 2 are evaluated as follows:*

1. if $f(x)$ is continuous on $[a,b)$ and not continuous at b then we define

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha = \lim_{t\to b^-}\frac{1}{\Gamma(1+\alpha)}\int_a^t f(x)(dx)^\alpha, \tag{3.1}$$

provided the limit exists as a finite number. In this case $\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha$ *is said to be convergent (or to converge). Otherwise,* $\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha$ *is said to be divergent (or to diverge).*

2. if $f(x)$ is continuous on $(a,b]$ and not continuous at a then we define

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha = \lim_{t\to a^+}\frac{1}{\Gamma(1+\alpha)}\int_t^b f(x)(dx)^\alpha,$$

provided the limit exists as a finite number. In this case $\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha$ *is said to be convergent (or to converge). Otherwise,* $\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha$ *is said to be divergent (or to diverge).*

3. if $f(x)$ is not continuous at c where $a < c < b$ and both $\frac{1}{\Gamma(1+\alpha)}\int_a^c f(x)(dx)^\alpha$ *and* $\frac{1}{\Gamma(1+\alpha)}\int_c^b f(x)(dx)^\alpha$ *converge then we define*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_a^c f(x)(dx)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_c^b f(x)(dx)^\alpha.$$

The local fractional integrals on the right are evaluated as shown in 1. and 2.

All theorems of Section 2 have exact analogues for the local fractional improper integrals of second kind whose wordings differ only slightly from the statements given in Section 2.

For the existence of the left side of (3.1) it is necessary and sufficient that the conditions of Cauchy's criterion hold: Given any $\varepsilon > 0$, there is $b_0 < b$ such that

$$\left|\frac{1}{\Gamma(1+\alpha)}\int_{c_1}^{c_2} f(x)(dx)^\alpha\right| \le \varepsilon^\alpha,$$

for any c_1; $c_2 \in [a,\infty)$ satisfying the inequalities $b_0 < c_1 < b$ and $b_0 < c_2 < b$.

Suppose in local fractional integral (3.1) we have $f(x) \ge 0$. Then, for $c \in [a,b)$,

$$F(c) = \frac{1}{\Gamma(1+\alpha)}\int_a^c f(x)(dx)^\alpha,$$

does not decrease as c increases, and the local fractional integral (3.1) is convergent if and only if $F(c)$ is bounded, in which case the value of the integral is $\lim_{c\to b^-} F(c)$. This result enables us to prove a comparison test strictly parallel to Theorem 5, from which in turn we deduce limit tests in which the convergence or divergence of two integrals of the same type,

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha \quad \text{and} \quad \frac{1}{\Gamma(1+\alpha)}\int_a^b g(x)(dx)^\alpha,$$

are related by an examination of the limit

$$\lim_{x \to b^-} \frac{f(x)}{g(x)}.$$

Similar definitions are made and entirely similar results are obtained for integrals of the second kind improper at the lower limit of integration.

Remark 3.1. Many local fractional improper integrals occurring in practice are of mixed type. If singularities occur within the interval of integration, or at both ends of an interval of integration, the local fractional integral must be separated into several local fractional integrals, each of which is a pure type of either first or second kind. The local fractional integral is called divergent if any one of the constituent pure types is divergent.

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