Dirac Singletons in a Quantum Theory over a Galois Field

Felix M. Lev

Artwork Conversion Software Inc., 1201 Morningside Drive, Manhattan Beach, CA 90266, USA (Email: felixlev314@gmail.com)

Abstract

Dirac singletons are exceptional irreducible representations (IRs) of the so(2,3) algebra found by Dirac. As shown in a seminal work by Flato and Fronsdal, the tensor product of singletons can be decomposed into massless IRs of the so(2,3) algebra and therefore each massless particle (e.g. the photon) can be represented as a composite state of singletons. This poses a fundamental problem of whether only singletons can be treated as true elementary particles. However, in standard quantum theory (based on complex numbers) such a possibility encounters difficulties since one has to answer the following questions: a) why singletons have not been observed and b) why the photon is stable and its decay into singletons has not been observed. We show by direct calculations that in a quantum theory over a Galois field (GFQT), the decomposition of the tensor product of singletons IRs contains not only massless IRs but also special massive IRs which have no analogs in standard theory. In the case of supersymmetry we explicitly construct a complete set of IRs taking part in the decomposition of the tensor product of supersingletons. Then in GFQT one can give natural explanations of a) and b).

Key words: quantum theory; Galois fields; elementary particles; Dirac singletons

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1 Introduction: the notion of elementary particle in quantum theory

Although theory of elementary particles exists for a rather long period of time, there is no commonly accepted definition of elementary particle in this theory. A discussion of numerous controversial approaches can be found, for example, in Ref. [1]. In the spirit of quantum field theory (QFT), fields are more fundamental than particles and some authors even claim that particles do not exist. From the point of view of QFT, a possible definition follows [2]. *It is simply a particle whose field appears in the Lagrangian. It does not matter if it is stable, unstable, heavy, light. If its field appears in the Lagrangian then it is elementary, otherwise it is composite.*

Another possible approach is as follows. Suppose that the space-time background is invariant under the action of a group G which is called the symmetry group. Then the operators describing a system under consideration should act in a representation space of G. In that case an elementary particle can be defined such that its representation space is a space of an irreducible representation (IR) of the group G. This approach is in the spirit of the well known Klein's Erlangen program in mathematics. In particular, when the space-time background is Minkowski space, elementary particles are described by IRs of the Poincare group discussed for the first time by Wigner.

However, as we argue in Refs. [3, 4], quantum theory should not be based on classical spacetime background and the approach should be the opposite. Each system is described by a set of independent operators. By definition, the rules how these operators commute with each other define the symmetry algebra. Then the elementary particle can be defined such that its representation space is a space of an IR of the symmetry algebra.

From the point of view of our definition of symmetry on quantum level, de Sitter (dS) symmetry or anti-de Sitter (AdS) symmetry is more fundamental than Poincare symmetry since the Poincare algebra is a special case of the dS or AdS algebras obtained from them by contraction. For example, the representation operators of the AdS algebra should satisfy the commutation relations

$$[M^{ab}, M^{cd}] = -i(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \tag{1}$$

where $a, b = 0, 1, 2, 3, 4$, $M^{ab} = -M^{ba}$ and η^{ab} is the diagonal metric tensor such that $\eta^{00} = \eta^{44} = -\eta^{11} = -\eta^{22} = -\eta^{33} == 1$. All the operators M^{ab} are dimensionless while in the case of Poincare symmetry only the operators of the Lorentz algebra are dimensionless while the momentum operators have the dimension $1/length$. If R is a parameter with the dimension *length* and the operators P^μ are *defined* as $P^\mu = M^{4\mu}/2R$ ($\mu = 0, 1, 2, 3$) then in the formal limit $R \to \infty$ one gets the commutation relations of the Poincare algebra from Eq. (1). This contraction procedure is well known.

The relations between unitary IRs of Lie groups and IRs of their Lie algebras by Hermitian operators are well known and in typical cases these IRs are the same. In particular, a classification of IRs of the AdS algebra can be obtained from the well known results by Fronsdal and Evans [5, 6]. By analogy with IRs of the Poincare algebra, such IRs also can be characterized by the values of mass and spin. Among those IRs, there are ones which are analogous to massive and massless IRs of the Poincare algebra and therefore in the AdS case they also can be called massive and massless, respectively. However, a very interesting feature of the AdS case is that here there also exist two special positive energy IRs discovered by Dirac [7], which have no analogs with IRs of the Poincare algebra. These IRs are called the Dirac singletons. In the literature, the singleton with the spin 1/2 is often called Di and the one with the spin zero - Rac.

Since each massive IR of the AdS algebra can be constructed as a tensor product of another massive or massless IRs, a question arises whether states described by massive IRs should be called elementary. In Standard Model (which is based on Poincare invariance) only massless IRs are treated as elementary. On the other hand, as shown in a seminal paper by Flato and Fronsdal [8] (see also Ref. [9]), each massless IR can be constructed as a tensor product of singletons IRs. At the same time, the singleton IRs cannot be represented as tensor products of other IRs. These observations give grounds to think that in the AdS theory only the Dirac singletons should be treated as true elementary particles. However, in that case the following questions arise. Each massless boson (e.g. the photon) can be constructed as a tensor product of either two Dis or two Racs. Which of those possibilities (if any) is physically preferable? A natural answer is as follows. If the theory is supersymmetric then the AdS algebra should be extended to the superalgebra osp(1,4) which has only one positive energy IR combining Di and Rac into the Dirac supermultiplet. For the first time, this possibility has been discussed probably in Refs. [10, 11]. Therefore in that case there exists only one Dirac superparticle which could be treated as the only elementary superparticle. Nevertheless, one still should answer the following questions:

- a) Why singletons have not been observed yet.
- b) Why such massless particles as photons and others are stable and their decays into singletons have not been observed.

In the present paper we argue that in a quantum theory over a Galois field (GFQT) proposed in our earlier works (see e.g. Refs. [12, 13, 14]) one can give natural explanations of a) and b). All the results of the paper are obtained by direct calculations and therefore no special knowledge is needed for understanding those results. We define GFQT as a theory where

- *Quantum states are represented by elements of a linear projective space over a Galois field and physical quantities are represented by linear operators in that space.*

For any new theory, there should exist a correspondence principle that at some conditions the predictions of this theory are close to ones given by the old well established theory. In Refs. [12, 13, 14] it is discussed in detail that a correspondence between GFQT and standard quantum theory takes place if the characteristic p of the Galois field is rather large. Representations of Lie algebras over a Galois field in spaces over a Galois field are called modular representations. Let $F_p = Z/Zp$ be the residue field modulo p and F_{p^2} be a Galois field containing p^2 elements. As argued in Refs. [12, 13, 14], a natural version of GFQT is that the operators corresponding to physical quantities act in spaces over F_{p^2} where $p = 3\ (mod\ 4)$ since in that case each element of F_{p^2} can be written as $a + bi,\ a, b \in F_p$ and therefore the field F_{p^2} can be treated as a finite analog of complex numbers. The results of the present paper do not depend on the choice of the Galois field in GFQT.

The paper is organized as follows. In Secs. 2-5 we explicitly construct IRs describing massive and massless particles, and Dirac singletons. Standard and modular IRs are discussed in parallel, and we indicate their common and distinct features. In Sec. 6 we discuss in detail how usual particles and singletons should be discussed in the Poincare and semiclassical limits of standard theory. In Sec. 7 it is shown that, in contrast to standard theory, the tensor products of singleton IRs in GFQT contain not only massless IRs but also special massive IRs, which have no analogs in standard theory. Beginning from Sec. 8 we proceed to the supersymmetric case, and the main result of the paper is described in Sec. 10. Here we explicitly find a complete list of IRs taking part in the decomposition of the tensor product of two supersingletons. In standard theory the well known results are recovered while in GFQT this list also contains special supersymmetric IRs which have no analogs in standard theory. Finally, Sec. 11 is a discussion.

2 Modular IRs of the sp(2) and su(2) algebras

The key role in constructing modular IRs of the so(2,3) algebra is played by modular IRs of the sp(2) subalgebra. They are described by a set of operators $(a', a", h)$ satisfying the commutation relations

$$[h, a'] = -2a' \quad [h, a"] = 2a" \quad [a', a"] = h \tag{2}$$

The Casimir operator of the second order for the algebra (2) has the form

$$K = h^2 - 2h - 4a"a' = h^2 + 2h - 4a'a" \tag{3}$$

We first consider representations with the vector e_0 such that

$$a'e_0 = 0, \quad he_0 = q_0 e_0 \tag{4}$$

where $q_0 \in F_p$. We will denote q_0 by the numbers $0, 1, ...p-1$. Denote $e_n = (a")^n e_0$. Then it follows from Eq. (3) and (4), that

$$he_n = (q_0 + 2n)e_n, \quad Ke_n = q_0(q_0 - 2)e_n \tag{5}$$

$$a'a"e_n = (n+1)(q_0+n)e_n \tag{6}$$

One can consider analogous representations in standard theory. Then q_0 is a positive real number, $n = 0, 1, 2, ...$ and the elements e_n form a basis of the IR. In this case e_0 is a vector with a minimum eigenvalue of the operator h (minimum weight) and there are no vectors with the maximum weight. The operator h is positive definite and bounded below by the quantity q_0. For these reasons the above modular IRs can be treated as modular analogs of such standard IRs that h is positive definite.

Analogously, one can construct modular IRs starting from the element e_0' such that

$$a"e_0' = 0, \quad he_0' = -q_0e_0' \tag{7}$$

and the elements e_n' can be defined as $e_n' = (a')^n e_0'$. Such modular IRs are analogs of standard IRs where h is negative definite. However, in the modular case Eqs. (4) and (7) define the same IR. This is clear from the following consideration.

The set $(e_0, e_1, ...e_N)$ will be a basis of IR if $a"e_i \neq 0$ for $i < N$ and $a"e_N = 0$. These conditions must be compatible with $a'a"e_N = 0$. The case $q_0 = 0$ is of no interest since, as follows from Eqs. (5-7), all the representation operators are null operators, the representation is one-dimensional and e_0 is the only basis vector in the representation space. If $q_0 = 1, ...p-1$, it follows from Eq. (6) that N is defined by the condition $q_0 + N = 0$. Hence $N = p - q_0$ and the dimension of IR equals

$$Dim(q_0) = p - q_0 + 1 \tag{8}$$

This result is formally valid for all the values of q_0 if we treat q_0 as one of the numbers $1, ...p-1, p$. It is easy to see that e_N satisfies Eq. (7) and therefore it can be identified with e_0'.

In standard theory, IRs are discussed in Hilbert spaces, i.e. the space of the IR is supplied by a positive definite scalar product. It can be defined such that $(e_0, e_0) = 1$, the operator h is selfadjoint and the operators a' and $a"$ are adjoint to each other: $(a')^* = a"$. Then, as follows from Eq. (6),

$$(e_n, e_n) = n!(q_0)_n \tag{9}$$

where we use the Pocchammer symbol $(q_0)_n = q_0(q_0+1)\cdots(q_0+n-1)$. Usually the basis vectors are normalized to one but this is only a matter of convention but not a matter of principle since not the probability itself but only ratios of probabilities have a physical meaning (see e.g. the discussion in Ref. [14]). In GFQT one can formally define the scalar product by the same formulas but in that case this scalar product cannot be positive definite since in Galois fields the notions of positive and negative numbers can be only approximate. Therefore in GFQT the probabilistic interpretation cannot be universal. However, if the quantities q_0 and n are such that the r.h.s. of Eq. (9) is much less than p then the probabilistic interpretation is (approximately) valid if the IR is discussed in a space over F_{p^2} (see Refs. [12, 13, 14] for a detailed discussion). Therefore if p is very large, then for a large number of elements there is a correspondence between standard theory and GFQT.

Representations of the su(2) algebra are defined by a set of operators (L_+, L_-, L_3) satisfying the commutations relations

$$[L_3, L_+] = 2L_+ \quad [L_3, L_-] = 2L_- \quad [L_+, L_-] = 2L_3 \tag{10}$$

In the case of representations over the field of complex numbers, these relations can be formally obtained from Eq. (2) by the replacements $h \to L_3$, $a' \to iL_-$ and $a" \to iL_+$.

The difference between the representations of the sp(2) and su(2) algebras in Hilbert spaces is that in the latter case the Hermiticity conditions are $L_3^* = L_3$ and $L_+^* = L_-$. The Casimir operator for the algebra (10) is

$$K = L_3^2 - 2L_3 + 4L_+L_- = L_3^2 + 2L_3 + 4L_-L_+ \tag{11}$$

For constructing IRs, we assume that the representation space contains a vector e_0 such that

$$L_3e_0 = se_0 \quad L_+e_0 = 0 \tag{12}$$

where $s \geq 0$ for standard IRs and $s \in F_p$ for modular IRs. In the letter case we will denote s by the numbers $0, 1, ...p-1$. If $e_k = (L_-)^k e_0$ $(k = 0, 1, 2, ...)$ then it is easy to see that

$$L_3e_k = (s-2k)e_k \quad Ke_k = s(s+2)e_k \quad L_+L_-e_k = (k+1)(s-k)e_k \tag{13}$$

The IR will be finite dimensional if there exists $k = k_{max}$ such that $L_+L_-e_k = 0$ for this value of k. As follows from the above expression, for modular IRs such a value of k always exists, $k_{max} = s$ and the dimension of the IR is $Dim(s) = s + 1$. For standard IRs the same conclusion is valid if s iz zero or a natural number.

In standard quantum theory, the representation operators of the su(2) algebra are associated with the components of the angular momentum operator $\mathbf{L} = (L_x, L_y, L_z)$ such that $L_3 = L_z$ and $L_\pm = (L_x \pm iL_y)/2$. The commutation relations for the components of $\mathbf{L}$ are usually written in units where $\hbar = 1$. Then s can be only an integer or a half-integer and $Dim(s) = 2s + 1$. As argued in Ref. [3], fundamental quantum theory should not involve dimensionful physical quantities at all. For correspondence between GFQT and standard theory, we write the commutation relations in the form (10) which corresponds to commutation relations in standard quantum theory in units $\hbar/2 = 1$. The matter is that if p is very large then $1/2$ in a Galois field is a very large number $(p+1)/2$.

3 Tensor product of modular IRs of the sp(2) algebra

Consider two IRs of the sp(2) algebra in spaces H_j $(j = 1, 2)$. Each IR is defined by a set of operators $(h^{(j)}, a^{(j)\prime}, a^{(j)"})$ satisfying the commutation relations (2) and by a vector $e_0^{(j)}$ such that (see Eq. (4))

$$a^{(j)\prime}e_0^{(j)} = 0, \quad h^{(j)}e_0 = q_0^{(j)}e_0^{(j)} \tag{14}$$

As follows from the results of the preceding section, the vectors $e_n^{(j)} = (a^{(j)"})^n e_0^{(j)}$ where $k = 0, 1, ...N^{(j)}$ and $N^{(j)} = p - q_0^{(j)}$ form a basis in H_j.

The tensor product of such IRs is defined as follows. The basis of the representation space is formed by the elements $e_{kl} = e_k^{(1)} \times e_l^{(2)}$ and the independent representation operators are $(h, a', a")$ such that $h = h^{(1)} + h^{(2)}$, $a' = a^{(1)'} + a^{(2)'}$ and $a" = a^{(1)"} + a^{(2)"}$. Here it is assumed that the operator with the superscript (j) acts on the elements $e_k^{(j)}$ in the same way as in the IR in H_j while on the elements $e_l^{(j')}$ where $j' \neq j$ it acts as the identity operator. For example,

$$h \sum_{kl} c_{kl}(e_k^{(1)} \times e_l^{(2)}) = \sum_{kl} c_{kl}[(h^{(1)}e_k^{(1)}) \times e_l^{(2)} + e_k^{(1)} \times (h^{(2)}e_l^{(2)})]$$

Then the operators $(h, a', a")$ satisfy the same commutation relations as in Eq. (2) and hence they implement a representation of the sp(2) algebra in the space $H_1 \times H_2$. Our goal is to find a decomposition of this representation into irreducible components.

It is obvious that the cases when $q_0^{(1)} = 0$ or $q_0^{(2)} = 0$ are trivial and therefore we will assume that $q_0^{(1)} \neq 0$ and $q_0^{(2)} \neq 0$. Suppose that $q_0^{(1)} \geq q_0^{(2)}$ and consider the vector

$$e(k) = \sum_{i=0}^{k} c(i,k)(e_i^{(1)} \times e_{k-i}^{(2)}) \tag{15}$$

As follows from Eq. (5) and the definition of h,

$$he(k) = (q_0^{(1)} + q_0^{(2)} + 2k)e(k) \tag{16}$$

Therefore if $a'e(k) = 0$ then the vector $e(k)$ generates a modular IR with the dimension $Dim(q_0^{(1)}, q_0^{(2)}, k) = p + 1 - (q_0^{(1)} - q_0^{(2)} - 2k)$ where $q_0^{(1)} - q_0^{(2)} - 2k$ is taken modulo p. As follows from Eqs. (6) and (15),

$$a'e(k) = \sum_{i=0}^{k} c(i,k)[i(q_0^{(1)}+i-1)(e_{i-1}^{(1)} \times e_{k-i}^{(2)}) + (k-i)(q_0^{(2)}+k-i-1)(e_i^{(1)} \times e_{k-i-1}^{(2)})] \tag{17}$$

This condition will be satisfied if

$$c(i,k) = \frac{(k+1-i)(q_0^{(2)}+k-i)c(i-1,k)}{i(q_0^{(1)}+i-1)} \quad (i = 1, ...k) \tag{18}$$

It is clear from this expression that in standard case the possible values of k are $0, 1, ...\infty$ while in modular case $k = 0, 1, ...k_{max}$ where $k_{max} = p - q_0^{(1)}$.

It is obvious that at different values of k, the IRs generated by $e(k)$ are linearly independent and therefore the tensor product of the IRs generated by $e_0^{(1)}$ and $e_0^{(2)}$ contains all the IRs generated by $e(k)$. A question arises whether the latter IRs give a full decomposition of the tensor product. This is the case when the dimension of the tensor product equals the sum of dimensions of the IRs generated by $e(k)$. Below we will be interested in the tensor product of singleton IRs and, as shown in Sec. 5, in that case $q_0^{(1)} + q_0^{(2)} > p$. Therefore $q_0^{(1)} + q_0^{(2)} + 2k \in [q_0^{(1)} + q_0^{(2)}, 2p - q_0^{(1)} + q_0^{(2)}]$

and for all values of k, $q_0^{(1)}+q_0^{(2)}+2k$ is in the range $(p,2p]$. Then, as follows from Eq. (8), the fact that the IRs generated by $e(k)$ give a full decomposition of the tensor product follows from the relation

$$\sum_{k=0}^{p-q_0^{(1)}}(2p+1-q_0^{(1)}-q_0^{(2)}-2k)=(p+1-q_0^{(1)})(p+1-q_0^{(2)}) \tag{19}$$

4 Modular IRs of the so(2,3) algebra

Standard IRs of the so(2,3) algebra relevant for describing elementary particles have been considered by several authors. The description in this section is a combination of two elegant ones given in Ref. [6] for standard IRs and Ref. [15] for modular IRs. In standard theory, the commutation relations between the representation operators are given by Eq. (1)

If a modular IR is considered in a linear space over F_{p^2} with $p=3\ (mod\ 4)$ then Eq. (1) is also valid. However, in the general case it is convenient to work with another set of ten operators. Let $(a'_j, a_j", h_j)$ $(j=1,2)$ be two independent sets of operators satisfying the commutation relations for the sp(2) algebra

$$[h_j,a'_j]=-2a'_j \quad [h_j,a_j"]=2a_j" \quad [a'_j,a_j"]=h_j \tag{20}$$

The sets are independent in the sense that for different j they mutually commute with each other. We denote additional four operators as $b', b", L_+, L_-$. The operators $L_3=h_1-h_2, L_+, L_-$ satisfy the commutation relations (10) of the su(2) algebra while the other commutation relations are as follows

$$\begin{gathered}
[a'_1,b']=[a'_2,b']=[a_1",b"]=[a_2",b"]=[a'_1,L_-]=[a_1",L_+]=[a'_2,L_+]= \\
[a_2",L_-]=0 \quad [h_j,b']=-b' \quad [h_j,b"]=b" \quad [h_1,L_\pm]=\pm L_\pm \quad [h_2,L_\pm]=\mp L_\pm \\
[b',b"]=h_1+h_2 \quad [b',L_-]=2a'_1 \quad [b',L_+]=2a'_2 \quad [b",L_-]=-2a_2" \\
[b",L_+]=-2a_1" \quad [a'_1,b"]=[b',a_2"]=L_- \quad [a'_2,b"]=[b',a_1"]=L_+ \\
[a'_1,L_+]=[a'_2,L_-]=b' \quad [a_2",L_+]=[a_1",L_-]=-b"
\end{gathered} \tag{21}$$

At first glance these relations might seem rather chaotic but in fact they are very natural in the Weyl basis of the so(2,3) algebra.

In spaces over F_{p^2} with $p=3\ (mod\ 4)$ the relation between the above sets of ten operators is

$$\begin{gathered}
M_{10}=i(a_1"-a'_1-a_2"+a'_2) \quad M_{14}=a_2"+a'_2-a_1"-a'_1 \\
M_{20}=a_1"+a_2"+a'_1+a'_2 \quad M_{24}=i(a_1"+a_2"-a'_1-a'_2) \\
M_{12}=L_3=h_1-h_2 \quad M_{23}=L_++L_- \quad M_{31}=-i(L_+-L_-) \\
M_{04}=h_1+h_2 \quad M_{34}=b'+b" \quad M_{30}=-i(b"-b')
\end{gathered} \tag{22}$$

and therefore the sets are equivalent. However, the relations (10,20,21) are more general since they can be used when the representation space is a space over any Galois field. It is also obvious that such a *definition* of the operators M_{ab} is not unique. For example, any cyclic permutation of the indices $(1,2,3)$ gives a new set of operators satisfying the same commutation relations.

In standard theory, the Casimir operator of the second order for the representation of the so(2,3) algebra is given by

$$I_2 = \frac{1}{2}\sum_{ab} M_{ab}M^{ab} \tag{23}$$

As follows from Eqs. (10,20-22), I_2 can be written as

$$I_2 = 2(h_1^2 + h_2^2 - 2h_1 - 4h_2 - 2b"b' + 2L_-L_+ - 4a_1"a_1' - 4a_2"a_2') \tag{24}$$

We use the basis in which the operators (h_j, K_j) $(j = 1,2)$ are diagonal. Here K_j is the Casimir operator (3) for algebra $(a_j', a_j", h_j)$. For constructing IRs we need operators relating different representations of the sp(2)×sp(2) algebra. By analogy with Refs. [6, 15], one of the possible choices is as follows

$$\begin{aligned} A^{++} &= b"(h_1-1)(h_2-1) - a_1"L_-(h_2-1) - a_2"L_+(h_1-1) + a_1"a_2"b' \\ A^{+-} &= L_+(h_1-1) - a_1"b' \quad A^{-+} = L_-(h_2-1) - a_2"b' \quad A^{--} = b' \end{aligned} \tag{25}$$

We consider the action of these operators only on the space of minimal sp(2)×sp(2) vectors, *i.e.*, such vectors x that $a_j'x = 0$ for $j = 1,2$, and x is the eigenvector of the operators h_j. Then by using Eqs. (10,20-22), one can directly verify that if x is a minimal vector such that $h_jx = \alpha_jx$ then $A^{++}x$ is the minimal eigenvector of the operators h_j with the eigenvalues α_j+1, $A^{+-}x$ - with the eigenvalues (α_1+1, α_2-1), $A^{-+}x$ - with the eigenvalues (α_1-1, α_2+1), and $A^{--}x$ - with the eigenvalues α_j-1.

By analogy with Refs. [6, 15], we require the existence of the vector e_0 satisfying the conditions

$$a_j'e_0 = b'e_0 = L_+e_0 = 0 \quad h_je_0 = q_je_0 \quad (j = 1,2) \tag{26}$$

where $q_j \in F_p$. As follows from Eq. (24), in the IR characterized by the quantities (q_1, q_2), all the nonzero elements of the representation space are the eigenvectors of the operator I_2 with the eigenvalue

$$I_2 = 2(q_1^2 + q_2^2 - 2q_1 - 4q_2) \tag{27}$$

Since $L_3 = h_1 - h_2$ then, as follows from the results of Sec. 2, if q_1 and q_2 are characterized by the numbers $0, 1, ...p-1$, $q_1 \geq q_2$ and $q_1 - q_2 = s$ then the elements $(L_+)^ke_0$ $(k = 0, 1, ...s)$ form a basis of the IR of the su(2) algebra with the spin s such that the dimension of the IR is $s+1$. Therefore in the theory over a

Galois field the case when $q_1 < q_2$ should be treated such that $s = p + q_1 - q_2$. IRs with $q_1 < q_2$ have no analogs in standard theory and we will call them special massive IRs.

It is well known (see e.g., Ref. [6]) that $M^{05} = h_1 + h_2$ is the AdS analog of the energy operator. As follows from Eqs. (20) and (21), the operators (a'_1, a'_2, b') reduce the AdS energy by two units. Therefore e_0 is an analog of the state with the minimum energy which can be called the rest state. For this reason we use m to denote $q_1 + q_2$. In standard classification [6], the massive case is characterized by the condition $q_2 > 1$ and the massless one—by the condition $q_2 = 1$. There also exist two exceptional IRs discovered by Dirac [7] (Dirac singletons). They are characterized by the conditions $m = 1,\ s = 0$ and $m = 2,\ s = 1$. In this section we consider the massive case while the cases of singleton, massless and special massive IRs will be considered in the next section.

As follows from the above remarks, the elements

$$e_{nk} = (A^{++})^n (A^{-+})^k e_0 \tag{28}$$

represent the minimal sp(2)×sp(2) vectors with the eigenvalues of the operators h_1 and h_2 equal to $Q_1(n,k) = q_1 + n - k$ and $Q_2(n,k) = q_2 + n + k$, respectively.

Consider the element $A^{--}A^{++}e_{nk}$. In view of the properties of the A operators mentioned above, this element is proportional to e_{nk} and therefore one can write $A^{--}A^{++}e_{nk} = a(n,k)e_{nk}$. One can directly verify that the actions of the operators A^{++} and A^{-+} on the space of minimal $sp(2)\times sp(2)$ vectors are commutative and therefore $a(n,k)$ does not depend on k. A direct calculation gives

$$\begin{aligned}(A^{--}A^{++} - A^{++}A^{--})e(n,k) = \{(Q_2 - 1)[Q_1 - 1)(Q_1 + Q_2) - (Q_1 - Q_2)] + \\ (Q_1 + Q_2 - 2)(\frac{1}{2}Q_1^2 + \frac{1}{2}Q_2^2 - Q_1 - 2Q_2 - \frac{1}{4}I_2)\}e(n,k)\end{aligned} \tag{29}$$

where $Q_1 \equiv Q_1(n,k)$ and $Q_2 \equiv Q_2(n,k)$. As follows from this expression,

$$\begin{aligned}a(n) - a(n-1) = q_1(q_2 - 1)(m - 2) + 2n(q_1^2 + q_2^2 + \\ 3q_1q_2 - 5q_1 - 4q_2 + 4) + 6n^2(m - 2) + 4n^3\end{aligned} \tag{30}$$

Since $b'e_0 = 0$ by construction, we have that $a(-1) = 0$ and a direct calculation shows that, as a consequence of Eq. (30)

$$a(n) = (n + 1)(m + n - 2)(q_1 + n)(q_2 + n - 1) \tag{31}$$

Analogously, one can write $A^{+-}A^{-+}e_{nk} = b(k)e_{nk}$ and the result of a direct calculation is

$$b(k) = -\frac{1}{4}(Q_1 - 2)(Q_2 - 1)(2Q_1^2 + 2Q_2^2 - 8Q_1 - 4Q_2 - I_2) + a(n - 1) \tag{32}$$

Then, as a consequence of Eqs. (27) and (31)

$$b(k) = (k+1)(s-k)(q_1 - k - 2)(q_2 + k - 1) \quad (33)$$

As follows from these expressions, in the massive case k can assume only the values $0, 1, ...s$ and in standard theory $n = 0, 1, ...\infty$. However, in the modular case $n = 0, 1, ...n_{max}$ where n_{max} is the first number for which the r.h.s. of Eq. (31) becomes zero in F_p. Therefore $n_{max} = p + 2 - m$.

The full basis of the representation space can be chosen in the form

$$e(n_1 n_2 nk) = (a_1")^{n_1}(a_2")^{n_2} e_{nk} \quad (34)$$

In standard theory n_1 and n_2 can be any natural numbers. However, as follows from the results of Sect. 2, Eq. (20) and the properties of the A operators,

$$\begin{aligned} n_1 = 0, 1, ...N_1(n,k) \quad & n_2 = 0, 1, ...N_2(n,k) \\ N_1(n,k) = p - q_1 - n + k \quad & N_2(n,k) = p - q_2 - n - k \end{aligned} \quad (35)$$

As a consequence, the representation is finite dimensional in agreement with the Zassenhaus theorem [16] (moreover, it is finite since any Galois field is finite).

In the case of standard IR of the so(2,3) algebra, one can assume additionally that the representation space is supplied by a scalar product. The element e_0 can always be chosen such that $(e_0, e_0) = 1$ and then one can explicitly calculate all the scalar products $(e(n_1 n_2 nk), e(n_1 n_2 nk))$. If the representation operators satisfy the Hermiticity conditions $L_+^* = L_-$, $a_j^{'*} = a_j"$, $b^{'*} = b"$ and $h_j^* = h_j$ then, as follows from Eq. (22), the operators M^{ab} are Hermitian as it should be. However, by analogy with the discussion in Sect. 2, one can conclude that for modular representations in a special case when the representation space is a space over F_{p^2} with $p = 3 \ (mod\ 4)$, the probabilistic interpretation can be (approximately) valid only if $q_1, q_2 \ll p$ and we consider only a subset of elements, which are linear combinations of the elements $e(n_1 n_2 nk)$ such that $n_1, n_2, n, k \ll p$ and the coefficients are much less than p (see Refs. [12, 13, 14] for a detailed discussion).

In standard Poincare and AdS theories there also exist IRs with negative energies. They can be constructed by analogy with positive energy IRs. Instead of Eq. (26) one can require the existence of the vector e_0' such that

$$a_j" e_0' = b" e_0' = L_- e_0' = 0 \quad h_j e_0' = -q_j e_0' \quad (e_0', e_0') \neq 0 \quad (j = 1, 2) \quad (36)$$

where the quantities q_1, q_2 are the same as for positive energy IRs. It is obvious that positive and negative energy IRs are fully independent since the spectrum of the operator M^{05} for such IRs is positive and negative, respectively. However, *the modular analog of a positive energy IR characterized by q_1, q_2 in Eq. (26), and the modular analog of a negative energy IR characterized by the same values of q_1, q_2 in Eq. (36) represent the same modular IR.* This is the crucial difference between standard quantum theory and GFQT, and a proof is given below.

Let e_0 be a vector satisfying Eq. (26). Denote $N_1 = p - q_1$ and $N_2 = p - q_2$. Our goal is to prove that the vector $x = (a_1")^{N_1}(a_2")^{N_2}e_0$ satisfies the conditions (36), *i.e.*, x can be identified with e_0'.

As follows from the definition of N_1 and N_2, the vector x is the eigenvector of the operators h_1 and h_2 with the eigenvalues $-q_1$ and $-q_2$, respectively, and, in addition, it satisfies the conditions $a_1"x = a_2"x = 0$. Let us prove that $b"x = 0$. Since $b"$ commutes with the $a_j"$, we can write $b"x$ in the form

$$b"x = (a_1")^{N_1}(a_2")^{N_2}b"e_0 \tag{37}$$

As follows from Eqs. (21) and (26), $a_2'b"e_0 = L_+e_0 = 0$ and $b"e_0$ is the eigenvector of the operator h_2 with the eigenvalue $q_2 + 1$. Therefore, $b"e_0$ is the minimal vector of the sp(2) IR which has the dimension $p - q_2 = N_2$. Therefore $(a_2")^{N_2}b"e_0 = 0$ and $b"x = 0$.

The next stage of the proof is to show that $L_-x = 0$. As follows from Eq. (21) and the definition of x,

$$L_-x = (a_1")^{N_1}(a_2")^{N_2}L_-e_0 - N_1(a_1")^{N_1-1}(a_2")^{N_2}b"e_0 \tag{38}$$

We have already shown that $(a_2")^{N_2}b"e_0 = 0$, and therefore it suffices to prove that the first term in the r.h.s. of Eq. (38) is equal to zero. As follows from Eqs. (21) and (26), $a_2'L_-e_0 = b'e_0 = 0$, and L_-e_0 is the eigenvector of the operator h_2 with the eigenvalue $q_2 + 1$. Therefore $(a_2")^{N_2}L_-e_0 = 0$ and the proof is completed.

In standard theory, negative energy IRs are associated with antiparticles and their energy becomes positive after quantization. However, in GFQT the fact that positive and negative energy states belong to the same IR implies that very notion of particle and antiparticle is only approximate (see Refs. [14, 4] for a detailed discussion). As shown in Ref. [14], a modular IR splits into independent IRs corresponding to a particle and its antiparticle only in the approximation when at energies of order p the so(2,3) symmetry is broken.

5 Massless particles, Dirac singletons and special massive IRs

Those cases can be considered by analogy with the massive one. The case of Dirac singletons is especially simple. As follows from Eqs. (31) and (33), if $m = 1,\ s = 0$ then the only possible value of k is $k = 0$ and the only possible values of n are $n = 0, 1$ while if $m = 2,\ s = 1$ then the only possible values of k are $k = 0, 1$ and the only possible value of n is $n = 0$. This result does not depend on the value of p and therefore it is valid in both, standard theory and GFQT. The only difference between standard and modular cases is that in the former $n_1, n_2 = 0, 1, ...\infty$ while in the latter the quantities n_1, n_2 are in the range defined by Eq. (35).

The singleton IRs are indeed exceptional since the value of n in them does not exceed 1 and therefore the impression is that singletons are two-dimensional objects, not three-dimensional ones as usual particles. However, the singleton IRs have been obtained in the so(2,3) theory without reducing the algebra. Dirac has entitled his paper [7] "A Remarkable Representation of the 3 + 2 de Sitter Group". Below we argue that in GFQT the singleton IRs are even more remarkable than in standard theory.

First of all, as noted above, in standard theory there exist independent positive and negative IRs and the latter are associated with antiparticles. In particular, in standard theory there exist four singleton IRs - two IRs with positive energies and the corresponding IRs with negative energies, which can be called antisingletons. However, at the end of the preceding section we have proved that in GFQT one IR contains positive and negative energy states simultaneously. This proof can be applied to the singleton IRs without any changes. As a consequence, in the modular case there exist only two singleton IRs.

If $m = 1,\ s = 0$ then $q_1 = q_2 = 1/2$. In GFQT these relations should be treated as $q_1 = q_2 = (p+1)/2$. Analogously, if $m = 2,\ s = 1$ then $q_1 = 3/2,\ q_2 = 1/2$ and in GFQT $q_1 = (p+3)/2,\ q_2 = (p+1)/2$. Therefore the values of q_1 and q_2 for the singleton IRs are extremely large since they are of order $p/2$. As a consequence, the singleton IRs do not contain states where all the quantum numbers are much less than p. Since some of the quantum numbers are necessarily of order p, this is a natural explanation of the fact that the singletons have not been observed. In addition, as follows from the discussion in Sects. 2 and 4 (see also Refs. [12, 13, 14] for details) the fact that some quantum numbers are of order p implies that the singletons cannot be described in terms of the probabilistic interpretation.

Note also that if we consider the singleton IRs as modular analogs of negative energy IRs then the singleton IRs should be characterized either by $q_1 = q_2 = -1/2$ or by $q_1 = -3/2,\ q_2 = -1/2$. However, since in Galois fields $-1/2 = (p-1)/2$ and $-3/2 = (p-3)/2$, those values are very close to ones characterizing modular analogs of positive energy IRs. As a consequence, there is no approximation when singleton states can be characterized as particles or antiparticles.

The Rac IR contains only minimal $sp(2) \times sp(2)$ vectors with $h_1 = h_2 = (p+1)/2$ and $h_1 = h_2 = (p+3)/2$ while the Di IR contains only minimal $sp(2) \times sp(2)$ vectors with $h_1 = (p+3)/2,\ h_2 = (p+1)/2$ and $h_1 = (p+1)/2,\ h_2 = (p+3)/2$. Hence it easily follows from Eq. (8) that the dimensions of these IRs are equal to

$$Dim(Rac) = \frac{1}{2}(p^2+1) \quad Dim(Di) = \frac{1}{2}(p^2-1) \tag{39}$$

Consider now the massless case. Note first that when $q_2 = 1$, it follows from Eqs. (31) and (33) that $a(0) = 0$ and $b(0) = 0$. Therefore $A^{++}e_0 = A^{-+}e_0 = 0$ and if the definition $e(n,k) = (A^{++})^n (A^{-+})^k e_0$ is used for $(n = 0, 1, ...)$ and $(k = 0, 1, ...)$ then all the $e(n,k)$ will be the null elements.

We first consider the case when $s \neq 0$ and $s \neq p-1$. In that case we define $e(1,0)$ not as $A^{++}e_0$ but as $e(1,0) = [b"(h_1-1) - a_1"L_-]e_0$. A direct calculation using Eq. (21) shows that when $q_2 = 1$, this definition is legitimate since $e(1,0)$ is the minimal $sp(2) \times sp(2)$ vector with the eigenvalues of the operators h_1 and h_2 equal to $2+s$ and 2, respectively. With such a definition of $e(1,0)$, a direct calculation using Eqs. (10) and (21) gives $A^{--}e(1,0) = b'e(1,0) = s(s+1)e_0$ and therefore $e(1,0) \neq 0$. We now define $e(n,0)$ at $n \geq 1$ as $e(n,0) = (A^{++})^{n-1}e(1,0)$. Then Eq. (29) remains valid when $n \geq 1$. Since $A^{++}b'e(1,0) = s(s+1)A^{++}e_0 = 0$, Eq. (30) remains valid at $n = 1, 2, ...$ and $a(0) = 0$. Hence we get

$$a(n) = n(n+1)(n+s+1)(n+s) \quad (n \geq 1) \tag{40}$$

As a consequence, the maximal value of n in the modular case is $n_{max} = p-1-s$. This result has been obtained in Ref. [17].

For analogous reasons, we now cannot define $e(0,k)$ as $(A^{-+})^k e_0$. However, if we define $e(0,k) = (L_-)^k e_0$ then, as follows from the discussion at the end in Sec. 2, the elements $e(0,k)$ $(k = 0, 1, ...s)$ form a basis of the IR of the su(2) algebra with the spin s. Therefore the new definition of $e(0,k)$ is legitimate since $e(0,k)$ is the minimal $sp(2) \times sp(2)$ vector with the eigenvalues of the operators h_1 and h_2 equal to $1+s-k$ and $1+k$, respectively.

A direct calculation using Eqs. (10) and (21) gives that with the new definition of $e(0,k)$, $A^{--}A^{++}e(0,k) = b'A^{++}e(0,k) = 0$ and therefore $A^{++}e(0,k) = 0$. When $1 \leq k \leq s-1$, there is no way to obtain nonzero minimal $sp(2) \times sp(2)$ vectors with the eigenvalues of the operators h_1 and h_2 equal to $1+s-k+n$ and $1+k+n$, respectively, when $n > 0$. However, when $k = s$, such vectors can be obtained by analogy with the case $k = 0$. We define $e(1,s) = [b"(h_2-1) - a_2"L_+]e(0,s)$. Then a direct calculation gives $b'e(1,s) = s(s+1)e(0,s)$ and therefore $e(1,s) \neq 0$. We now define $e(n,s) = (A^{++})^{n-1}e(1,s)$ for $n \geq 1$. Then by analogy with the above discussion one can verify that if $A^{--}A^{++}e(n,s) = a(n)e(n,s)$ then $a(n)$ for $n \geq 1$ is again given by Eq. (40) and therefore in the modular case the maximal value of n is the same.

If $s = 0$ then the only possible value of k is $k = 0$ and for the vectors $e(n,0)$ we have the same results as above. In particular, Eq. (40) is valid with $s = 0$. When $s = p-1$, we can define $e(n,0)$ and $e(n,s)$ as above but since $s+1 = 0$ $(mod\ p)$, we get that $e(1,0) = e(1,s) = 0$. This is in agreement with the above discussion since $n_{max} = 0$ when $s = p-1$.

According to Standard Model (based on Poincare invariance), only massless Weyl particles can be fundamental elementary particles in Poincare invariant theory. Therefore a problem arises whether the above results can be treated as analogs of Weyl particles in standard and modular versions of AdS invariant theory. In view of the relation $P^\mu = M^{4\mu}/2R$ noted in Sect. 1, the AdS mass m and the Poincare mass m_P are related as $m_P = m/2R$. Since $m = 2q_2 + s$, the corresponding Poincare mass will be zero when $R \rightarrow \infty$ not only when $q_2 = 1$ but when q_2 is any finite number.

So a question arises why only the case $q_2 = 1$ is treated as massless. In Poincare invariant theory, Weyl particles are characterized not only by the condition that their mass is zero but also by the condition that they have a definite helicity. In standard case the minimum value of the AdS energy for massless IRs with positive energy is $E_{min} = 2 + s$ when $n = 0$. In contrast to the situation in Poincare invariant theory, where massless particles cannot be in the rest state, the massless particles in the AdS theory do have rest states and, as shown above, the value of the z projection of the spin in such states can be $-s, -s+2, ...s$ as usual. However, we have shown that for any value of energy greater than E_{min}, when $n \neq 0$, the spin state is characterized only by helicity, which can take the values either s when $k = 0$ or $-s$ when $k = s$, i.e. we have the same result as in Poincare invariant theory. Note that in contrast with IRs of the Poincare and dS algebras, standard IRs describing particles in AdS invariant theory belong to the discrete series of IRs and the energy spectrum in them is discrete: $E = E_{min}, E_{min} + 2, ...\infty$. Therefore, strictly speaking, rest states do not have measure zero as in Poincare and dS invariant theories. Nevertheless, the probability that the energy is exactly E_{min} is extremely small and therefore the above results show that the case $q_2 = 1$ indeed describes AdS analogs of Weyl particles.

Consider now dimensions of massless IRs. If $s = 0$ then, as follows from the above results, there exist only minimal $sp(2) \times sp(2)$ vectors with $h_1 = h_2 = 1+n$, $n = 0, 1, ...p-1$. Therefore, as follows from Eq. (8), the dimension of the massless IR with $s = 0$ equals

$$Dim(s=0) = \sum_{n=0}^{p-1} (p-n)^2 = \frac{1}{6}p(p+1)(2p+1) \tag{41}$$

If $s = 1$, there exist only minimal $sp(2) \times sp(2)$ vectors with $(h_1 = 2+n, h_2 = 1+n)$ and $(h_1 = 1+n, h_2 = 2+n)$ where $n = 0, 1, ...p-2$. Therefore

$$Dim(s=1) = 2\sum_{n=0}^{p-2} (p-n)(p-n-1) = \frac{2}{3}p(p-1)(p+1) \tag{42}$$

If $s \geq 2$, there exist only minimal $sp(2) \times sp(2)$ vectors with $(h_1 = 1+s+n, h_2 = 1+n)$, $(h_1 = 1+n, h_2 = 1+s+n)$ where $n = 0, 1, ...p-s$ and the minimal $sp(2) \times sp(2)$ vectors with $(h_1 = 1+s-k, h_2 = 1+k)$ where $k = 1, ...s-1$. Therefore, as follows from Eq. (8)

$$Dim(s \geq 2) = 2\sum_{n=0}^{p-s} (p-n)(p-n-s) + \sum_{k=1}^{s-1} (p-k)(p-s+k) =$$
$$\frac{p}{3}(2p^2 - 3s^2 + 1) + \frac{1}{2}s(s-1)(s+1) \tag{43}$$

As noted in Sec. 4, the case of special massive IRs corresponds to the situation where q_1 and q_2 are represented by the numbers $0, 1, ..p-1$ and $q_1 < q_2$.

This case can be investigated by analogy with massive IRs in Sec. 4. We will see below that the only special massive IRs taking part in the decomposition of the tensor product of singletons are those with $q_1 = 0$. Then $s = p - q_2$. If $q_2 = 2, 3, ...p - 1$ then, as follows from Eq. (31), the quantum number n can take only the value $n = 0$. If $q_2 = 1$ then the special massive IR can also be treated as the massless IR with $s = p - 1$. As noted above, in this case the quantity n also can take only the value $n = 0$. Let $Dim(q_1, q_2)$ be the dimension of the IR characterized by q_1 and q_2. Then, as follows from Eq. (8)

$$Dim(0, q_2) = \sum_{k=0}^{p-q_2}(1+p-q_2-k)(1+k) = (1+p-q_2)^2 + \frac{1}{2}(p-q_2)^2(1+p-q_2) \quad (44)$$

6 Semiclassical approximation and Poincare limit

As already noted, the Flato-Fronsdal result [8] poses a fundamental question of whether only singletons can be true elementary particles. In particular one has to understand why singletons have not been observed yet and why the photon is stable and its decay into singletons has not been observed. These questions have been widely discussed in the literature (see e.g. a review [18] and references therein) but, in our opinion, the explanation of the above facts in standard theory (based on complex numbers) encounters serious difficulties. In the present paper we consider singletons from the point of view of a quantum theory over a Galois field (GFQT) but the approach is applicable in standard theory (over the complex numbers) as well. As already noted in the preceding sections, the properties of singletons in standard theory and GFQT are considerably different. In Sect. 11 we argue that in GFQT the singleton physics is even more interesting than in standard theory. However, since there exists a wide literature on singleton properties in standard theory, in the present section we discuss what conclusions can be made about semiclassical approximation and Poincare limit for singletons in this theory.

The first step is to obtain expressions for matrix elements of representation operators. Since spin is a pure quantum phenomenon, one might expect that in semiclassical approximation it suffices to consider the spinless case. Then, as shown in Sec. 4, the quantum number k can take only the value $k = 0$, the basis vectors of the IR can be chosen as $e(n_1n_2n) = (a_1")^{n_1}(a_2")^{n_2}e_n$ (compare with Eq. (34)) where (see Eq. (6)) $e_n = (A^{++})^n e_0$. In the spinless case, $q_1 = q_2 = m/2$. A direct calculation using Eqs. (20,21,25,26) gives the following expressions for the matrix elements:

$$h_1e(n_1n_2n) = [Q + 2n_1]e(n_1n_2n) \quad h_2e(n_1n_2n) = [Q + 2n_2]e(n_1n_2n) \quad (45)$$

$$a_1'e(n_1n_2n) = n_1[Q + n_1 - 1]e(n_1 - 1, n_2n) \ a_1"e(n_1n_2n) = e(n_1 + 1, n_2n)$$
$$a_2'e(n_1n_2n) = n_2[Q + n_2 - 1]e(n_1, n_2 - 1, n) \ a_2"e(n_1n_2n) = e(n_1, n_2 + 1, n) \quad (46)$$

$$b"e(n_1n_2n) = \frac{Q-2}{Q-1}n(m+n-3)e(n_1+1,n_2+1,n-1) + \frac{1}{(Q-1)^2}e(n_1,n_2,n+1) \quad (47)$$

$$b'e(n_1n_2n) = \frac{Q-2}{Q-1}n(m+n-3)(Q+n_1-1)(Q+n_2-1)e(n_1,n_2,n-1) + \frac{n_1n_2}{(Q-1)^2}e(n_1-1,n_2-1,n+1) \quad (48)$$

$$L_+e(n_1n_2n) = \frac{Q-2}{Q-1}n(m+n-3)(Q+n_2-1)e(n_1+1,n_2,n-1) + \frac{n_2}{(Q-1)^2}e(n_1,n_2-1,n+1) \quad (49)$$

$$L_-e(n_1n_2n) = \frac{Q-2}{Q-1}n(m+n-3)(Q+n_1-1)e(n_1,n_2+1,n-1) + \frac{n_1}{(Q-1)^2}e(n_1-1,n_2,n+1) \quad (50)$$

where $Q = Q(n) = m/2 + n$.

The basis elements $e(n_1n_2n)$ are not normalized to one and their norm can be calculated by using Eqs.(9,20,21,25,26):

$$||e(n_1n_2n)|| = F(n_1n_2n) = \{n!(m-2)_n[(\frac{m}{2})_n]^3(\frac{m}{2}-1)_n n_1!n_2!(\frac{m}{2}+n)_{n_1}(\frac{m}{2}+n)_{n_2}\}^{1/2} \quad (51)$$

By using this expression, Eqs. (45-50) can be rewritten in terms of the matrix elements of representation operators with respect to the normalized basis $\tilde{e}(n_1n_2n) = e(n_1n_2n)/F(n_1n_2n)^{1/2}$.

Each element of the representation space can be written as

$$x = \sum_{n_1n_2n} c(n_1n_2n)\tilde{e}(n_1n_2n)$$

where $c(n_1n_2n)$ can be called the wave function in the (n_1n_2n) representation. It is normalized as

$$\sum_{n_1n_2n} |c(n_1n_2n)|^2 = 1$$

In standard theory the quantum numbers n_1 and n_2 are in the range $[0,\infty)$. For massive and massless particles the quantum number n also is in this range while, as shown in the preceding section, the only possible values of n for the spinless Rac singleton are $n = 0,1$. By using Eqs. (45-51), one can obtain the action of the representation operator on the wave function $c(n_1n_2n)$:

$$h_1c(n_1n_2n) = [m/2+n+2n_1]c(n_1n_2n) \quad h_2c(n_1n_2n) = [m/2+n+2n_2]c(n_1n_2n) \quad (52)$$

$$\begin{aligned} a_1'c(n_1n_2n) &= [(n_1+1)(m/2+n+n_1)]^{1/2}c(n_1+1,n_2n) \\ a_1"c(n_1n_2n) &= [n_1(m/2+n+n_1-1)]^{1/2}c(n_1-1,n_2n) \\ a_2'c(n_1n_2n) &= [(n_2+1)(m/2+n+n_2)]^{1/2}c(n_1,n_2+1,n) \\ a_2"c(n_1n_2n) &= [n_2(m/2+n+n_2-1)]^{1/2}c(n_1,n_2-1,n) \end{aligned} \quad (53)$$

$$b"c(n_1n_2n) = [\frac{n(m+n-3)(m/2+n+n_1-1)(m/2+n+n_2-1)}{(m/2+n-1)(m/2+n-2)}]^{1/2}c(n_1,n_2,n-1) +$$
$$[\frac{n_1n_2(n+1)(m+n-2)}{(m/2+n)(m/2+n-1)}]^{1/2}c(n_1-1,n_2-1,n+1) \quad (54)$$

$$b'c(n_1n_2n) = [\frac{(n+1)(m+n-2)(m/2+n+n_1)(m/2+n+n_2)}{(m/2+n)(m/2+n-1)}]^{1/2}c(n_1,n_2,n+1) +$$
$$[\frac{(n_1+1)(n_2+1)n(m+n-3)}{(m/2+n-1)(m/2+n-2)}]^{1/2}c(n_1+1,n_2+1,n-1) \quad (55)$$

$$L_+c(n_1n_2n) = [\frac{(n+1)(m+n-2)n_1(m/2+n+n_2)}{(m/2+n)(m/2+n-1)}]^{1/2}c(n_1-1,n_2,n+1) +$$
$$[\frac{(n_2+1)n(m+n-3)(m/2+n+n_1-1)}{(m/2+n-1)(m/2+n-2)}]^{1/2}c(n_1,n_2+1,n-1) \quad (56)$$

$$L_-c(n_1n_2n) = [\frac{n(m+n-3)(n_1+1)(m/2+n+n_2-1)}{(m/2+n-1)(m/2+n-2)}]^{1/2}c(n_1+1,n_2,n-1) +$$
$$[\frac{n_2(n+1)(m+n-2)(m/2+n+n_1)}{(m/2+n)(m/2+n-1)}]^{1/2}c(n_1,n_2-1,n+1) \quad (57)$$

Consider first the case of massive and massless particles. As noted in Sec. 1, the contraction to the Poincare invariant case can be performed as follows. If R is a parameter with the dimension *length* and the operators P_μ ($\mu = 0,1,2,3$) are defined as $P_\mu = M_{\mu 4}/2R$ then in the formal limit when $R \to \infty$, $M_{\mu 4} \to \infty$ but the ratio $M_{\mu 4}/R$ remains finite, one gets the commutation relations of the Poincare algebra from the commutation relations of the so(2,3) algebra. Therefore in situations where the Poincare limit is valid with a high accuracy, the operators $M_{\mu 4}$ are much greater than the other operators. The quantum numbers (m,n_1,n_2,n) should be very large since in the formal limit $R \to \infty$, $m/2R$ should become the standard Poincare mass and the quantities $(n_1/2R, n_2/2R, n/2R)$ should become continuous momentum variables.

A typical form of the semiclassical wave function is

$$c(n_1,n_2,n) = a(n_1,n_2,n)exp[i(n_1\varphi_1 + n_2\varphi_2 + n\varphi)]$$

where the amplitude $a(n_1,n_2,n)$ has a sharp maximum at semiclassical values of (n_1,n_2,n). Since the numbers (n_1,n_2,n) are very large, when some of them change by one, the major change of $c(n_1,n_2,n)$ comes from the rapidly oscillating exponent. As a consequence, in semiclassical approximation each representation operator becomes the operator of multiplication by a function and, as follows from Eqs. (22,52-57)

$$M_{04} = m + 2(n_1+n_2+n) \quad M_{12} = 2(n_1-n_2) \quad (58)$$

$$M_{10} = 2[n_1(m/2+n+n_1)]^{1/2}sin\varphi_1 - 2[n_2(m/2+n+n_2)]^{1/2}sin\varphi_2 \quad (59)$$

$$M_{20} = 2[n_1(m/2+n+n_1)]^{1/2}cos\varphi_1 + 2[n_2(m/2+n+n_2)]^{1/2}cos\varphi_2 \quad (60)$$

$$M_{14} = -2[n_1(m/2+n+n_1)]^{1/2}cos\varphi_1 + 2[n_2(m/2+n+n_2)]^{1/2}cos\varphi_2 \quad (61)$$

$$M_{24} = 2[n_1(m/2+n+n_1)]^{1/2}sin\varphi_1 + 2[n_2(m/2+n+n_2)]^{1/2}sin\varphi_2 \quad (62)$$

$$M_{23} = 2\frac{[n(m+n)]^{1/2}}{m/2+n}\{[n_1(m/2+n+n_2)]^{1/2}cos(\varphi-\varphi_1) + [n_2(m/2+n+n_1)]^{1/2}cos(\varphi-\varphi_2)\} \quad (63)$$

$$M_{31} = 2\frac{[n(m+n)]^{1/2}}{m/2+n}\{[n_1(m/2+n+n_2)]^{1/2}sin(\varphi-\varphi_1) - [n_2(m/2+n+n_1)]^{1/2}sin(\varphi-\varphi_2)\} \quad (64)$$

$$M_{34} = 2\frac{[n(m+n)]^{1/2}}{m/2+n}\{[(m/2+n+n_1)(m/2+n+n_2)]^{1/2}cos\varphi + (n_1n_2)^{1/2}cos(\varphi-\varphi_1-\varphi_2)\} \quad (65)$$

$$M_{30} = -2\frac{[n(m+n)]^{1/2}}{m/2+n}\{[(m/2+n+n_1)(m/2+n+n_2)]^{1/2}sin\varphi - (n_1n_2)^{1/2}sin(\varphi-\varphi_1-\varphi_2)\} \quad (66)$$

We now consider what restrictions follow from the fact that in the Poincare limit the operators $M_{\mu 4}$ ($\mu = 0,1,2,3$) should be much greater than the other operators. The first conclusion is that, as follows from Eq. (58), the quantum numbers n_1 and n_2 should be such that $|n_1 - n_2| \ll n_1, n_2$. Therefore in the main approximation in $1/R$ we have that $n_1 \approx n_2$. Then it follows from Eq. (66) that $sin\varphi$ should be of order $1/R$ and hence φ ahould be close either to zero or to π. Then it follows from Eqs. (63-62) that the operators $M_{\mu 4}$ will be indeed much greater than the other operators if $\varphi_2 \approx \pi - \varphi_1$ and in the main approximation in $1/R$

$$M_{04} = m + 2(2n_1+n) \quad M_{14} = -4[n_1(m/2+n+n_1)]^{1/2}cos\varphi_1$$
$$M_{14} = 4[n_1(m/2+n+n_1)]^{1/2}sin\varphi_1 \quad M_{34} = \pm 2[n(m+n)]^{1/2} \quad (67)$$

where M_{34} is positive if φ is close to zero and negative if φ is close to π. In this approximation we have that $M_{04}^2 - \sum_{i=1}^3 M_{i4}^2 = m^2$ which ensures that in the Poincare limit we have the correct relation between the energy and momentum.

Consider now the case of the spinless Rac singleton. Then $m = 1$ and the quantity n can take only the values 0 and 1. Since Eqs. (52-57) are exact, we can use them in the given case as well. However, since the quantum number n cannot be large, we now cannot consider the n dependence of the wave function in semiclassical approximation. At the same time, if the numbers (n_1, n_2) are very large, the dependence of the wave function on (n_1, n_2) still can be considered in this approximation assuming that the wave function contains the rapidly oscillating exponent $exp[i(n_1\varphi_1+n_2\varphi_2)]$. Hence Eqs. (58-62) remain valid but for calculating the operators M_{a3} ($a=0,1,2,4$) one can use the fact that Eqs. (54-57) can now be written as

$$b"c(n_1,n_2,n)=2(n_1n_2)^{1/2}\{c(n_1,n_2,0)\delta_{n1}+exp[-i(\varphi_1+\varphi_2)]c(n_1,n_2,1)\delta_{n0}\} \quad (68)$$

$$b'c(n_1,n_2,n)=2(n_1n_2)^{1/2}\{c(n_1,n_2,1)\delta_{n0}+exp[i(\varphi_1+\varphi_2)]c(n_1,n_2,0)\delta_{n1}\} \quad (69)$$

$$L_+c(n_1,n_2,n)=2(n_1n_2)^{1/2}\{exp(-i\varphi_1)c(n_1,n_2,1)\delta_{n0}+exp(i\varphi_2)c(n_1,n_2,0)\delta_{n1}\} \quad (70)$$

$$L_-c(n_1,n_2,n)=2(n_1n_2)^{1/2}\{exp(i\varphi_1)c(n_1,n_2,0)\delta_{n1}+exp(-i\varphi_2)c(n_1,n_2,1)\delta_{n0}\} \quad (71)$$

where δ is the Kronecker symbol. Then the mean values of these operators can be written as

$$<b">=A\{exp(i\varphi)+exp[-i(\varphi+\varphi_1+\varphi_2)]\} \quad <b'>=<b">^* \quad (72)$$

$$<L_+>=A\{exp[-i(\varphi+\varphi_1)]+exp[i(\varphi+\varphi_2)]\} \quad <L_->=<L_+>^* \quad (73)$$

where

$$\sum_{n_1,n_2}2(n_1n_2)^{1/2}c(n_1,n_2,1)^*c(n_1,n_2,0)=Aexp(i\varphi)$$

and we use * to denote the complex conjugation. By analogy with the above discussion, we conclude that the Poincare limit exists only if $\varphi_2\approx\pi-\varphi_1$ and φ is close either to zero or π. Then

$$M_{04}\approx 4n_1 \quad M_{14}\approx -4n_1cos(\varphi_1) \quad M_{24}\approx 4n_1sin(\varphi_1) \quad (74)$$

and the mean value of the operator M_{34} is much less than M_{14} and M_{24}.

Consider now the case of the Di singleton. It is characterized by $q_1=3/2$, $q_2=1/2$. Then, as shown in the preceding sections, $s=1$, the quantum number n can take only the value $n=0$ and the quantum number k can take only the values $k=0,1$. We denote $e_0=e(n=0,k=0)$ and $e_1=e(n=0,k=1)$. Then, as shown in the preceding section, $e_1=L_-e_0$ and the basis of the IR in standard theory consists of elements $e_0(n_1,n_2)=(a_1")^{n_1}(a_2")^{n_2}e_0$ and $e_1(n_1,n_2)=(a_1")^{n_1}(a_2")^{n_2}e_1$ ($n_1,n_2=0,1,...\infty$).

As explained in the preceding section, $e(n=1,k=0)$ should be defined as $[b"(h_1-1)-a_1"L_-]e_0$ and $e(n=1,k=1)$ should be defined as $[b"(h_2-1)-a_2"L_+]e_1$. Since in the case of the Di singleton $e(n=1,k=0)=e(n=1,k=1)=0$, it follows from Eq. (13) that

$$L_+e_0=L_-e_1=0 \quad L_-e_0=e_1 \quad L_+e_1=e_0 \quad b"e_0=a_1"e_1 \quad b"e_1=a_2"e_0 \quad (75)$$

Now it follows from Eq. (21) that

$$\begin{aligned}
&b"e_0(n_1,n_2)=e_1(n_1+1,n_2) \quad b"e_1(n_1,n_2)=e_0(n_1,n_2+1)\\
&b'e_0(n_1,n_2)=(n_1+1)n_2e_1(n_1,n_2-1) \quad b'e_1(n_1,n_2)=n_1(n_2+1)e_0(n_1-1,n_2)\\
&L_+e_0(n_1,n_2)=n_2e_1(n_1+1,n_2-1) \quad L_+e_1(n_1,n_2)=(n_2+1)e_0(n_1,n_2)\\
&L_-e_0(n_1,n_2)=(n_1+1)e_1(n_1,n_2) \quad L_-e_1(n_1,n_2)=n_1e_0(n_1-1,n_2+1)
\end{aligned} \quad (76)$$

As follows from Eqs. (9) and (13)

$$||e_0(n_1,n_2)|| = (n_1+1)!n_2!(n_2+1)^{1/2} \quad ||e_1(n_1,n_2)|| = n_1!(n_2+1)!(n_1+1)^{1/2} \quad (77)$$

Hence one can define the normalized basis elements $\tilde{e}_j(n_1,n_2)$ $(j=0,1)$ and any element in the representation space can be written as $x=\sum_{j=0}^{1} c_j(n_1,n_2)\tilde{e}_j(n_1,n_2)$. By analogy with the above discussion, one can show that a necessary condition for the Poincare limit in semiclassical approximation is that the quantities (n_1,n_2) are very large, $n_1 \approx n_2$, the functions $c_j(n_1,n_2)$ contain a rapidly oscillating exponents $exp[i(n_1\varphi_1+n_2\varphi_2)]$ and $\varphi_2 \approx \pi-\varphi_1$. In this approximation one can obtain the results (58-62) while calculating the operators M_{a3} $(a=0,1,2,4)$ can be performed as follows.

One can represent the wave function as $(c_0(n_1,n_2),c_1(n_1,n_2))$ and then, as follows from Eqs. (76) and (77)

$$\begin{aligned} b"(c_0(n_1,n_2),c_1(n_1,n_2)) &\approx n_1(exp(-i\varphi_1)c_1(n_1,n_2),exp(-i\varphi_2)c_0(n_1,n_2)) \\ b'(c_0(n_1,n_2),c_1(n_1,n_2)) &\approx n_1(exp(i\varphi_2)c_1(n_1,n_2),exp(i\varphi_1)c_0(n_1,n_2)) \\ L_+(c_0(n_1,n_2),c_1(n_1,n_2)) &\approx n_1(exp[-i(\varphi_1-\varphi_2)]c_1(n_1,n_2),c_0(n_1,n_2)) \\ L_-(c_0(n_1,n_2),c_1(n_1,n_2)) &\approx n_1(c_1(n_1,n_2),exp[i(\varphi_1-\varphi_2)]c_0(n_1,n_2)) \end{aligned} \quad (78)$$

Now it follows from Eqs. (22) and (78) that the mean values of the operators M_{a3} are given by

$$\begin{aligned} <M_{34}> &\approx 2A[cos(\varphi-\varphi_1)+cos(\varphi+\varphi_2)] \\ <M_{30}> &\approx 2A[sin(\varphi-\varphi_1)-sin(\varphi+\varphi_2)] \\ <M_{23}> &\approx 2A[cos(\varphi-\varphi_1+\varphi_2)+cos\varphi] \\ <M_{31}> &\approx 2A[sin(\varphi-\varphi_1+\varphi_2)-sin\varphi] \end{aligned} \quad (79)$$

where

$$\sum_{n_1 n_2} n_1 c_1(n_1,n_2)^* c_0(n_1,n_2) = Aexp(i\varphi)$$

If $\varphi_2 \approx \pi-\varphi_1$ then it is easy to see that the Poincare limit for $<M_{23}>$ and $<M_{31}>$ exists if $\varphi \approx \varphi_1$ or $\varphi \approx \varphi_1+\pi$. In that case the Poincare limit for $<M_{34}>$ and $<M_{30}>$ exists as well and $<M_{34}>$ disappears in the main approximation.

We have shown that if the operators M_{ab} are defined by Eq. (22) then in the Poincare limit the z component of the momentum is negligible for both, the Di and Rac singletons. This result could be expected from Eq. (67) since for them neither m nor n can be large numbers. As noted in the remark after Eq. (22), the definition (22) is not unique and, in particular, any definition obtained from Eq. (22) by cyclic permutation of the indices $(1,2,3)$ is valid as well. Therefore we conclude that in standard theory, the Di and Rac singletons have the property that in the Poincare limit they are characterized by two independent components of the momentum, not three as usual particles. This is a consequence of the fact that for singletons only the quantum numbers n_1 and n_2 can be very large.

The properties of singletons in the Poincare limit have been discussed by several authors, and their conclusions are not in agreement with each other (a detailed list of references can be found e.g. in Refs. [18, 19]). In particular, there are statements that the Poincare limit for singletons does not exist or that in this limit all the components of the four-momentum become zero. The above consideration shows that the Poincare limit for singletons can be investigated in full analogy with the Poincare limit for usual particles. In particular, the statement that the singleton energy in the Poincare limit becomes zero is not in agreement with the fact that each massless particle (for which the energy in the Poincare limit is not zero) can be represented as a composite state of two singletons. The fact that the standard singleton momentum can have only two independent components does not contradict the fact that the momentum of a massless particle has three independent components since, as noted above, the independent momentum components of two singletons can be in different planes.

The following important remark is now in order. Our consideration proceeds only from the commutation relations (1). As discussed in Refs. [3, 4] and Sec. 1, these relations should be treated as a *definition of the AdS symmetry on quantum level.* This definition is a must in any quantum AdS theory, regardless of whether the theory involves local fields on the AdS space or does not involve the AdS space at all. The goal of any quantum theory is to construct Hilbert spaces and operators for a system under consideration and, in the case of IRs, such spaces and operators have been described in Secs. 4 and 5. This is the maximal possible information for describing a particle in quantum theory. In particular, this information does not involve the AdS space. One might pose a question about the space-time description of a particle in this approach. According to quantum theory, any physical quantity can be discussed only in conjunction with the operator defining this quantity. In the given case a problem arises of how one can construct position operators from the operators M_{ab}.

A well-known analogy is nonrelativistic quantum mechanics. Here a particle is described by an IR of the Galilei algebra, which can be implement in the Hilbert space of functions $\psi(\mathbf{p})$ depending on the momentum $\mathbf{p}$, and the momentum operator $\mathbf{P}$ is the operator of multiplication by $\mathbf{p}$. Then the position operator $\mathbf{r}$ can be *defined* as $i\hbar\partial/\partial\mathbf{p}$. At the same time, in quantum theory there is no operator corresponding to time and the problem of how time should be understood on quantum level is discussed in a vast literature. It is also well known since the 1930s that when quantum theory is combined with relativity, there is no operator having all the properties of the position operator (see e.g. Ref. [20]). As a consequence, in relativistic and de Sitter theories, a space-time description has a physical meaning only in semiclassical approximation. In particular, local fields defined on Minkowski, dS or AdS spaces do not have the probabilistic interpretation. Those fields are only auxiliary notions for constructing operators describing a system of *interacting* particles in QFT.

7 Tensor products of singleton IRs

We now return to the presentation when the properties of singletons in standard and modular approaches are discussed in parallel. The tensor products of singleton IRs can be defined by analogy with tensor products of sp(2) IRs discussed in Sec. 3. If $e^{(j)}(n_1^{(j)},n_2^{(j)},n^{(j)},k^{(j)})$ $(j=1,2)$ are the basis elements of the IR for singleton j then the basis elements in the representation space of the tensor product can be chosen as

$$e(n_1^{(1)},n_2^{(1)},n^{(1)},k^{(1)},n_1^{(2)},n_2^{(2)},n^{(2)},k^{(2)}) = e^{(1)}(n_1^{(1)},n_2^{(1)},n^{(1)},k^{(1)})\times e^{(2)}(n_1^{(2)},n_2^{(2)},n^{(2)},k^{(2)}) \quad (80)$$

Each representation operator B is a sum of the corresponding operators for the singleton IRs, $B=B^{(1)}+B^{(2)}$ such that the operator $B^{(j)}$ acts on the vectors $e^{(j)}$ as in the IRs for singleton j while on the vectors $e^{(j')}$ $(j'\neq j)$ it acts as the identity operator. In the case of the tensor product of singleton IRs of different types, we assume that singleton 1 is Di and singleton 2 is Rac.

Consider a vector

$$e(q)=\sum_{i=0}^{q} c(i,q)e^{(1)}(i,0,0,0)\times e^{(2)}(q-i,0,0,0) \quad (81)$$

where the coefficients $c(i,q)$ are given by Eq. (18) such that the $q_0^{(j)}$ should be replaced by $q_1^{(j)}$ $(j=1,2)$. Since $h_2^{(j)}e^{(j)}(i,0,0,0)=((p+1)/2)e^{(j)}(i,0,0,0)$ $(j=1,2)$ then the vector $e(q)$ is the eigenvector of the operator $h_2=h_2^{(1)}+h_2^{(2)}$ with the eigenvalue $q_2=1$ and satisfies the condition $a_2'e(q)=0$ where $a_2'=a_2^{(1)'}+a_2^{(2)'}$. As follows from the results of Sec. 3, $e(q)$ is the eigenvector of the operator $h_1=h_1^{(1)}+h_1^{(2)}$ with the eigenvalue $q_1=q_1^{(1)}+q_1^{(2)}+2q$ and satisfies the condition $a_1'e(q)=0$ where $a_1'=a_1^{(1)'}+a_1^{(2)'}$. It is obvious that the value of q_1 equals $3+2q$ for the tensor product $Di\times Di$, $2+2q$ for the tensor product $Di\times Rac$ and $1+2q$ for the tensor product $Rac\times Rac$.

As follows from Eqs. (21) and (34), in the case of IRs

$$b'e(n_1n_2nk)=[(a_1")^{n_1}(a_2")^{n_2}b'+n_1(a_1")^{n_1-1}(a_2")^{n_2}L_+ + n_2(a_1")^{n_1}(a_2")^{n_2-1}L_- + n_1n_2(a_1")^{n_1-1}(a_2")^{n_2-1}b"]e(0,0,n,k)$$
$$L_+e(n_1n_2nk)=[(a_1")^{n_1}(a_2")^{n_2}L_+ + n_2(a_1")^{n_1}(a_2")^{n_2-1}b"]e(0,0,n,k) \quad (82)$$

Therefore, $e(q)$ satisfies the conditions $b'e(q)=L_+e(q)=0$ where $b'=b^{(1)'}+b^{(2)'}$ and $L_+=L_+^{(1)}+L_+^{(2)}$. Hence, $e(q)$ is an analog of the vector e_0 in Eq. (26) and generates an IR corresponding to the quantum numbers $(q_1,q_2=1)$.

We conclude that the tensor product of singleton IRs contains massless IRs corresponding to $q_1 = q_1^{(1)} + q_1^{(2)} + 2q$. As follows from the results of Sect. 3 (see the remark after Eq. (18)), q can take the values $0, 1, ..., p - q_1^{(1)}$. Therefore $Rac \times Rac$ contains massless IRs with $s = 0, 2, 4, ..., (p-1)$, $Di \times Rac$ contains massless IRs with $s = 1, 3, 5, ...(p-2)$ and $Di \times Di$ contains massless IRs with $s = 2, 4, ...(p-1)$. In addition, as noted in Ref. [8]), $Di \times Di$ contains a spinless massive IR corresponding to $q_1 = q_2 = 2$. This question will be discussed in Sec. 10

Our next goal is to investigate whether or not all those IRs give a complete decomposition of the corresponding tensor products. For example, as follows from Eq. (39), for the product $Rac \times Rac$ this would be the case if the sum $\sum_{k=0}^{(p-1)/2} Dim(2k)$ equals $(p^2+1)^2/4 = p^4/4 + O(p^2)$. However, as follows from Eqs. (41) and (43), this sum can be easily estimated as $11p^4/48 + O(p^3)$ and hence, in contrast to the Flato-Fronsdal result in standard theory, in the modular case the decomposition of $Rac \times Rac$ contains not only massles IRs. Analogously, the sum of dimensions of massless IRs entering into the decompositions of $Di \times Rac$ and $Di \times Di$ also can be easily estimated as $11p^4/48 + O(p^3)$ what is less than $p^4/4 + O(p^2)$. The reason is that in the modular case the decompositions of the tensor products of singletons contain not only massles IRs but also special massive IRs. We will not investigate the modular analog of the Flato-Fronsdal theorem [8] but concentrate our efforts on finding a full solution of the problem in the supersymmetric case.

8 Modular IRs of the osp(1,4) superalgebra

If one accepts supersymmetry then the results on modular IRs of the so(2,3) algebra can be generalized by considering modular IRs of the osp(1,4) superalgebra. Representations of the osp(1,4) superalgebra have several interesting distinctions from representations of the Poincare superalgebra. For this reason we first briefly mention some well known facts about the latter representations (see e.g Ref. [21] for details).

Representations of the Poincare superalgebra are described by 14 operators. Ten of them are the well known representation operators of the Poincare algebra—four momentum operators and six representation operators of the Lorentz algebra, which satisfy the well known commutation relations. In addition, there also exist four fermionic operators. The anticommutators of the fermionic operators are linear combinations of the momentum operators, and the commutators of the fermionic operators with the Lorentz algebra operators are linear combinations of the fermionic operators. In addition, the fermionic operators commute with the momentum operators.

From the formal point of view, representations of the osp(1,4) superalgebra are also described by 14 operators — ten representation operators of the so(2,3) algebra and four fermionic operators. There are three types of relations: the operators of the so(2,3) algebra commute with each other as usual (see Sec. 4), anticommutators of the fermionic operators are linear combinations of the so(2,3) operators and commutators of the latter with the fermionic operators are their linear combinations. However, in fact representations of the osp(1,4) superalgebra can be described exclusively in terms of the fermionic operators. The matter is as follows. In the general case the anticommutators of four operators form ten independent linear combinations. Therefore, ten bosonic operators can be expressed in terms of fermionic ones. This is not the case for the Poincare superalgebra since the Poincare algebra operators are obtained from the so(2,3) one by contraction. One can say that the representations of the osp(1,4) superalgebra is an implementation of the idea that supersymmetry is the extraction of the square root from the usual symmetry (by analogy with the well known treatment of the Dirac equation as a square root from the Klein-Gordon one).

We use $(d'_1, d'_2, d_1", d_2")$ to denote the fermionic operators of the osp(1,4) superalgebra. They should satisfy the following relations. If (A, B, C) are any fermionic operators, [...,...] is used to denote a commutator and $\{..., ...\}$ to denote an anticommutator then

$$[A, \{B, C\}] = F(A, B)C + F(A, C)B \tag{83}$$

where the form $F(A, B)$ is skew symmetric, $F(d'_j, d_j") = 1$ $(j = 1, 2)$ and the other independent values of $F(A, B)$ are equal to zero. The fact that the representation of the osp(1,4) superalgebra is fully defined by Eq. (83) and the properties of the form $F(.,.)$, shows that osp(1,4) is a special case of the superalgebra.

We can now **define** the so(2,3) generators as follows:

$$b' = \{d'_1, d'_2\} \quad b" = \{d_1", d_2"\} \quad L_+ = \{d'_2, d_1"\} \quad L_- = \{d'_1, d_2"\}$$
$$a'_j = (d'_j)^2 \quad a_j" = (d_j")^2 \quad h_j = \{d'_j, d_j"\} \quad (j = 1, 2) \tag{84}$$

Then by using Eq. (83) and the properties of the form $F(.,.)$, one can show by direct calculations that so defined operators satisfy the commutation relations (10,20,21). This result can be treated as a fact that the operators of the so(2,3) algebra are not fundamental, only the fermionic operators are.

By analogy with the construction of IRs of the osp(1,4) superalgebra in standard theory [11], we require the existence of the generating vector e_0 satisfying the conditions :

$$d'_j e_0 = d'_2 d_1" e_0 = 0 \quad d'_j d_j" e_0 = q_j e_0 \quad (j = 1, 2) \tag{85}$$

These conditions are written exclusively in terms of the d operators. As follows from Eq. (84), they can be rewritten as (compare with Eq. (26))

$$d'_j e_0 = L_+ e_0 = 0 \quad h_j e_0 = q_j e_0 \quad (j = 1, 2) \tag{86}$$

The full representation space can be obtained by successively acting by the fermionic operators on e_0 and taking all possible linear combinations of such vectors.

We use E to denote an arbitrary linear combination of the vectors $(e_0, d_1"e_0, d_2"e_0, d_2"d_1"e_0)$. Our next goal is to prove a statement analogous to that in Ref. [11]:

Statement 1: Any vector from the representation space can be represented as a linear combination of the elements $O_1O_2...O_nE$ where $n = 0, 1, ...$ and O_i is an operator of the so(2,3) algebra.

The first step is to prove a simple

Lemma: If D is any fermionic operator then DE is a linear combination of elements E and OE where O is an operator of the so(2,3) algebra.

The proof is by a straightforward check using Eqs. (83-86). For example,

$$d_1"(d_2"d_1"e_0) = \{d_1", d_2"\}d_1"e_0 - d_2"a_1"e_0 = b"d_1"e_0 - a_1"d_2"e_0$$

To prove Statement 1 we define the height of a linear combination of the elements $O_1O_2...O_nE$ as the maximum sum of powers of the fermionic operator in this element. For example, since each operator of the so(2,3) algebra is composed of two fermionic operator, the height of the element $O_1O_2...O_nE$ equals $2n + 2$ if E contains $d_2"d_1"e_0$, equals $2n + 1$ if E does not contain $d_2"d_1"e_0$ but contains either $d_1"e_0$ or $d_2"e_0$ and equals $2n$ if E contains only e_0.

We can now prove Statement 1 by induction. The elements with the heights 0, 1 and 2 obviously have the required form since, as follows from Eq. (84), $d_1"d_2"e_0 = b"e_0 - d_2"d_1"e_0$. Let us assume that Statement 1 is correct for all elements with the heights $\leq N$. Every element with the height $N + 1$ can be represented as Dx where x is an element with the height N. If $x = O_1O_2...O_nE$ then by using Eq. (83) we can represent Dx as $Dx = O_1O_2...O_nDE + y$ where the height of the element y is $N - 1$. As follows from the induction assumption, y has the required form, and, as follows from Lemma, DE is a linear combination of the elements E and OE. Therefore Statement 1 is proved.

As follows from Eqs. (83) and (84),

$$[d_j', h_j] = d_j' \quad [d_j", h_j] = -d_j" \quad [d_j', h_l] = [d_j", h_l] = 0 \quad (j, l = 1, 2\ j \neq l) \qquad (87)$$

It follows from these expressions that if x is such that $h_jx = \alpha_jx$ $(j = 1, 2)$ then $d_1"x$ is the eigenvector of the operators h_j with the eigenvalues $(\alpha_1 + 1, \alpha_2)$, $d_2"x$ - with the eigenvalues $(\alpha_1, \alpha_2 + 1)$, $d_1'x$ - with the eigenvalues $(\alpha_1 - 1, \alpha_2)$, and $d_2'x$ - with the eigenvalues $\alpha_1, \alpha_2 - 1$.

By analogy with the case of IRs of the so(2,3) algebra (see Sec. 4), we assume that q_1 and q_2 are represented by the numbers $0, 1, ...p - 1$. We first consider the case when $q_2 \geq 1$ and $q_1 \geq q_2$. We again use m to denote $q_1 + q_2$ and s to denote $q_1 - q_2$. We first assume that $m \neq 2$ and $s \neq p - 1$. Then Statement 1 obviously remains valid if we now assume that E contains linear combinations of (e_0, e_1, e_2, e_3) where

$$e_1 = d_1"e_0 \quad e_2 = [d_2" - \frac{1}{s+1}L_-d_1"]e_0$$

$$e_3 = (d_2"d_1"e_0 - \frac{q_1-1}{m-2}b" + \frac{1}{m-2}a_1"L_-)e_0 \tag{88}$$

As follows from Eqs. (83-87), e_0 satisfies Eq. (26) and e_1 satisfies the same condition with q_1 replaced by q_1+1. We see that the representation of the osp(1,4) superalgebra defined by Eq. (86) necessarily contains at least two IRs of the so(2,3) algebra characterized by the values of the mass and spin (m,s) and $(m+1,s+1)$ and the generating vectors e_0 and e_1, respectively.

As follows from Eqs. (83-87), the vectors e_2 and e_3 satisfy the conditions

$$h_1e_2 = q_1e_2 \quad h_2e_2 = (q_2+1)e_2 \quad h_1e_3 = (q_1+1)e_3 \quad h_2e_3 = (q_2+1)e_3$$
$$a_1'e_j = a_2'e_j = b'e_j = L_+e_j = 0 \quad (j=2,3) \tag{89}$$

and therefore (see Eq. (26)) they will be generating vectors of IRs of the so(2,3) algebra if they are not equal to zero.

If $s=0$ then, as follows from Eqs. (83,84,88), $e_2=0$. In the general case, as follows from these expressions,

$$d_1'e_2 = \frac{1-q_2}{s+1}L_-e_0 \quad d_2'e_2 = \frac{s(q_2-1)}{s+1}e_0 \tag{90}$$

Therefore e_2 is also a null vector if e_0 belongs to the massless IR (with $q_2=1$) while $e_2 \neq 0$ if $s \neq 0$ and $q_2 \neq 1$. As follows from direct calculation using Eqs. (83,84,88)

$$d_1'e_3 = \frac{m-1}{m-2}[L_-d_1" - (2q_2+s-1)d_2"]e_0 \quad d_2'e_3 = (q_2 - \frac{q_1-1}{m-2})e_0 \tag{91}$$

If $q_2=1$ then $d_1'e_3$ is proportional to e_2 (see Eq. (88)) and hence $d_1'e_3=0$. In this case $q_1-1=m-2$ and hence $d_2'e_3=0$. Therefore we conclude that $e_3=0$. It is also clear from Eq. (91) that $e_3=0$ if $m=1$, $s=0$, i.e. when e_0 is the generating vector of the Rac singleton. In all other cases $e_3 \neq 0$.

Consider now the case $m=2$. If $s=0$ then $q_1=q_2=1$. The condition $e_2=0$ is still valid for the same reasons as above but if e_3 is defined as $[d_2", d_1"]e_0/2$ then e_3 is the minimal $sp(2)\times sp(2)$ vector with $h_1=h_2=2$ and, as a result of direct calculations using Eqs. (83,84,88)

$$d_1'e_3 = \frac{1}{2}(1-2q_1)d_2"e_0 \quad d_2'e_3 = \frac{1}{2}(2q_2-1)e_0 \tag{92}$$

Hence in this case $e_3 \neq 0$ and the IR of the osp(1,4) superalgebra corresponding to $(q_1,q_2)=(1,1)$ contains IRs of the so(2,3) algebra corresponding to $(1,1)$, $(2,1)$ and $(2,2)$. Therefore this IR of the osp(1,4) superalgebra should be treated as massive rather than massless.

At this point the condition that q_1 and q_2 are taken modulo p has not been explicitly used and, as already mentioned, our considerations are similar to those in Ref. [11]. Therefore when $q_1 \geq q_2$, modular IRs of the osp(1,4) superalgebra can be characterized in the same way as conventional IRs [11, 10]:

- If $q_2 > 1$ and $s \neq 0$ (massive IRs), the osp(1,4) supermultiplets contain four IRs of the so(2,3) algebra characterized by the values of the mass and spin $(m,s), (m+1,s+1), (m+1,s-1), (m+2,s)$.

- If $q_2 \geq 1$ and $s = 0$ (collapsed massive IRs), the osp(1,4) supermultiplets contain three IRs of the so(2,3) algebra characterized by the values of the mass and spin $(m,s), (m+1,s+1), (m+2,s)$.

- If $q_2 = 1$ and $s = 1,2,...p-2$ (massless IRs) the osp(1,4) supermultiplets contains two IRs of the so(2,3) algebra characterized by the values of the mass and spin $(2+s,s), (3+s,s+1)$.

- Dirac supermultiplet containing two Dirac singletons (see Section 5).

The first three cases have well known analogs of IRs of the super-Poincare algebra (see e.g., Ref. [21]) while there is no super-Poincare analog of the Dirac supermultiplet.

Since the space of IR of the superalgebra osp(1,4) is a direct sum of spaces of IRs of the so(2,3) algebra, for modular IRs of the osp(1,4) superalgebra one can prove results analogous to those discussed in the preceding sections. In particular, one modular IR of the osp(1,4) algebra is a modular analog of both standard IRs of the osp(1,4) superalgebra with positive and negative energies. This implies that one modular IR of the osp(1,4) superalgebra contains both, a superparticle and its anti-superparticle.

At the same time, the cases when $q_1 < q_2$ (special massive IRs) have no analogs in standard theory. We will see below that the decomposition of the supersingleton tensor product can contain only special massive IRs of the osp(1,4) superalgebra with $q_1 = 0$. In this case we have that $d_1'd_1"e_0 = q_1e_0 = 0$, $d_2'd_1"e_0 = L_+e_0 = 0$ and hence $d_1"e_0 = 0$. Since $L_+d_2"e_0 = d_1"e_0 = 0$ and $d_2'd_2"e_0 = q_2e_0$, the vector $d_2"e_0$ is not zero and if e_0 is the generating vector for the IR of the so(2,3) algebra with $(q_1 = 0, q_2)$ then $d_2"e_0$ is the generating vector for the IR of the so(2,3) algebra with $(0, q_2+1)$. The IR of the osp(1,4) superalgebra does not contain other IRs of the so(2,3) algebra since $d_2"d_1"e_0 = 0$ and $d_1"d_2"e_0 = (d_1"d_2" + d_2"d_1")e_0 = b"e_0$.

By analogy with Sect. 5, we use $SDim(s)$ to denote the dimension of the IR of the osp(1,4) superalgebra in the massless case with the spin s and $SDim(q_1,q_2)$ to denote the dimension of the IR of the osp(1,4) superalgebra characterized by the quantities q_1 and q_2. Then as follows from the above discussion

$$
\begin{aligned}
&SDim(0,q_2) = Dim(0,q_2) + Dim(0,q_2+1) \quad (q_2 = 1,2,...p-1)\\
&SDim(s) = Dim(s) + Dim(s+1) \quad (s = 1,2,...p-2)\\
&SDim(1,1) = Dim(1,1) + Dim(2,1) + Dim(2,2)
\end{aligned}
\tag{93}
$$

and $Dim(p-1) = Dim(0,1)$.

9 Supersingleton IR

In this section we consider the supersingleton IR exclusively in terms of the fermionic operators without decomposing the IR into the Di and Rac IRs. As a preparatory step, we first consider IRs of a simple superalgebra generated by two fermionic operators $(d', d")$ and one bosonic operator h such that

$$h = \{d', d"\} \quad [h, d'] = -d' \quad [h, d"] = d" \tag{94}$$

Here the first expression shows that, by analogy with the osp(1,4) superalgebra, the relations (94) can be formulated only in terms of the fermionic operators.

Consider an IR of the algebra (94) generated by a vector e_0 such that

$$d'e_0 = 0 \quad d'd"e_0 = q_0 e_0 \tag{95}$$

and define $e_n = (d")^n e_0$. Then $d'e_n = a(n)e_{n-1}$ where, as follows from Eq. (95), $a(0) = 0$, $a(1) = q_0$ and $a(n) = q_0 + n - 1 - a(n-1)$. It is easy to prove by induction that

$$a(n) = \frac{1}{2}\{(q_0 - \frac{1}{2})[1 - (-1)^n] + n\} \tag{96}$$

The maximum possible value of n can be found from the condition that $a(n_{max}) \neq 0$, $a(n_{max} + 1) = 0$. In the special case of the supersingleton, we will be interested in the case when $q_0 = (p + 1)2$. Then, as follows from Eq. (96), $a(n) = n/2$. Therefore $n_{max} = p - 1$ and the dimension of the IR is p. In the general case, if $q_0 \neq 0$ then $a(n) = 0$ if $n = 2p + 1 - 2q_0$ and the dimension of the IR is $D(q_0) = 2p + 1 - 2q_0$.

Consider now the supersingleton IR. Let $x = (d_1^"d_2^" - d_2^"d_1^")e_0$. Then, as follows from Eq. (83), $d_1'x = (2q_1 - 1)d_2^"e_0$ and $d_2'x = (1 - 2q_2)d_1^"e_0$. Since $q_1 = q_2 = (p + 1)/2$ we have that $d_1'x = d_2'x = 0$ and therefore $x = 0$. Hence the actions of the operators $d_1^"$ and $d_2^"$ on e_0 commute with each other. If n is even then $d_1^"(d_2^")^n e_0 = (d_2^")^n d_1^" e_0$ as a consequence of Eq. (83) and if n is odd then $d_1^"(d_2^")^n e_0 = (d_2^")^{n-1}d_1^"d_2^"e_0 = (d_2^")^n d_1^" e_0$ in view of the fact that $x = 0$. Analogously one can prove that $d_2^"(d_1^")^n e_0 = (d_1^")^n d_2^" e_0$. We now can prove that $d_1^"(d_2^")^n(d_1^")^k e_0 = (d_2^")^n(d_1^")^{k+1}e_0$. Indeed, if n is even, this is obvious while if n is odd then

$$d_1^"(d_2^")^n(d_1^")^k e_0 = (d_2^")^{n-1}d_1^"d_2^"(d_1^")^k e_0 = (d_2^")^{n-1}(d_1^")^{k+1}d_2^"e_0 = (d_2^")^n(d_1^")^{k+1}e_0$$

and analogously $d_2^"(d_1^")^n(d_2^")^k e_0 = (d_1^")^n(d_2^")^{k+1}e_0$. Therefore the supersingleton IR is distinguished among other IRs of the osp(1,4) superalgebra by the fact that the operators $d_1^"$ and $d_2^"$ commute in the representation space of this IR. Hence the basis of the representation space can be chosen in the form $e(nk) = (d_1^")^n(d_2^")^k e_0$. As a consequence of the above consideration, $n, k = 0, 1, ...p - 1$ and the dimension of the IR is p^2 in agreement with Eq. (39).

10 Tensor product of supersingleton IRs

We first consider the tensor product of IRs of the superalgebra (94) with $q_0 = (p+1)/2$. It can be defined as follows. The representation space of the tensor product consists of all linear combinations of elements $x^{(1)} \times x^{(2)}$ where $x^{(j)}$ is an element of the representation space for the IR j ($j = 1, 2$). The representation operators of the tensor product are linear combinations of the operators $(d', d")$ where $d' = d^{(1)'} + d^{(2)'}$ and $d" = d^{(1)"} + d^{(2)"}$. Here $d^{(j)'}$ and $d^{(j)"}$ mean the operators acting in the representation spaces of IRs 1 and 2, respectively. We also assume that if $d^{(j)}$ is some of the d-operators for the IR j then $\{d^{(1)}, d^{(2)}\} = 0$.

Let $e_0^{(j)}$ be the generating vector for IR j and $e_i^{(j)} = (d^{(j)"})^i e_0^{(j)}$. Consider the following element of the representation space of the tensor product

$$e(k) = \sum_{i=0}^{k} c(i)(e_i^{(1)} \times e_{k-i}^{(2)}) \tag{97}$$

where $c(i)$ is some function. This element will be the generating vector of the IR of the superalgebra (94) if $d'e(q) = 0$. As follows from the above results and Eq. (97)

$$d'e(q) = \frac{1}{2}\sum_{i=1}^{k} ic(i)(e_{i-1}^{(1)} \times e_{k-i}^{(2)}) + \frac{1}{2}\sum_{i=0}^{k-1}(-1)^i(k-i)c(i)(e_i^{(1)} \times e_{k-i-1}^{(2)}) \tag{98}$$

Therefore $d'e(k) = 0$ is satisfied if $k = 0$ or

$$c(i+1) = (-1)^{i+1}\frac{k-i}{i+1}c(i) \quad i = 0, 1, ...k-1 \tag{99}$$

when $k \neq 0$. As follows from this expression, if $c(0) = 1$ then

$$c(i) = (-1)^{\frac{i(i+1)}{2}} C_k^i \tag{100}$$

where $C_k^i = k!/i!(k-i)!$ is the binomial coefficient. As follows from Eq. (96), the possible values of k are $0, 1, ...p-1$ and, as follows from Eq. (97), $he(k) = q_0 e(k)$ where $q_0 = 1 + k$. The fact that the tensor product is fully decomposable into IRs with the different values of k follows from the relation $\sum_{q_0=1}^{p} D(q_0) = p^2$.

The tensor product of the supersingleton IRs can be defined as follows. The representation space of the tensor product consists of all linear combinations of elements $x^{(1)} \times x^{(2)}$ where $x^{(j)}$ is an element of the representation space for the supersingleton j ($j = 1, 2$). The fermionic operators of the representation are linear combinations of the operators $(d_1', d_2', d_1", d_2")$ where $d_1' = d_1^{(1)'} + d_1^{(2)'}$ and analogously for the other operators. Here $d_k^{(j)'}$ and $d_k^{(j)"}$ ($k = 1, 2$) mean the operators d_k' and $d_k"$ acting in the representation spaces of supersingletons 1 and 2, respectively. We also assume that if $d^{(j)}$ is some of the d-operators for supersingleton j then $\{d^{(1)}, d^{(2)}\} = 0$. The action of the bosonic operators in the tensor product can be defined by Eq. (84).

Let $e_0^{(j)}$ be the generating vector for supersingleton j (see Eq. (86)) and $e_0 = e_0^{(1)} \times e_0^{(2)}$. Consider the following element of the representation space of the tensor product:

$$x(k_1,k_2) = \sum_{i=0}^{k_1}\sum_{j=0}^{k_2}(-1)^{[\frac{i(i+1)}{2}+\frac{j(j+1)}{2}+k_1 j]}C_{k_1}^i C_{k_2}^j$$
$$(d_1^{(1)"})^i (d_1^{(2)"})^{k_1-i}(d_2^{(1)"})^j (d_2^{(1)"})^{k_2-j} e_0 \quad (k_1,k_2 = 0,1,...p-1) \qquad (101)$$

By using Eq. (83) and the results of this section, one can explicitly verify that all the $x(k_1,k_2)$ are the nonzero vectors and

$$d_1' x(k_1,k_2) = d_2' x(k_1,k_2) = 0 \quad d_2' d_1" x(k_1,k_2) = x(k_1+1,k_2-1) \qquad (102)$$

Since the $e_0^{(j)}$ $(j = 1,2)$ are the generating vectors of the IRs of the osp(1,4) superalgebra with $(q_1,q_2) = ((p+1)/2,(p+1)/2)$, it follows from Eq. (85) that $x(k_1,k_2)$ is the generating vector of the IRs of the osp(1,4) superalgebra with $(q_1,q_2) = (1+k_1,1+k_2)$ if $d_2' d_1" x(k_1,k_2) = 0$. Therefore, as follows from Eq. (102), this is the case if $k_2 = 0$. Hence the tensor product of the supersingleton IRs contains IRs of the osp(1,4) algebra corresponding to $(q_1,q_2) = (1+k_1,1)$ $(k_1 = 0,1,...p-1)$. As noted in Sect. 8, the case $(0,1)$ can be treated either as the massless IR with $s = p-1$ or as the special massive IR; the case $(1,1)$ can be treated as the massive IR of the osp(1,4) superalgebra and the cases when $k_1 = 1,...p-2$ can be treated as massless IRs with $s = k_1$.

The results of standard theory follow from the above results in the formal limit $p \to \infty$. Therefore in standard theory the decomposition of tensor product of supersingletons contains the IRs of the osp(1,4) superalgebra corresponding to $(q_1,q_2) = (1,1), (2,1), ...(\infty,1)$ in agreement with the results obtained by Flato and Fronsdal [8] and Heidenreich [9].

As noted in Sect. 7, the Flato-Fronsdal result for the tensor product $Di \times Di$ is that it also contains a massive IR corresponding to $q_1 = q_2 = 2$. In terms of the fermionic operators this result can be obtained as follows. If $y = (d_1^{(1)"}d_2^{(2)"} - d_2^{(1)"}d_1^{(2)"})e_0$ then, as follows from Eqs. (83) and (84),

$$h_1 y = h_2 y = 2y \quad L_+ y = L_- y = 0 \quad d_1^{(1)'} y = \frac{p+1}{2} d_2^{(2)"} e_0 \quad d_1^{(2)'} y = \frac{p+1}{2} d_2^{(1)"} e_0$$
$$d_2^{(1)'} y = -\frac{p+1}{2} d_1^{(2)"} e_0 \quad d_2^{(2)'} y = -\frac{p+1}{2} d_1^{(1)"} e_0 \qquad (103)$$

Since $a_j' = (d_j')^2$ for $j = 1,2$ (see Eq. (84)), it follows from these expressions that $a_1' y = a_2' y = 0$, i.e. y indeed is the generating vector for the IR of the so(2,3) algebra characterized by $q_1 = q_2 = 2$. However, y is not a generating vector for any IR of the osp(1,4) superalgebra since it does not satisfy the condition $d_1' y = d_2' y = 0$.

The vector $x(k_1,k_2)$ defined by Eq. (101) becomes the null vector when $k_1 = p$. Indeed, since $C_{k_1}^i = k_1!/[i!(k_1-i)!]$, the sum over i in Eq. (101) does not contain terms with $i \neq 0$ and $i \neq p$. At the same time, if $i = 0$ or $i = p$ the corresponding terms are also the null vectors since, as follows from the results of the preceding section, $(d_1')^p e_0 = (d_2')^p e_0 = 0$. It is obvious that this result is valid only in the modular case and does not have an analog in standard theory. Therefore, as follows from Eq. (102), the decomposition of the tensor products of two supersingletons also contains IRs of the osp(1,4) superalgebra characterized by $(q_1, q_2) = (0,0), (0,1), (0,2), ...(0, p-1)$.

We have shown that the decomposition of the tensor products of two supersingletons contains IRs of the osp(1,4) superalgebra characterized by the following values of (q_1, q_2):

$$(0,0),\ (0,1),\ (0,2), ...(0, p-1), (1,1),\ (2,1), ...(p-1, 1)$$

The question arises whether this set of IRs is complete, i.e. the decomposition of the tensor products of two supersingletons does not contain other IRs of the osp(1,4) superalgebra. Since the dimension of the supersigleton IR is p^2 (see the preceding section), this is the case if

$$\sum_{k=0}^{p-1} SDim(0,k) + \sum_{k=1}^{p-1} SDim(1,k) = p^4 \tag{104}$$

It is obvious that $SDim(0,0) = 1$ since the IR characterized by $(q_1, q_2) = (0,0)$ is such that all the representation operators acting on the generating vector give zero. Therefore, as follows from Eq. (93), the condition (104) can be rewritten as

$$2 + Dim(0) + Dim(2,2) + 2\sum_{s=1}^{p-2} Dim(s) + 2\sum_{q_2=1}^{p-1} Dim(0, q_2) = p^4 \tag{105}$$

since $Dim(1,1) = Dim(0)$. The expressions for $Dim(s)$ and $Dim(0, q_2)$ are given in Eqs. (41-44) and hence the only quantity which remains to be calculated is $Dim(2,2)$.

The IR of the so(2,3) algebra characterized by $(q_1, q_2) = (2,2)$ is the massive IR with $m = 4$ and $s = 0$. Therefore, as follows from the results of Sect. 4, the quantity k in Eq. (28) can take only the value $k = 0$ and the quantity n can take the values $0, 1, ...n_{max}$ where $n_{max} = p - 2$. Hence, as follows from Eqs. (8) and (35)

$$Dim(2,2) = \sum_{n=0}^{p-2} (p-1-n)^2 = \frac{1}{6} p(p-1)(2p-1) \tag{106}$$

The validity of Eq. (105) now follows from Eqs. (41-44,106).

The main result of the present paper can now be formulated as follows:

In a quantum theory over a Galois field, the tensor product of two supersingletons is fully decomposable into the following IRs of the osp(1,4) superalgebra:

- *Massive IR characterized by* $(q_1 = 1, q_2 = 1)$

- *Massless IRs characterized by* $(q_1 = 2, ...p - 1, q_2 = 1)$
- *Special massive IRs characterized by* $(q_1 = 0, q_2 = 0, 1, ...p - 1)$

and the multiplicity of each IR in the decomposition is equal to one.

11 Discussion

As it has been noted throughout the paper, the seminal result by Flato and Fronsdal [8] poses a fundamental problem of whether only Dirac singletons can be true elementary particles. In this case one has to answer the questions a) and b) in Sec. 1. In the literature, a typical explanations of a) are that singletons are not observable because they cannot be considered in the Poincare limit or because in this limit the singleton four-momentum becomes zero or because the singleton field lives on the boundary of the AdS bulk or as a consequence of other reasons. As shown in Sec. 6, semiclassical approximations for singletons in the Poincare limit can be discussed in full analogy with the case of massive and massless particles. As a result, in the general case the energy of singletons in the Poincare limit is not zero but, in contrast to the case of usual particles, singletons can have only two independent components of standard momentum, not three as usual particles. A problem arises about whether such objects can be detected by standard devices, whether they have a coordinate description etc. At the same time, in standard theory there is no natural explanation of b).

In Refs. [12, 13] we have proposed a new approach to quantum theory where Hilbert spaces are replaced by spaces over a Galois field, and operators of physical quantities become operators acting in such spaces. We call this approach a quantum theory over a Galois field (GFQT). A detailed motivation of this approach can be found in Ref. [14]. We believe that there are several reasons of why GFQT is a more natural and physical approach to quantum theory than standard one (see Refs. [14, 4]).

For any new theory there should exist a correspondence principle describing conditions when the new and old theories give close predictions. In the given case, standard theory can be considered as a formal limit of GFQT when the characteristic p of the Galois field in GFQT becomes infinitely large. In the case of dS or AdS theories this implies that GFQT and standard theory give close predictions when dS or AdS energies, momenta and angular momenta are much less than p (note that all those quantities are dimensionless). A rough estimation of p in the framework of our approach to gravity gives a value of order $exp(10^{80})$ [22]. One might think that this value is so huge that GFQT should be practically indistinguishable from standard theory. However, the fact that p is finite has far reaching practical and theoretical consequences. In our approach the gravitational constant is proportional to $1/lnp$, so in the formal limit $p \to \infty$ gravity disappears, i.e. in our approach gravity is a consequence of finiteness of nature. Another consequence of the finiteness of p is that the very notion of particle-antiparticle becomes only approximate since states treated as particle and antiparticle in standard theory belong to the same IR [14, 4]. Note that this property takes place even in standard theory if the symmetry algebra is dS rather than AdS [23, 3]. As a consequence, while in standard theory there are four singleton IRs describing the Di and Rac singletons and their antiparticles, in GFQT only two IRs remain since standard Di and anti-Di now belong to the same IR and the same is true for standard Rac and anti-Rac. We use Di and Rac to call the corresponding modular IRs, respectively. Nevertheless, since each massless boson can be represented as a composite state of two Dis or two Racs, a problem remains of what representation (if any) is preferable. This problem has a natural solution if the theory is supersymmetric. Then the only IR is the (modular) Dirac supermultiplet combining (modular) Di and (modular) Rac into one IR.

The main result of the paper is described in Sec. 10 where we explicitly describe a complete set of supersymmetric modular IRs taking part in the decomposition of the tensor product of two modular supersingleton IRs. In particular, by analogy with the Flato-Fronsdal result, each massless superparticle can be represented as a composite state of two supersingletons and one again can pose a question of whether only (super)singletons can be true elementary (super)particles.

This question is also natural in view of the following observation. As shown in Refs. [5, 6] (see also Sec. 4 of the present paper), the AdS mass m and standard Poincare mass m_P are related as $m = 2Rm_P$ where R is the radius of the Universe. In view of the present data on the cosmological constant, one might think that R is of order $10^{26}m$. Then the AdS mass of the electron is of order 10^{39}. It is natural to think that a particle with such an AdS mass cannot be elementary. Moreover, the present upper level for the photon mass is $10^{-18}ev$ which seems to be an extremely tiny quantity. However, the corresponding AdS mass is of order 10^{15} and so even the mass which is treated as extremely small in Poincare invariant theory might be very large in AdS invariant theory. Nevertheless, assuming that only (super)singletons can be true elementary (super)particles, one still has to answer the questions a) and b).

As explained in Sec. 5, a crucial difference between singletons in standard theory and GFQT follows. Since 1/2 in the Galois field is $(p+1)/2$, the eigenvalues of the operators h_1 and h_2 for singletons in GFQT are $(p+1)/2, (p+3)/2, (p+5)/2...$, i.e. huge numbers if p is huge. Hence the Poincare limit and semiclassical approximation for singletons in GFQT have no physical meaning and singletons cannot be observable. In addition, as noted in Sec. 4 (see also Refs. [12, 13, 14]), the probabilistic interpretation for a particle can be meaningful only if the eigenvalues of all the operators M_{ab} are much less than p. Since for singletons this is not the case, their state vectors do not have a probabilistic interpretation. These facts give a natural answer to the question a).

For answering the question b) we note the following. In standard theory the notion of binding energy (or mass deficit) means that if a state with the mass M

is a bound state of two objects with the masses m_1 and m_2 then $M < m_1 + m_2$ and the quantity $|M - (m_1 + m_2)|c^2$ is called the binding energy. The binding energy is a measure of stability: the greater the binding energy is, the greater is the probability that the bound state will not decay into its components under the influence of external forces. On the contrary, if two free particles with the masses m_1 and m_2 can create a bound state with the mass M then the greater the quantity $|M - (m_1 + m_2)|$ is, the greater is the probability of the a reaction where this bound state is created.

If a massless particle is a composite state of two singletons, and the eigenvalues of the operators h_1 and h_2 for the singletons in GFQT are $(p+1)/2, (p+3)/2, (p+5)/2...$ then, since in GFQT the eigenvalues of these operators should be taken modulo p, the corresponding eigenvalues for the massles particle are $1, 2, 3....$ Hence an analog of the binding energy for the operators h_1 and h_2 is p, i.e. a huge number. This phenomenon can take place only in GFQT: although, from the formal point of view, the singletons comprising the massless state do not interact with each other, the analog of the binding energy for the operators h_1 and h_2 is huge. In other words, the fact that all the quantities in GFQT are taken modulo p implies a very strong effective interactions between the singletons. It explains why the massless state does not decay into singletons and why free singletons effectively interact pairwise for creating their bound state.

Another interesting feature of singleton physics in GFQT follows. In standard theory the difference between the dS and AdS symmetries on quantum level is in the choice of the scalar product in the Hilbert space of states. If in Eq. (1) $\eta^{44} = 1$ is replaced by $\eta^{44} = -1$ then the Hermitian operators M_{a4} become anti-Hermitian and vice versa. The singleton IRs are the implementation of the IRs of the so(2,3) algebra by Hermitian operators but there are no singleton IRs in the implementation of the IRs of the so(1,4) algebra by such operators. In addition, the dS theory does not have a supersymmetric generalization. On the other hand, the fact that the cosmological constant is positive can be naturally explained from dS symmetry on quantum level [3, 4] while in standard AdS theory this constant is negative.

As already noted, in GFQT the notions of positive definite scalar product and Hermiticity can be only approximate. Hence the relations (1) in GFQT can be treated as the GFQT generalization of dS and AdS symmetries simultaneously. In different situations, a description of a physical system can be close to a description in standard theory for the dS or AdS cases. The main results of the present paper are purely algebraical and no choice of the scalar product has been involved. As already noted, since in GFQT some quantum numbers characterizing singletons are of order p, the singleton state vectors do not have the probabilistic interpretation and therefore a choice of the scalar product for singletons does not have a physical meaning. Hence the existence of singletons and supersingletons in GFQT does not contradict the fact that the cosmological constant is positive.

The above discussion shows that singleton physics in GFQT is even more interesting than in standard theory.

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