ENERGY LAWS , FOLLOW MOULDS OF EUCLIDEAN GEOMETRY . AND THE SIX , TRIPLE - CONCURRENCY POINTS , LINE .

A geometrical attempt was developed for approaching the Open question 1 [The Orthic triangle] concerning the the locus of centerpoints $H_{1 \ldots n}$ *and the point of convergency . A further Analysis in Euclidean geometry follows .*

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The present article **`Energy Laws , follow moulds of Euclidean Geometry′ ,** *demonstrates the correlation of* ***< Changeable Areas in a closed or Isolated Figure*** (*Any triangle or Square in a circle*) ***to a Totally constant Area >*** *to the* ***< Constant Total Energy in a closed or Isolated System >*** *The article* **, Six , [Triple - Points] , Line ,** *shows correspondences of lengths and angles* **, of Projective and other Geometries , under Euclidean First dimensional Unit ,** *which is one , Duality , Triple and Multiple .* P.12 , P19 .

Limits of Recursive Triangle and Polygon Tunnels

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Abstract.
In this paper we present unsolved problems that involve infinite tunnels of recursive triangles or recursive polygons, either in a decreasing or in an increasing way. The "nedians or order i in a triangle" are generalized to "nedians of ratio r" and "nedians of angle α" or "nedians at angle β", and afterwards one considers their corresponding "nedian triangles" and "nedian polygons".
This tunneling idea came from physics. Further research would be to construct similar tunnel of 3-D solids (and generally tunnels of n-D solids).

A) Open Question 1 (Decreasing Tunnel).

1. Let ΔABC be a triangle and let $\Delta A_1B_1C_1$ be its **orthic triangle** (i.e. the triangle formed by the feet of its altitudes) and H_1 its orthocenter (the point on intersection of its altitudes).
Then, let's consider the triangle $\Delta A_2B_2C_2$, which is the orthic triangle of triangle $\Delta A_1B_1C_1$, and H_2 its orthocenter.
And the recursive tunneling process continues in the same way.
Therefore, let's consider the triangle $\Delta A_nB_nC_n$, which is the orthic triangle of triangle $\Delta A_{n-1}B_{n-1}C_{n-1}$, and H_n its orthocenter.

a) What is the locus of the orthocenter points $H_1, H_2, \ldots, H_n, \ldots$? {Locus means the set of all points satisfying some condition.}

b) Is this limit:

$$\lim_{n \to \infty} \Delta A_nB_nC_n$$

convergent to a point? If so, what is this point?

c) Calculate the sequences

$$\alpha_n = \frac{area(\Delta A_nB_nC_n)}{area(\Delta A_{n-1}B_{n-1}C_{n-1})} \text{ and } \beta_n = \frac{perimeter(\Delta A_nB_nC_n)}{perimeter(\Delta A_{n-1}B_{n-1}C_{n-1})}.$$

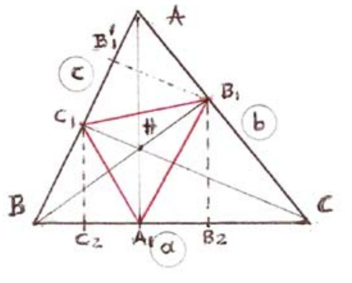
d) We generalize the problem from triangles to polygons. Let $AB\ldots M$ be a polygon with $m \geq 4$ sides. From A we draw a perpendicular on the next polygon's side BC, and note its intersection with this side by A_1. And so on. We get another polygon $A_1B_1\ldots M_1$.
We continue the recursive construction of this tunnel of polygons and we get the polygon sequence $A_nB_n\ldots M_n$.
d1) Calculate the limit:

$$\lim_{n \to \infty} \Delta A_nB_n\ldots M_n$$

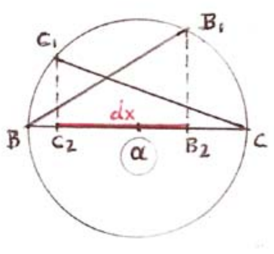
d2) And the ratios of areas and perimeters as in question *c)*.

e) A version of this polygonal extension *d)* would be to draw a perpendicular from A not necessarily on the next polygon's side, but on another side (say, for example, on the third polygon's side) – and keep a similar procedure for the next perpendiculars from all polygon vertices B, C, etc.

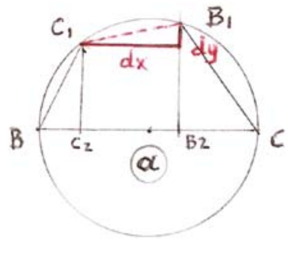
In order to tackle the problem in a easier way, one can start by firstly studying particular initial triangles ΔABC, such as the equilateral and then the isosceles.



F.1(1)



F.1(2)



F.1(3)

THE ORTHOCENTER (H) OF A TRIANGLE (ABC) 15 / 1 / 2012

Let Δ ABC be any triangle with vertices A , B , C and let **Δ $A_1B_1C_1$** be its orthic triangle .

AA_1 , BB_1 , CC_1 are the altitudes of the vertices and points A_1 , B_1 , C_1 the feet of them .
Altitudes $AA_1 \perp BC$, $BB_1 \perp AC$, $CC_1 \perp AB$ and triangles AA_1B , AA_1C , BB_1A , BB_1C CC_1A , CC_1B are all right angled . It is also holding $B_1B'_1 \perp AB$

E_{ABC} = E = Area of triangle ABC and let be
$E_A = E_{AB_1C_1}$ = Area of triangle AB_1C_1 ,
$E_B = E_{BC_1A_1}$ = Area of triangle BC_1A_1 ,
$E_C = E_{CA_1B_1}$ = Area of triangle AC_1B_1 .

For shortness , we use the known ratios on any right angle triangle ABC at C :
The ratio of opposite side BC = a to the hypotenuse AB = c is called , sine of angle A , (sinA = a / c) and the ratio of the adjacent side AC = b to the hypotenuse AB is called cosine of angle A (cosA = a / c) . It is known ,

$E_A = E_{AB_1C_1} = ½ . AC_1. B_1B'_1 = ½ . AC_1 .[AB_1 . \sin A] = ½ . AC_1 . AB_1 .\sin A$ .... (1)
$E_{ABC} = ½ . AB . CC_1 = ½ . AB.AC. \sin A$ …………………….. (2)

By division : $E_A / E_{ABC} = [AC_1 . AB_1 .\sin A] / [AB .AC. \sin A] = [AC_1 . AB_1.] / [AB .AC]$

and for $AC_1 = AC. \cos A = [\mathbf{b} .\cos A]$, $AB_1 = AB. \cos A = [\mathbf{c} .\cos A]$ and also

Using rotational similarity then :

E_A / E_{ABC} = [b .cosA . c .cosA] / [b . c] = cos² A ……….. (3a)

E_B / E_{ABC} = [c .cosA . a .cosA] / [c . a] = cos² B ……….. (3b)

E_C / E_{ABC} = [a .cosA . b .cosA] / [a . b] = cos² C ……….. (3c)

The area of the three triangles is ,

$E_A = E_{ABC} . \cos^2 A = E . \cos^2 A$
$E_B = E_{ABC} . \cos^2 B = E . \cos^2 B$
$E_C = E_{ABC} . \cos^2 C = E . \cos^2 C$ by summation :

$E_A+E_B+E_C = E.\cos^2A + E.\cos^2B + E.\cos^2C = E . [\cos^2A + \cos^2B + \cos^2C]$, so

Area of triangle **$A_1B_1C_1$** is **E_1** = E – [$E_A+E_B+E_C$] = **$E . [1 - \cos^2A - \cos^2B - \cos^2C]$** or

$E_1 = E . [1 - \cos^2A - \cos^2B - \cos^2C] = E. [1 – (AB_1/c)^2 - (BC_1/a)^2 - (CA_1/b)^2]$ …. (4)

Replacing the results of (6.abc) in (4) , then using ,

$$\left|\frac{AB_1}{c}\right|^2 = \frac{|b^2+c^2- a^2|^2}{4. b^2.c^2} = \frac{a^4 + b^4 +c^4 -2. a^2.b^2 -2.a^2.c^2 + 2.b^2.c^2}{4. b^2.c^2}$$

$$\left|\frac{BC_1}{a}\right|^2 = \frac{|a^2+c^2- b^2|^2}{4. a^2.c^2} = \frac{a^4 + b^4 +c^4 -2. a^2.b^2 +2.a^2.c^2 - 2.b^2.c^2}{4. a^2.c^2}$$

$$\left|\frac{CA_1}{b}\right|^2 = \frac{|a^2+b^2-c^2|^2}{4.a^2.b^2} = \frac{a^4+b^4+c^4+2.a^2.b^2-2.a^2.c^2-2.b^2.c^2}{4.a^2.b^2}$$

and rewrite equation (4) as :

$4.a^2b^2c^2.[1-\cos^2A-\cos^2B-\cos^2C] = 4.a^2b^2c^2-[a6+a^2b^4+a^2c^4-2.a^4b^2-2.a^4c^2+2.a^2.b^2.c^2]$

$-[b6-2.a^2b^4+b^2c^4-2.b^4c^2+a^4b^2+2.a^2.b^2.c^2] - [c6-2.a^2c^4+b^4c^2-2.b^2c^4+a^4c^2+2.a^2.b^2.c^2]$

$= a^4.b^2+a^4.c^2+a^2.b^4+a^2.c^4+b^4.c^2+b^2.c^4-2.a^2.b^2.c^2-a6-b6-c6$.

then the ratio of area **E1** of the Orthic triangle to the area **E** of the triangle ABC is :

$$\frac{E_1}{E} = \frac{a^4.b^2+a^4.c^2+a^2.b^4+a^2.c^4+b^4.c^2+b^2.c^4-2.a^2.b^2.c^2-a6-b6-c6}{4.a^2.b^2.c^2} \quad (5)$$

$$\frac{E_1}{E} = \frac{4.a^2.b^2.c^2 - a^2[b^2+c^2-a^2]^2 - b^2[a^2+c^2-b^2]^2 - c^2[a^2+b^2-c^2]^2}{4.a^2.b^2.c^2} \quad \ldots\ldots\ldots\ldots(5)$$

Verification :

For equilateral triangle $a = b = c \rightarrow$ **E1 / E** $= (6.a6-2.a6-3.a6)/4.a6 = a6/4.a6 = ¼$

For isosceles triangle $a, b = c \rightarrow$ **E1 / E** $= [2.a^4.b^2+2.a^2.b^4-2.a^2.b^4-a6]/4.a^2.b^4$

$= [2.a^4.b^2-a6]/4.a^2.b^4 = [2.a^2.b^2-a^4]/4.b^4 = a^2.(2.b^2-a^2)/4.b^4$

Extensions to Pythagoras`s theorem :

Using the general conversions of Pythagoras`s theorem is easy to measure altitude
$la = AA_1$, $lb = BB_1$, $lc = CC_1$

In the right angled triangle **ABA1** $\rightarrow AA_1^2 = AB^2 - BA_1^2 = AB^2 - (BC - A_1C)^2$
$AA_1^2 = AB^2 - (BC^2 + A_1C^2 - 2.BC.A_1C) = AB^2 - BC^2 - A_1C^2 + 2.BC.A_1C$
In the right angled triangle **AA1C** $\rightarrow AA_1^2 = AC^2 - A_1C^2$ and by subtraction ,

$0 = AB^2 - BC^2 - A_1C^2 + 2.BC.A_1C - AC^2 + A_1C^2 = AB^2 - BC^2 - AC^2 + 2.BC.A_1C$ or

$2.BC.A_1C = AC^2 + BC^2 - AB^2$, so$\rightarrow$ **A1C** $= [AC^2+BC^2-AB^2]/2.BC =$ $\mathbf{[a^2+b^2-c^2]/2a}$

$\mathbf{A_1C = [a^2+b^2-c^2]/2.a}$ **…… (6ac)**

also

$A_1B = a - A_1C = a - [a^2+b^2+c^2]/2.a = [2a^2-b^2-a^2+c^2]/2.a =$ $\mathbf{[a^2+c^2-b^2]/2a}$

$\mathbf{A_1B = [a^2+c^2-b^2]/2.a}$ **…… (6ab)**

exists also

$AA_1^2 = AC^2 - CA_1^2 = (AC+CA_1).(AC-CA_1)$ and by substitution (6ac)

$4.a^2(AA_1)^2 = [a^2+b^2+2.ab-c^2].[c^2-b^2-a^2+2.ab] = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4]$ or

$$4.a^2(AA_1)^2 = 4.a^2(la)^2 = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4] \quad .... (7a)$$

In right angled triangle $BB_1C \rightarrow BB_1{}^2 = BC^2 - B_1C^2$.
In right angled triangle $BB_1A \rightarrow BB_1{}^2 = BA^2-(AC- B_1C)^2 = BA^2- AC^2- B_1C^2+ 2.AC.B_1C$ and by subtraction , and

$0= BC^2 - B_1C^2- BA^2 +AC^2+ B_1C^2 - 2.AC.B_1C$ then calculating B_1C, B_1A

$$B_1C=[-AB^2+ AC^2+ BC^2] /2.AC = [a^2+b^2-c^2] / 2.b \quad ...... (6bc)$$

$$B_1A=b - [a^2+b^2-c^2] / 2.b =[2.b^2-b^2-a^2+ c^2] /2.b = [c^2+b^2-a^2] / 2.b \quad .... (6ba)$$

also is

$BB_1{}^2 = BC^2 - B_1C^2 = (BC+ B_1C) . (BC - B_1C) = (a+ B_1C).(a - B_1C)$ by substitution (6bc)

$4.b^2(BB_1)^2 =[a^2+b^2+2.ab - c^2].[c^2-b^2- a^2+ 2.ab] = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4]$ or

$$4.b^2(BB_1)^2 = 4.a^2(lb)^2 = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4] \quad ... (7b)$$

in the same way exists ,

$$C_1B= [CB^2+ AB^2- AC^2] /2.AC = [a^2+c^2-b^2] / 2.c \quad ...... (6cb)$$

$$C_1A= [AC^2+ AB^2-BC^2] / 2.c = [b^2+c^2-a^2] / 2.c \quad .... (6ca)$$

also

$$4.c^2(CC_1)^2 = 4.c^2(lc)^2 = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4] \quad ... (7c)$$

and altitudes la , lb , lc are

$$la = AA_1 = \frac{\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}}{2.a}$$

$$lb = BB_1 = \frac{\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}}{2.b} \qquad lc = CC_1 = \frac{\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}}{2.c}$$

Verification :

For $a = b$ then $C_1A = C_1B \rightarrow C_1A = [c^2 / 2c] = c / 2 \rightarrow C_1B = [c^2 / 2c] = c / 2$
For $a = c$ then $B_1A = B_1C \rightarrow B_1A = [b^2 / 2b] = b / 2 \rightarrow B_1C = [b^2 / 2b] = b / 2$
For $b = c$ then $A_1B = A_1C \rightarrow A_1B = [a^2 / 2a] = a / 2 \rightarrow A_1C = [a^2 / 2a] = a / 2$

Since angle $< BC_1C = BB_1C = 90°$ then :

Using Pythagoras`s theorem in F.1(2) $\rightarrow BB_1{}^2 = B B_2. BC \rightarrow BB_2 = BB_1{}^2/BC$

$$BB_2 = [BB_1{}^2] / BC = \frac{2.a^2.b^2 + 2.a^2.c^2 +2.b^2.c^2 - (a^4+ b^4+ c^4)}{4.b^2.a}$$

$$\mathbf{BC_2} = [\,BC_1^2\,] \;/\; BC = \frac{2.a^2.c^2 - 2.a^2.b^2 - 2.b^2.c^2 + (a^4+.b^4+c^4)}{4.c^2.a}$$

For $dx = [\,BB_2 - BC_2\,]$ then :

$4.\ ab^2c^2.\ dx = 2.a^2b^2c^2 + 2.a^2c^4 + 2.b^2c^4 - c^2a^4 - c^2b^4 - c6 - b6 - a^4b^2 - b^2c^4 + 2.a^2b^4 + 2.\ b^4c^2 - 2.a^2b^2c^2$ and

$4.\ a.b^2.c^2\ .\ [\ dx = (\ BB_2 - BC_2\ )\ ] = 2.a^2b^4 + 2.a^2c^4 + b^2c^4 + c^2b^4 - a^4b^2 - a^4c^2 - b6 - c6$

In the right angle triangle BB_1B_2 exists also ,

$(BB_2)^2 = BB_1^2 - B_1B_2^2 = [2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)] \;/\; 4.b^2$

$- [a^2+b^2-c^2\ ].\ [2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)] \;/ 16a^2b^4 =$

$[2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)][4.a^2b^2 - a^2 - b^2 + c^2] \;/ 16a^2b^4 =$

$[2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)][\ c^2 + 2.a^2b^2 - (a+b)^2] \;/\ 16a^2b^4$

and the previous equation is :

$4.\ a.b^2.c^2.\ dx = 2.a^2b^4 + 2.a^2c^4 - a^4b^2 - a^4c^2 + b^2c^4 + c^2b^4 - b6 - c6$ or

$$\mathbf{dx = [\ BB_2 - BC_2\ ] = [\ 2.a^2b^4 + 2.a^2c^4 - a^4b^2 - a^4c^2 + b^2c^4 + c^2b^4 - b6 - c6\ ] \;/\; 4.ab^2c^2} \quad ..(a)$$

Verification :

For $b = c \;\rightarrow\; BA_1 = BC_2 + [\ BB_2. - BC_2\ ] / 2 = a / 2$

$\mathbf{dx} = BB_2 - BC_2 = (\ 4.a^2b^4 - 2.a^4b^2) / 4.ab^4 = (4.ab^2 - 2.a^3) \;/ 4.b^2 = (\ 2.ab^2 - a^3) / 2.b^2$

$BC_2 = (a^4+b^4+b^4 + 2.a^2b^2 - 2.a^2b^2 - 2.b^2b^2) \;/ 4.ab^2 = (a^4 + 2.b^4 - 2b^4) / \ 4.ab^2 = a^3 / 4.b^2$

$BA_1 = (\ a^3) \;/\; 4.b^2 + (\ 2.ab^2 - a^3) / 4.b^2 = 2.ab^2 \;/\; 4.b^2 = a\ /2$

$\mathbf{dx} = BB_2 - BC_2 = (\ 4.a^2b^4 - 2.a^4b^2) / 4.ab^4 = (4.ab^2 - 2.a^3) \;/ 4.b^2 = (\ 2.ab^2 - a^3) / 2.b^2$

From similar triangles CAA_1, $C\,B_1B_2 \rightarrow [\ B_1B_2 / AA_1\ ] = CB_1/b$ and $(B_1B_2)^2 = [\ CB_1^2.AA_1^2] /b^2$ or

$B_1B_2^2 = (a^2+b^2-c^2\ )^2\ .\ [2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)] \;/\ 4.b^4.4a^2$ and

$$\mathbf{B_1B_2^2 = (a^2+b^2-c^2\ )^2.\ [\ (a+b)^2 - c^2]\ .\ [\ c^2 - (a-b)^2] \;/\; 16.\ a^2b^4} \quad ……………..$$

From similar triangles BAA_1, $BC_1C_2 \rightarrow [\ C_1C_2 / AA_1\ ] = BC_1/c$ and $(C_1C_2)^2 = [\ BC_1^2.AA_1^2] /c^2$

$$\mathbf{C_1C_2^2 = (a^2+c^2-b^2\ )^2\ .\ [\ (a+b)^2 - c^2]\ .\ [\ c^2 - (a-b)^2] \;/\; 16.\ b^2c^4} \quad ………….$$

and so

$$\mathbf{B_1B_2 = (a^2+b^2-c^2\ )\ .\ \sqrt{[\ c^2 - (a+b)^2\ ]\ .\ [\ (a-b)^2 - c^2]} \;/\; 4.\ ab^2} \quad …………….. (8)$$

$$\mathbf{C_1C_2 = (\ a^2+c^2-b^2\ )\ .\ \sqrt{[\ c^2 - (a+b)^2\ ]\ .\ [\ (a-b)^2 - c^2]} \;/\; 4.\ bc^2} \quad ………….. (8.a)$$

Verification :

For $b = c \;\rightarrow\; B_1B_2 = C_1C_2$

$$B_1B_2 = \frac{a^2}{4.ab^2}\ \sqrt{(b^2-a^2-b^2\ -2.ab\ ).(a^2+b^2\ -2.ab\ -b^2)} = \frac{a}{4.b^2}\ \sqrt{-\ (\ a^2+2.ab\ ).(\ a^2-2.ab\ )}$$

$$C_1C_2 = \frac{a^2}{4.ab^2}\sqrt{(b^2-a^2-b^2\ -2.ab\ ).(a^2+b^2\ -2.ab\ -b^2)} = \frac{a}{4.b^2}\sqrt{-(\ a^2+2.ab\ ).(\ a^2-2.ab\ )}$$

.

$[AA_1]^2 = b^2 - a^2\ /4 = (4.b^2-a^2)\ /4 \quad \rightarrow \quad$ **$AA_1 = |\sqrt{4.b^2-a^2}\ |\ /\ 4$**

Let $[B_1B_2 - C_1C_2] = dy$, then **dy** $= [CB_1.\ AA_1]/b - [BC_1.\ AA_1]/c = [\ AA_1].[\ c.CB_1 - b.BC_1]\ /\ bc$

$c.CB_1 - b.BC_1 = c.(a^2+b^2-c^2)/2.b - b.(a^2+c^2-b^2/2c = [\ a^2c^2 - a^2b^2+b^4+\ c^4]\ /\ 2.bc$

and

$$\mathbf{dy = [\ B_1B_2\ -C_1C_2\ ] = \frac{\sqrt{[c^2-(a+b)^2]\ .\ [(a-b)^2\ -\ c^2]}}{4.a\ b^2c^2}\ [\ a^2c^2\ -\ a^2b^2\ +\ b^4+\ c^4\ ]} \quad \textbf{or}$$

$$\mathbf{dy = [\ B_1B_2\ -C_1C_2\ ] = \frac{\sqrt{[\ (a+b)^2\ -\ c^2]\ .\ [c^2-(a-b)^2]}}{4.a\ b^2c^2}\ [\ a^2c^2\ -\ a^2b^2\ +\ b^4\ -\ c^4\ ]\ ...} \quad \textbf{(b)}$$

B_1C_1 is measured by using Pythagoras`s theorem in triangle (B_1B_1, dy , dx) and then by squaring equation (a) and (b) ,

$dx^2\ .\ [4.a\ b^2c^2]^2 = [4.a^4b^8 + 4.a^4c^8 + a^8b^4 + a^8c^4 + b^4c^8 + b^8c^4 + b12 + c12 - 4.a6.b6 - 4a6b^4c^2$
$+ 4.a^2b6.c^4 + 4.a^2b^8c^2 - 4.a^2b10 - 4.a^2b^4c6 + 8.a^4b^4c^4 - 4.a6b^2c^4 - 4.a6.c6 + 4.a^2b^2c^8 + 4.a^2b^4c6$
$- 4.a^2b6.c^4 - 4.a^2c10 + 2a^8b^2c^2 - 2.a^4b^4c^4 - 2.a^4b6c^2 + 2.a^4b^8 + 2.a^4b^2c6 - 2.a^4b^2c6 - 2.a^4b^4c^4 +$
$2.a^4b6c^2 + 2.a^4c^8 + 2.b6.c6 - 2.b^8c^4 - 2.b^2c10 - 2.b10c^2 - 2.b^4c^8 + 2.b6.c6\ ] =$

$[\ 4.a^2b^2c^8 - 4.a^2b10 - 4.a^2c10 + 4.a^2b^8c^2 + 6.\ a^4b^8 + 6.a^4c^8 + 4.a^4b^4c^4 - 4.a6.b^2c^4 - 4.a6.b^4c^2$
$- 4.a6.b6 - 4.a6.c6 + 2a^8b^2c^2 + a^8b^4 + a^8c^4 - 2.b^2c10 + 4.b6.c6 - b^8c^4 - 2.b10c^2 + b12 + c12\ ]$

$dy^2\ .\ [4.a\ b^2c^2]^2 = [\ (a+b)^2\ -\ c^2]\ .\ [c^2-(a-b)^2]\ .\ [\ a^2c^2\ -\ a^2b^2\ +\ b^4 - c^4\ ]^2 =$
$[a^2+b^2+2.ab\ -c^2].[\ c^2-a^2\ -b^2+\ 2.ab].[\ a^2c^2\ -\ a^2b^2\ +\ b^4-c^4\ ]^2 =$
$[a^2c^2\ -a^2b^2+2.a^3b\ +b^2c^2 - b^4 + 2.ab^3\ +\ 4.abc^2\ -c^4 +\ a^2c^2\ -a^2b^2-2.a^3b\ +b^2c^2 - 2.ab^3$
$-2.abc^2 +\ 4.a^2b^2].[\ a^2c^2 - a^2b^2+b^4-c^4\ ]^2 = [2.a^2c^2+2.a^2b^2\ +\ 2.b^2c^2\ -\ a^4\ -\ b^4 - c^4]\ .$
$[a^4c^4 +\ a^4b^4 +\ b^8 + c^8\ -\ 2.a^4b^2c^2\ + 2.a^2b^4c^2 - 2.a^2c6 - 2.a^2b6 + 2.a^2b^2c^4 - 2.b^4c^4\ ] =$

$2.a6.c6 + 2.a6.b^4c^2 + 2.a^2b^8c^2 + 2.a^2c10\ -4.a6b^2c^4 + 4.a^4\ b^4c^4 - 4.a^4c^8\ -4.a^4b6c^2 + 4.a^4b^2c6$
$-4.a^2b^4c6 + 2.a6.c6 - 4.a6.b^4c^2 + 2.a^2b^2c^8 + 4.a^2b10 + 2.a6.b^2c^4 + 4.a^4\ b^4c^4 - 4.a^4b^8 + 4.a^4b6c^2$
$- 4.a^4b^2c6 - 4.a^2b6.c^4 - 4.a6.c6 - 4.a^2b^2c^8\ + 2.b10c^2\ +\ c^4 + 4.a^2b^4c6 - 4.a^4\ b^4c^4 + 2.a^4b6c^2$
$+ 2.a^4b^2c6\ c6 + 2.a^2b6.c^4 - 4.a^2b^8c^2\ + 2.b^2c10\ - a^8c^4\ - a^8b^4\ - a^4b^8\ -\ a^4c^8 +\ 2.a^8b^2c^2$
$-2.a6.b^4c^2 +\ 4.a6.c6 + 2.a6.b6 - 2.a6.b^2c^4 + 2.a^4\ b^4c^4 +\ 2.a^8c^4\ -b^4c^8\ - a^4b^8\ - 2.a^2b6.c^4$
$2.a^4b6c^2\ -2.a^2b^8c^2 +\ 2.a^2b^4c6 + 2.b^2c10 - a^4\ b^4c^4 - b12 - b^8c^4 + 2.b^4c^8\ -a^4c^8 + 2.a^4b^2c6$
$-2.a^2b^4c6 + 2.a^2c10 + 2.a^2b6.c^4\ -2.a^2b^2c^8\ -\ a^4\ b^4c^4 - c12\ .$

and segment B_1C_1 is :

$(dx^2+dy^2)\ .[\ 4.ab^2c^2]^2 = [\ 4.a^4b^2c^2]\ .\ [\ a^4 +\ b^4 + c^4\ -2.a^2c^2 - 2.a^2b^2\ + 2.b^2c^2]$ and

$$(dx^2+dy^2) = [B_1C_1]^2 = \frac{[\ 4.a^4b^2c^2]\ .\ [\ b^2+c^2\ -\ a^2]^2}{[\ 4.ab^2c^2]^2}$$

$$[B_1C_1] = \frac{2.\,a^2bc.\,[\,b^2+c^2-a^2]}{4.\,ab^2c^2} = \frac{a}{2.bc}\,[\,b^2+c^2-a^2]$$

Similarly for the other sides is holding :

$$\mathbf{a_1 = B_1C_1 = \frac{a}{2.bc}\,[\,b^2+c^2-a^2] = \frac{a}{2.bc}\,[\,b^2+c^2-a^2]} \quad \text{... (9.a)}$$

$$\mathbf{b_1 = A_1C_1 = \frac{b}{2.ca}\,[\,c^2+a^2-b^2] = \frac{b}{2.ac}\,[\,a^2+c^2-b^2]} \quad \text{...(9.b)}$$

$$\mathbf{c_1 = A_1B_1 = \frac{c}{2.ab}\,[\,a^2+b^2-c^2] = \frac{c}{2.ab}\,[\,a^2+b^2-c^2]} \quad \text{...(9.c)}$$

Verification :

For **a = b** then :

$$A_1C_1 = \frac{a}{2.ac}\,[\,c^2] = c/2 \;\text{---}\; C_1B_1 = \frac{a}{2.ac}[\,c^2] = c/2 \quad \text{therefore } C_1A_1 = C_1B_1$$

For **a = c** then :

$$A_1B_1 = \frac{a}{2.ab}\,[\,b^2] = b/2 \;\text{---}\; B_1C_1 = \frac{a}{2.ab}[\,b^2] = b/2 \quad \text{therefore } B_1A_1 = B_1C_1$$

For **b = c** then :

$$A_1B_1 = \frac{b}{2.ab}\,[\,a^2] = a/2 \;\text{---}\; A_1C_1 = \frac{a}{2.ab}[\,b^2] = b/2 \quad \text{therefore } A_1B_1 = A_1C_1$$

For **a = b = c** then :

$$A_1B_1 = \frac{a}{2aa}\,a^2 = a/2 \;\text{---}\; A_1C_1 = \frac{a}{2aa}\,a^2 = a/2 \;\text{----}\; B_1C_1 = \frac{a}{2aa}\,a^2 = a/2$$

The Perimeter P_o of the orthic triangle $A_1B_1C_1$ is :

$$P_o = B_1C_1 + A_1C_1 + A_1B_1 = \frac{[\,a^2b^2+a^2c^2-a^4+a^2b^2+b^2c^2-b^4+a^2c^2+b^2c^2-c^4]}{2.abc} =$$

$$P_o = \frac{[\,2.a^2b^2+2.a^2c^2+2.b^2c^2-a^4-b^4-c^4]}{2.abc} = \frac{4.b^2c^2-[\,a^2-b^2-c^2]^2}{2.abc} =$$

$$P_o = \frac{[\,2.bc+a^2-b^2-c^2\,].[2.bc-a^2+b+c^2]}{2.abc} = \frac{[a^2-(b-c)^2].[(b+c)^2-a^2]}{2.abc} \quad \text{or}$$

$$\mathbf{P_o} = \frac{[\,a+b-c\,].[\,a-b+c].[\,a+b+c\,].[\,b+c-a]}{2.abc} = \mathbf{\frac{[a+b+c].[-a+b+c].[a-b+c].[a+b-c]}{2.abc}} \quad \text{... (10)}$$

Remarks :

a) Perimeter P_o becomes zero when . [- a+b+c] = 0 , [a - b+c] = 0 , [a+b -c] = 0 or when $a = b+c$, $b = a+c$, $c = a+b$ and

This is the property of any point A on line BC *where then* Segment BC = a is equal to the parts AB = c and AC = b . The same for points B and C respectively . [6]
Since perimeter is minimized by the orthic triangle then this triangle is the only one among all inscribed triangles in the triangle ABC . This property is very useful later on .

b) Since Perimeter Po of the orthic triangle is minimized at the three lengths a , b , c then is also at the three vertices A , B , C of the original triangle ABC .

c) The ratio (R_p) of the perimeter of the orthic triangle $A_1B_1C_1$ to the triangle ABC is :

$$R_p = \frac{\mathbf{Po}}{\mathbf{P}} = \frac{[a+b-c].[a-b+c].[b+c-a].[a+b+c]}{2.abc \ . \ [a+b+c]} = \frac{\mathbf{[-a+b+c] \,.\, [a-b+c] \,.\, [a+b-c]}}{\mathbf{2.abc}} \quad \ldots (11)$$

For a = b = c then → (P_o / P) = (a.a.a) / 2.a.a.a **= 1 / 2** which is holding . Let

P_o = a+b+c is the perimeter of triangle Δ ABC
$P_1 = a_1 + b_1 + c_1$ is the perimeter of orthic triangle Δ $A_1 B_1 C_1$
$P_n = a_n + b_n + c_n$ is the perimeter of **n-**th orthic triangle

The ratio R_p of the perimeters is depended only on the **n-1** sides of orthic triangles :

$$R_p = \frac{P_n}{P_{n-1}} = \frac{a_n + b_n + c_n}{a_{n-1} + b_{n-1} + c_{n-1}} = \frac{\mathbf{[- a_{n-1} + b_{n-1} + c_{n-1}].[a_{n-1} - b_{n-1} + c_{n-1}].[a_{n-1} + b_{n-1} - c_{n-1}]}}{\mathbf{2.\, a_{n-1}\, b_{n-1}\, c_{n-1}}}$$

d) The ratio R_a of the area of triangle Δ $A_n B_n C_n$ to the Δ $A_{n-1} B_{n-1} C_{n-1}$ is :

$$R_a = \frac{E_{n-1}}{E_n} = \frac{4.\mathbf{a}_{n-1}^2.\mathbf{b}_{n-1}^2.\mathbf{c}_{n-1}^2 - \mathbf{a}_{n-1}^2[\mathbf{b}_{n-1}^2+\mathbf{c}_{n-1}^2-\mathbf{a}_{n-1}^2]^2 - \mathbf{b}_{n-1}^2[\mathbf{a}_{n-1}^2+\mathbf{c}_{n-1}^2-\mathbf{b}_{n-1}^2]^2 - \mathbf{c}_{n-1}^2[\mathbf{a}_{n-1}^2+\mathbf{b}_{n-1}^2-\mathbf{c}_{n-1}^2]^2}{4.\mathbf{a}_{n-1}^2 .\ \mathbf{b}_{n-1}^2 .\ \mathbf{c}_{n-1}^2} \quad \ldots (11a)$$

The ratio R_a of the areas is also depended only on the **n-1** sides of orthic triangles :

e.a . It is well known that the nine point circle is the circumcircle of the orthic triangle and has circumradius R_{A1B1C1} which is one half of the radius R = R_{ABC} of the circumcircle of triangle Δ ABC .

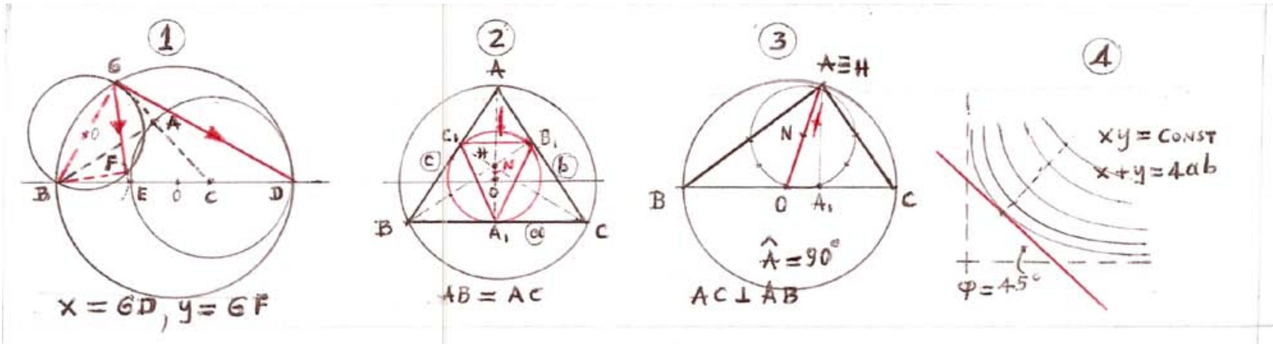
From F.4 , using Pythagoras theorem on triangle with hypotenuse equal to the diameter of the circle , then we have the law of sines as :

$$\frac{AB}{\sin C} = \frac{AC}{\sin B} = \frac{BC}{\sin A} = 2.R_{ABC} \quad \text{and} \rightarrow \quad R = \frac{AB}{2.\sin C} = \frac{AC}{2.\sin B} = \frac{BC}{2.\sin A} \quad \text{and R is}$$

$$R = \frac{BC}{2.\sin A} = \frac{a.c}{2.BB_1} = \frac{a.c\,[2.b] = abc}{2.\sqrt{[(a+b)^2-c^2].[c^2-(a-b)^2]}} = \frac{\mathbf{abc}}{\sqrt{\mathbf{[(a+b)^2-c^2].[c^2-(a-b)^2]}}} \quad \text{and}$$

$$R_{A1B1C1} = \frac{\mathbf{abc}}{2.\sqrt{\mathbf{[(a+b)^2-c^2].[c^2-(a-b)^2]}}} \quad \ldots. \text{ the circumradius of orthic triangle } \ldots. (12)$$

For $(a+b)^2 - c^2 = 0$ or $c^2 - (a-b)^2 = 0$ then $R_{A1B1C1} = \infty$, and it is $a + b = c$ and $a - b = c$ $a = b + c$ i.e. triangle ABC has the three vertices (the points) A , B , C on lines **c** and **a** respectively , a property of points on the two lines . [6]



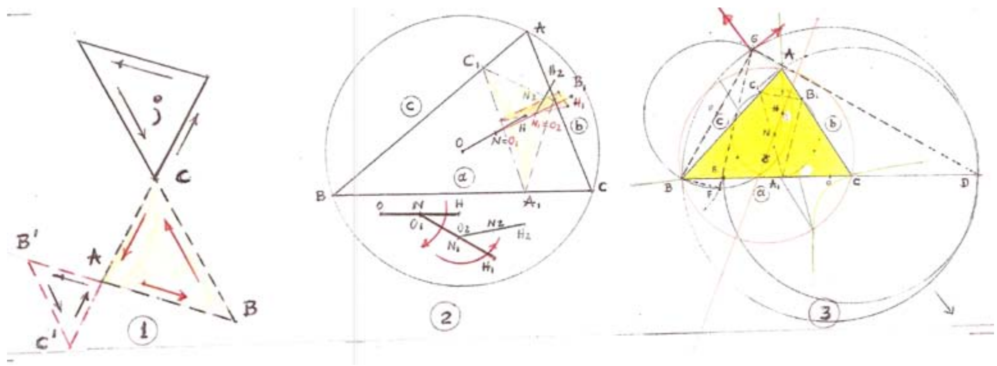
F.2(1) F.2(2) F.2(3) F.2(4)

e.b. The denominator of circum radius is of two variables $x = [(a+b)^2 - c^2]$ and $y = [c^2 - (a-b)^2]$ which have a fixed sum of lengths equal to ***4.ab*** and their product is to be made as large as possible. The product of the given variables, $x = [(a+b)^2 - c^2]$, $y = [c^2 - (a-b)^2]$ becomes maximum (this is proven) when ***x = y = 2.ab*** and exists on a family of hyperbolas curves where xy is constant for each of these curves . F.2(4)
All factors in denominator are conjugate and this is why orthogonal hyperbolas on any triangle are conjugate hyperbolas and follow Axial symmetry to their asymptotes .
The tangent on hyperbolas at point $x = 2.ab$ formulates 45 ° angle to x , y plane system .
Since Complex numbers spring from Euclidean geometry [11] then the same result follows from complex numbers also . It is holding ,
x.y $= \rho\rho' . [\cos(\varphi+\varphi') + i.\sin(\varphi+\varphi')]$ where ρ, ρ' are the modulus and φ, φ' the angles between the positive direction of the x- axis and x , y direction .
Product **x.y** becomes maximum when the derivative of the second part is zero as ,

$$-\sin(\varphi+\varphi') + i.\cos(\varphi+\varphi') = 0 \quad \rightarrow \quad \sin(\varphi+\varphi') = i.\cos(\varphi+\varphi') \quad \text{or}$$

$$\frac{\sin(\varphi+\varphi')}{\cos(\varphi+\varphi')} = \tan(\varphi+\varphi') = +i \ \text{ or } \rightarrow \ \tan^2(\varphi+\varphi') = -1 \quad \text{i.e.}$$

the slope of the tangent line (which is equal to the derivative) at point ***x = y = 2.ab*** is **- 1** and this happens for $(\varphi+\varphi') = 135°$ or 45°.



F.3(1) Central Symmetry

F.3(2) Vector AH

F.3(3) Central and Axial Symmetry

e.c. Let`s consider the length AH and the directions rotated through point A . F.1(1)
The right angles triangles HAB_1 , BB_1C are similar because all angles are equal respectively (angle $AHB_1 = B_1BC$ because $BB_1 \perp AB_1$ and $BC \perp AH$) , so

$$\frac{AH}{BC} = \frac{AB_1}{BB_1} \quad \text{or} \quad AH = \frac{BC . AB_1}{BB_1} = \frac{a.[c^2 + b^2 - a^2]\ 2.b}{2b.\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}} = \frac{a.[c^2 + b^2 - a^2]}{\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}}$$

In any triangle ABC is holding ---- F.1(1)

$a^2 = b^2+c^2 - 2.bc.\cos A \rightarrow 2.bc.\cos A = b^2+c^2 - a^2 \rightarrow \cos A = [b^2+c^2 - a^2] / 2.bc$
$b^2 = c^2+a^2 - 2.ac.\cos B \rightarrow 2.ac.\cos B = c^2+a^2 - b^2 \rightarrow \cos B = [c^2+a^2 - b^2] / 2.ca$
$c^2 = a^2+b^2 - 2.ab.\cos C \rightarrow 2.ab.\cos C = a^2+b^2 - c^2 \rightarrow \cos C = [a^2+b^2 - c^2] / 2.ab$ so

$$\mathbf{AH} = \frac{a.[c^2+b^2- a^2].[\sqrt{[(a+b)^2-c^2].[c^2-(a-b)^2]}}{[(a+b)^2-c^2].[c^2-(a-b)^2]} = \frac{a.[c^2+b^2-a^2].\sqrt{[x.y]}}{x.y} = \frac{2.abc.\sqrt{x.y}}{x.y} (\cos A)$$

where $x= (a+b)^2-c^2$, $y= c^2-(a-b)^2$

$$\mathbf{AH} = \left|\frac{2.abc.\sqrt{x.y}}{x.y}\right|. \cos A = \left|\frac{2.abc}{\sqrt{x.y}}\right|. \cos A = \textbf{Vector AH} \quad \ldots\ldots\ldots\ldots(13)$$

From F2(1) BC = a , AC = b , AB = c , CD = CE = CA = b , BD = BC+CD = a + b ,
BA = BG = c , BE = BF = BC - EC = a – b .

With point O as center draw circle with BD = a+b as diameter and with point B as center draw circle (B , BA = BG) intersecting circle (O , OB) at point G .
Using Pythagoras theorem in triangles BDG , BFG then $GD^2 = BD^2-BG^2 = [(a+b)^2-c^2] = \mathbf{x}$
and $GF^2 = GB^2 - BF^2 = c^2 - BE^2 = c^2 - (a-b)^2 = \mathbf{y}$ and it is holding
$x+y = a^2+b^2+2.a b -c^2 + c^2-a^2-b^2 + 2.a b = 4.a.b = GD + GF$ = constant
$x . y = [(a+b)^2 - c^2].[c^2 - (a-b)^2] = GD . GF$ = constant

Since Ortocenter H changes Position , following orthic triangles $A_1B_1C_1$, then Sector AH_n is altering magnitude and direction , so AH_n is a mathematical vector .

f. In F.2(1) GD ⊥ GB , GF ⊥ BF , therefore angle DGF = GBF so the sum of the pairs of two vectors [GD ,GE] , [BG , BE] is constant . Since addition of the two bounded magnitudes follows parallelogram rule , then their sum is the diagonal of the parallelogram .
In F2(2-4) the product of the pairs of the two vectors [GD ,GE] , [BG , BE] is constant in the direction perpendicular to diameter DE , so rectangular hyperbolas on the x , y system are 45 ° at the point x = y = 2.ab .

g. In F.3(2) orthic triangle $A_1B_1C_1$ of triangle ABC is circumscribed in Nine-point circle with center point N , *the middle of OH* , while point O is ABC`s circumcenter .
Since point N is the center of $A_1B_1C_1$ `s circumcenter , therefore Euler line OH is rotated through the middle point N of OH to all nested orthic triangles $A_nB_nC_n$ as is shown in F.3(2) , or are as

Sector (ONH) , ($N = O_1 N_1 H_1$) , ($N_1 = O_2 N_2 H_2$) → ($N_{n-1} = O_n N_n H_n$) → ($N_\infty = O_\infty N_\infty H_\infty$)

From vertices A,B,C and orthocenter H of triangle ABC passes rectangular circum-hyperbolas with point N as the center of the Nine-point circle . In Kiepert hyperbola , its center is the midpoint of Fermat points . In Jerabek rectangular hyperbola , its center is the intersection of Euler lines of the three triangles , of the four in triangle ABC . (the fourth is the orthic triangle).

h. In F.2(1) GD ⊥ GB and GF ⊥ BF therefore angle DGF = GBF or DGF + GBF = 360° [6] ie. angles DGF , GBF are conjugate . It is known that in an equilateral hyperbola , conjugate diameters make equal angles with the asymptotes , and because angle DGF = GBF then the sum of the two pairs of the two vectors (GD,GE) , (BG ,BE) is constant for all AH of orthic triangles .

To be vector **AH** a relative extreme (minimum or zero) , numerator of equation (13) must be zero in a fix direction to triangle ABC . Since a , b , c are constants , vector AH is getting extreme to the opposite direction by Central Symmetry through point A .

Verification :

C1. For a = b = c then $\cos A = [\,2a^2 - a^2\,] / 2.a^2 = a^2/2a^2 = 1/2$ therefore → A = 60°
$x = (a+b)^2 - c^2 = 4a^2 - a^2 = 3.a^2$, $y = c^2 - (a-b)^2 = a^2$
$x + y = 4.a^2$ and $x.y = 3.a^4$ __a
$AH = [\,2.a^3.\,a^2\,\sqrt{3}\,] / [\,3.a^4.2\,] =$ $\mathbf{a.\sqrt{3}/3}$ $= [a\sqrt{3}/2].(2/3) \rightarrow$ a | / = $a\sqrt{2}$ \ $a = a\sqrt{3}$ i.e.
Orthocenter Hn limits to → circumcenter O → (2 / 3) . AA1

C2. For b = c then $\cos A = [\,2b^2 - a^2\,] / 2.b^2 = [\,b.\sqrt{2}+a\,].[\,b\sqrt{2} - a\,] / 2.b^2$ ……. (13.a)

1. For cos A = 0 then $b\sqrt{2} - a = 0$ and this for A = 90° and **AH = 0** i.e.
Orthocenter Hn limits to Vertice A → AHn → A

Since also cos A becomes zero with $b\sqrt{2} = -a$ (The Central symmetry to a) then
Orthocenter Hn limits always to → Vertice A or AHn → A

2. For cos A = 1 / 2 then A = 60° , $x = 2.ab + a^2$, $y = 2.ab - b^2$ and b = a
$AH = [2.ab^2.\sqrt{x.y}].(1/2) / xy = [\,a.b^2 / \sqrt{x.y}\,] = (\,ab^2) / a\sqrt{3} = a/\sqrt{3} =$ $\mathbf{a\sqrt{3}/3}$ = AO i.e.
Orthocenter Hn limits to circumcenter O of triangle ABC or AHn → O .

3. For $\cos A = \sqrt{2}/2$ then A = 45° , $x = 2.ab + a^2$, $y = 2.ab - b^2$ and $4.la^2 = 4.b^2 - a^2$
$AH = [2.ab^2\sqrt{x.y}].(\sqrt{2}/2) / xy = [2.ab^2\sqrt{2}/\sqrt{x.y}] = [2.ab^2.\sqrt{2}] / \sqrt{(4.la^2)} = b^2/(la.\sqrt{2})$ i.e.

Orthocenter Hn limits to An on altitude la, or AHn → $b^2 / (la.\sqrt{2})$, and for $\cos A = la^2/b^2$
AHn = la i.e ***Orthocenter Hn limits to A1 , or AHn = la = AA1 = Altitude AA1 .***

4. For cos A = 1 then A = 0° , $x = a^2 + b^2 + 2.ab - b^2 = a.(\,a+2.b\,)$
$y = b^2 - a^2 + 2.ab - b^2 = a.(\,2.b - a\,)$ and

$AH = [2.ab^2] / [a.\sqrt{4.b^2 - a^2}] = [2.b^2] / [\sqrt{4.b^2 - a^2}\,]$ and for A = 0° then
$AH = [\,2.b^2] / 2.b = b$ i.e. Orthocenter Hn limits to vertices B or C (the two vertices coincide) . or ***Orthocenter Hn limits to B , C or AHn → B = C***

5. For cos A = - 1 then A = 180° , $x = a.(\,a+2.b\,)$ $y = a.(\,2.b - a\,)$ and a = 2.b

$AH = [2.b^2](-1) / [\sqrt{4.b^2 - (2.b)^2}] = [-2.b^2] / [\infty] = 0$ i.e point A coincides with the foot A_1 or AA1 = 0 , ***Orthocenter Hn limits to B , C or AHn → B = C***

C3. The first Numerator term of equation (13) and in F.2(3) is , $c^2 + b^2 - a^2$, and this *in order to be* on a right angle triangle ABC with angle < A = 90 ° , must be zero , or $a^2 = b^2 + c^2$.
The second term is $x = (a+b)^2 - c^2 \neq 0$, $a^2 + b^2 + 2.ab - c^2 = b^2 + c^2 + b^2 + 2.ab - c^2 =$ $2.[\,ab + b^2] \neq 0$ or $a + b \neq 0$ = constant → (b/a) < 0 i.e. H → A , and for $y = c^2 - (a-b)^2 \neq 0$, $c^2 - a^2 - b^2 + 2.ab = c^2 - b^2 - c^2 - b^2 + 2.ab = 2.[\,ab - b^2] \neq 0$ or $a - b \neq 0$ = constant , and (b/a) > 0 i.e. H → A .

Geometrically , since B_1C_1 of orthic triangle $A_1B_1C_1$ coincides with point A , so the center N of the nine-point circle on ABC is always on OA . Since point N is $A_1B_1C_1$ \` s circum-center and represents the new $O_1$ then the new $N_1$ of $A_1B_1C_1$ \`s is the center point on NA or $NH_1 = NA / 2$
Therefore point N is on OA direction at (AO/2) , (AO/4) , (AO/8) ,,,, ($AO/2^a$) points , i.e.

In any right-angle triangle ABC where angle $< A = 90°$ *, the locus of the orthocenter points* $H_1 \ldots\ldots H_n$ *of the orthic triangles* $A_n B_n C_n$ *, is on line OA , and for* $n = \infty$ *then convergent to point A (vertice A) of the triangle ABC .*

so , In an equilateral triangle ABC where $a = b = c$ ***, Orthocenter H is fixed at Centroid K which coincides with Circum center O , and the Nine-point center N .***
In an Isosceles triangle ABC where a , b = c , Orthocenter H moves on altitude AA_1 ***(this is the locus of the orthocenter points*** $H_1 \ldots H_n$ ***) and for angle A*** $= 90°$ ***then convergent to the point A of the triangle .***

Remark : In any triangle ABC rectangular hyperbolas follow Axial symmetry to their Asymptotes , in contradiction to orthocenter H , which follows Central Symmetry and Rotation through point A , the vertice opposite to the greatest side of the triangle .
This Springs out of the logic of Spaces , Anti-Spaces , Sub-Spaces of any first dimentional Unit $\mathbf{ds} > \mathbf{0}$. [11] , therefore vector AH_n is limiting to the Orthocenters $H_1 \ldots H_n$ of orthic triangles in triangle ABC . (Equation 13) . Conics through the vertices of triangle ABC that are not passing through Orthocenter H are not rectangular hyperbolas .
A further geometrical analysis follows .

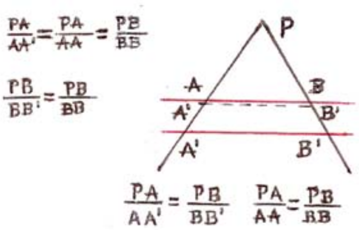
Extremum Principle or Extrema : 20 / 1 / 2012

All Principles are holding on any Point A .
For two points A , B not coinciding , exists Principle of Inequality which consists another quality . Any two Points exist in their Position under one Principle , *Equality and Stability in Virtual displacement which presupposes Work in a Restrain System*) . [11]
This Equilibrium presupposes homogenous Space and Symmetrical Anti-Space .
For two points A , B which coincide , exists *Principle of Superposition* which is a Steady State containing Extrema for each point separately .
Extrema , for a *point A* is the Point , for a *straight line* the infinite points on line , either these coincide or not or these are in infinite , and for *a Plane* the infinite lines and points with all combinations and Symmetrical ones ***, i.e. all Properties of Euclidean geometry , compactly exist in Extrema Points , Lines , Planes , circles .***
Since Extrema is holding on Points , lines , Surfaces etc , therefore all their compact Properties (Principles of Equality , Arithmetic and Scalar , Geometric and Vectors , Proportionality , Qualitative , Quantities , Inequality , Perspectivity etc) , exist in a common context .
Since a quantity is either a vector or a scalar and by their distinct definitions are ,

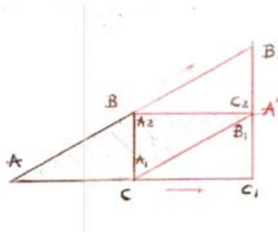
Scalars , are quantities that are fully described by a magnitude (or numerical value) alone ,
Vectors , are quantities that are fully described by both magnitude and a direction , therefore

In Superposition magnitude AB is equal to zero and direction , *any direction* , $\neq 0$ i.e.

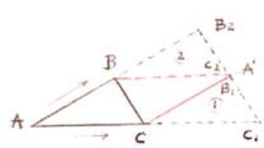
Any Segment AB between two points A , B consist a Vector described by the magnitude AB and directions ÃB , BÃ and in case of Superposition ÃA , AÃ . i.e. Properties of Vectors , Proportionality , Symmetry , etc exists either on edges A , B or on segment AB as follows :



A . Thales Theorem . F4.(1) F4.(1)



F4.(2)



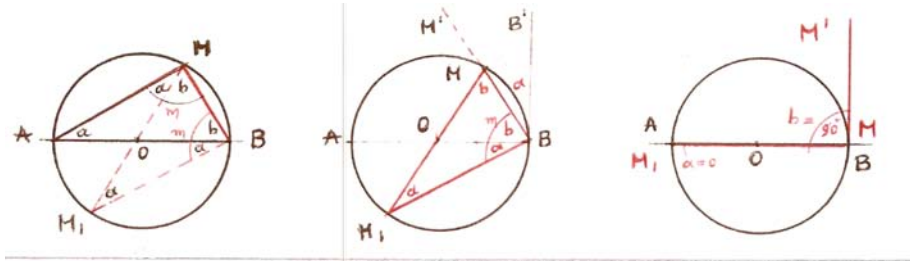
F4.(3)

According to Thales F4.(1), if two intersecting lines PA , PB are intercepted by a pair of Parallels AB // A′B′ , then ratios PA/AA′ , PB/ BB′ , PA/PA′ , PB/ PB′ of lines , or ratios in similar triangles PAB , PA′B′ are equal or λ = [PA / AA′] = [PB / BB′] .
In case line A′ B′ coincides with AB , then AA′ = AA , BB′ = BB , i.e. exist **Extrema** and then λ = [PA / AA] = [PB / BB] , (Principle of Superposition).

B. Rational Figured Numbers of Figures F4.(2 - 3) .

The definition of " Heron " that gnomon is as that which , when added to anything , a number or figure , makes the whole similar to that to which it is added . (Proportional Principle) .
It has been proven that the triangle with sides twice the length of the initial , preserves the same angles of the triangle . [6] i.e. exists **Extrema** for Segment BC at B_1C_1 .

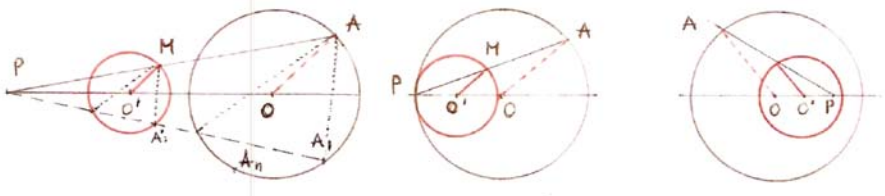
C. A given Point **M** and any circle with AB as diameter . [6] .



F5.(1) F5.(2) F5.(3)

It has been proved [6] that angle AMB = 90° for all points of the circle . This when point M is on B where then exists **Extrema** at point B for angle AMB . (Segments MA , MB) i.e. Angle AMB = AMO + OMB = $AMM_1 + M_1MB$ = ABA + ABB = 0 + 90° = 90° …. F5.(3)

D. A given Point **P** and any circle (O , OA) . [6]



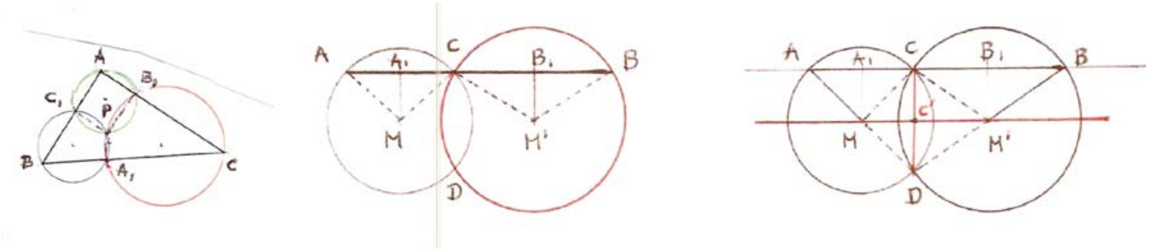
F6.(1) F6.(2) F6.(3)

It has been proved that the locus of midpoints M of any Segments PA , is a circle with center O′ at the middle of PO and radius O′ M = OA / 2 . (F6)
Extending the above for point M to be any point on PA such that PM / PA = a constant ratio equal to λ , then the locus of point M is a circle with center O′ on line PO , PO′ = λ .PO , and radius O′ M = λ . OA . **Extrema** exists for points on the circle .

Remarks :
1. In case PO′ = λ . PO and O′M = λ . PO and this is holding for every point on circle , so also for 2 , 3 ,.. n points i.e. any Segment AA_1 , triangle AA_1A_2 , Polygon $AA_1 \ldots A_n$,… circle, is represented as a Similar Figure (*Segment , triangle , Polygon or circle*) .

2. In case PO′ = λ . PO and $PM_1 = PA . \lambda_1$, then for every point A on the circle (O , OA) , in or out the circle , exist a point M_1 such that line AM_1 passes through point P . i.e. every segment AA_1 Triangle AAA_2 , Polygon $AA_1 \ldots\ldots A_n$, or circle , is represented as segment MM_1M_2 , Polygon $MM_1 \ldots\ldots M_n$ or a closed circle [Perspective or Homological shape] . F6.(1)

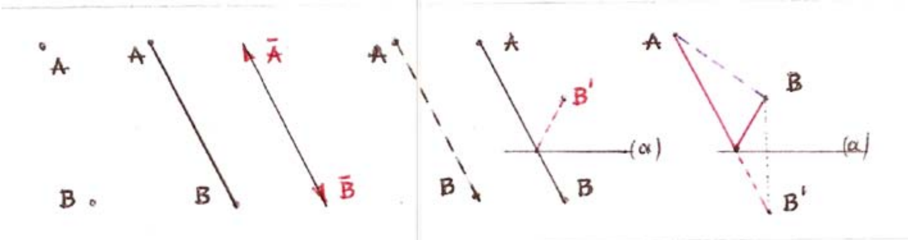
E. The Parallel Postulate . [6] .



F7.(1) F7.(2) F7.(3)

Any point M , not coinciding with points A , B consists a Plane (the Plane MAB) and from point M passes only one Parallel to AB .This parallel and the symmetrical to AB is the **Extrema** because all altitudes are equal and of minimum distance . When point M lies on line AB then parallel is on AB and all properties of point P are carried on AB and all of M on line AB . F7.(2-3)

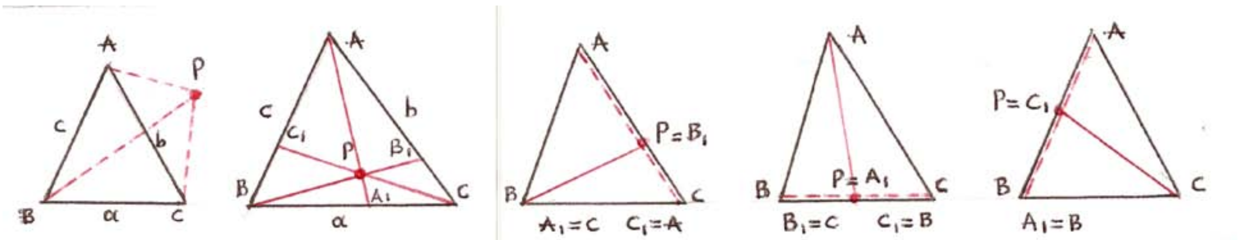
F. Symmetry (Central , Axial) is Extrema F.8



F8.(1) F8.(2) F8.(3) F8.(4)

Since any two points A,B consist the first dimentional Unit (magnitude AB and direction ÃB ,B Ã) F8.(1-2) equilibrium at the middle point of A , B , *Central Symmetry* . Since the middle point belongs to a Plane with infinite lines passing through and in case of altering central to axial symmetry ,then equilibrium also at *axial Symmetry* F8.(3), so Symmetrical Points are in **Extrema.** *Nature* follows this property of points of Euclidean geometry (*common context*) as Fermat`s Principle for Reflection and Refraction . F8.(4) .

G. A point P and a triangle ABC . (F9)



F9.(1) F9.(2) F9.(3) F9.(4) F9.(5)

If a point P is in Plane ABC and lines AP , BP , CP intersect sides BC , AC , AB , of the triangle ABC at points A_1 ,B_1 ,C_1 respectively F9. (1–2) , then the product of ratios λ is , $\lambda = (AB_1 / B_1C) . (CA_1 / A_1B) . (BC_1 / C_1A) = 1$, and the opposite . Proof :

Since **Extrema of Vectors** exist on edge Points A ,B ,C , and lines AC , BC , AB then for ,

a. Point P on line AC , F9.(3) → $A_1 = C$, $B_1 = P$, $C_1 = A$ and for
$\lambda = (AB_1 / B_1C).(CA_1 / A_1B).(BC_1 / C_1A) \rightarrow \lambda_1 = (PA / PC).(CC / BC).(AB / AA)$

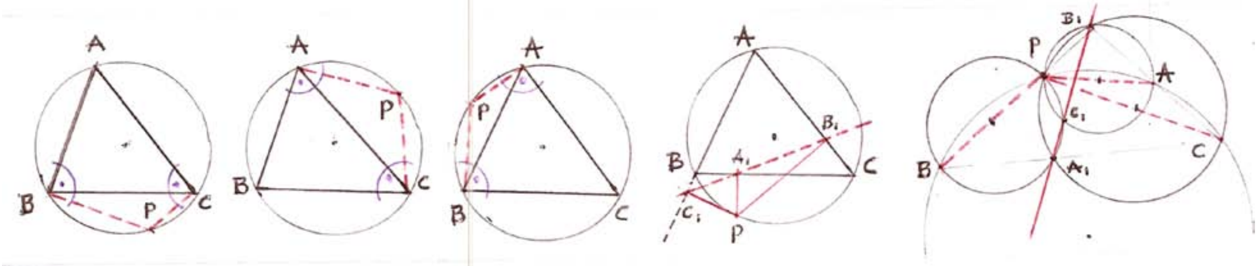
b. Point P on line BC , F9.(4) → $A_1 = P$, $B_1 = C$, $C_1 = B$ and for
$\lambda = (AB_1 / B_1C).(CA_1 / A_1B).(BC_1 / C_1A) \rightarrow \lambda_2 = (AC / CC).(PC / PB).(BB / AB)$

c. Point P on line BA , F9.(5) → A_1 = B , B_1 = A , C_1 = P and for
$\lambda = (AB_1 / B_1C).(CA_1 / A_1B).(BC_1 / C_1A)$ → $\lambda_3 = (AA / AC).(BC / BB).(PB / PA)$ and for

$\lambda_1 . \lambda_2 . \lambda_3$ = [PA.CC.AB.AC.PC.BB.AA.BC.PB] : [PC.BC.AA .CC.PB.AB .AC.BB.PA] = 1

i.e. Menelaus *Theorem* , and for Obtuse triangle $\lambda_1 . \lambda_2 . \lambda_3 = -1$ → Ceva`s Theorem

H. A point P a triangle ABC and the circumcircle .



F10.(1) F10.(2) F10.(3) F10.(4) F10.(5)

a. Let be A_1 ,B_1 ,C_1 the feet of the perpendiculars (altitudes are the Extrema) from any point P to the side lines BC , AC , AB of the triangle ABC , F10.(4)
Since Properties of Vectors exist on lines AA_1 , BB_1 ,CC_1 then for **Extrema Points** ,

F10.(1) → Point A_1 on points B , C of line BC respectively , formulates perpendicular lines BP ,CP which are intersected at point P . Since Sum of opposite angles PBA+PCA = 180° therefore the quadrilateral PBAC is cyclic on circumcircle of triangle ABC and also the feet of the perpendiculars of point P on sides of triangle ABC are points B , C .

F10.(2) → Point B_1 on points A , C of line AC respectively , formulates perpendicular lines AP ,CP which are intersected at point P . Since Sum of opposite angles PAB+PCB = 180° therefore the quadrilateral PABC is cyclic on circumcircle of triangle ABC and also the feet of the perpendiculars of point P on sides of triangle ABC are points A , C .

F10.(3) → Point C_1 on points A , B of line AB respectively , formulates perpendicular lines AP ,BP which are intersected at point P . Since Sum of opposite angles PAC+PBC = 180° therefore the quadrilateral PACB is cyclic on circumcircle of triangle ABC and also the feet of the perpendiculars of point P on sides of triangle ABC are points A , B .

So **Extrema** for the three sides of the triangle is point P which is on the circumcircle of the triangle and the feet A_1 ,B_1 ,C_1 of the perpendiculars of point P on sides of triangle ABC are respectively on sides BC , CA , AB i.e. on a line **. [** F10.4] *Simsons line.*
Since altitudes PA_1 , PB_1 , PC_1 are also perpendiculars on the sides BC,AC,AB and segments PA,PB,PC are diameters of the circles,then points A_1,B_1,C_1 are collinear.[F10.5] . *Saimon Theorem*

b. Let be A_1 ,B_1 ,C_1 any three points on sides of a triangle ABC (these points are considered **Extrema** because they maybe on vertices A , B , C or to ∞) . F7.(1) . If point P is the common point of the two circumcircles of triangles AB_1C_1 , BC_1A_1 , (a vertex and two adjacent sides) then circumcircle of the third triangle CA_1B_1 passes through the same point P .
Proof :
Since points A , B_1 , P , C_1 are cyclic then the sum of opposite angles BAC + B_1PC_1 = 180°.
Since points B , C_1 , P , A_1 are cyclic then the sum of opposite angles ABC + C_1PA_1 = 180°.
and by Summation BAC + ABC + (B_1PC_1 + C_1PA_1) = 360 ° ……….. (1)
Since BAC+ABC+ACB = 180 ° then BAC+ABC = 180° - ACB and by substitution to (1)
(180° - ACB) + 360 - A_1PB_1 = 360° or , the sum of angles ACB + A_1PB_1 = 180° , therefore points C , B_1 , P , C_1 are cyclic i.e.

Any three points A_1, B_1, C_1 on sides of triangle ABC, forming three circles determined by a Vertex and the two Adjacent sides, meet at a point P. (Miguel`s Theorem)

c. In case angle $PA_1B = 90°$ then also $PA_1C = 90°$. Since angle $PA_1C + PB_1C = 180°$ then also $PB_1C = 90°$ (*angles PA_1B, PC_1B, PB_1C are extreme of point P*).

d. ABC is any triangle and A_1, B_1, C_1 any three points on sides opposite to vertices A, B, C Show that Perimeter $C_1B_1 + B_1A_1 + A_1C_1$ is minimized at Orthic triangle $A_1B_1C_1$ of ABC. F7(1)

Since point P gets Extrema on circumcircle of triangle ABC, so sides A_1B_1, B_1C_1, C_1A_1 are **Extrema** at circumcircle determined by a vertex and the two adjacent sides. Since adjacent sides are determined by sides AA_1, BB_1, CC_1 then maximum exists on these sides. Proof :

In triangle AA_1B, AA_1C the sum of sides p1
$p1 = (AA_1+A_1B+AB) + (AA_1+AC+A_1C) = 2.(AA_1) + AB+AC+BC = 2.(AA_1) + a+b+c$

In triangle BB_1A, BB_1C the sum of sides p2
$p2 = (BB_1+B_1A+AB) + (BB_1+BC+B_1C) = 2.(BB_1) + AB+AC+BC = 2.(BB_1) + a+b+c$

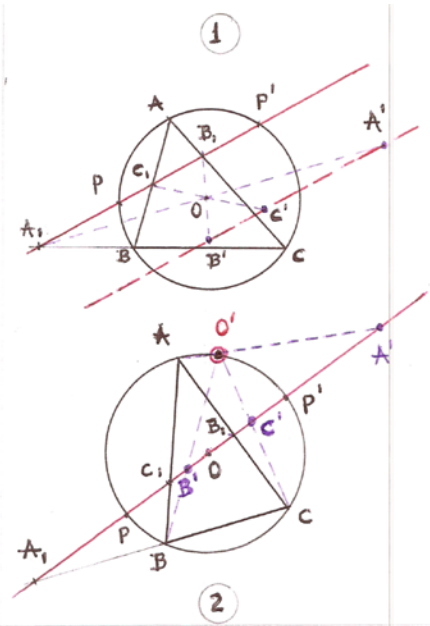
In triangle CC_1A, CC_1B the sum of sides p3
$p3 = (CC_1+C_1A+AB) + (CC_1+AC+C_1C) = 2.(CC_1) + AB+AC+BC = 2.(CC_1) + a+b+c$

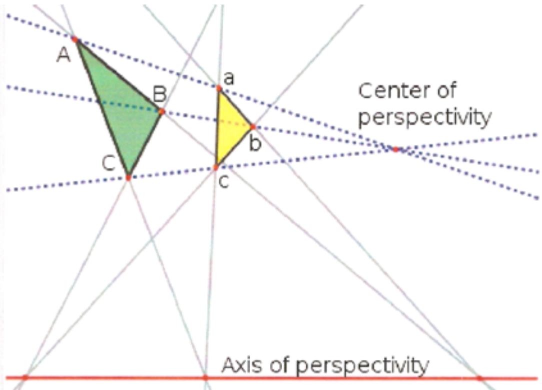
The sum of sides **p** $= p1 + p2 + p3 = 2.[AA_1+BB_1+CC_1] + 3.[a+b+c]$ and since $a+b+c$ is constant then **p** becomes minimum when $AA_1+BB_1+CC_1$ or when these are the altitudes of the triangle ABC, where then are the vertices of orthic triangle. ***i.e.***

The Perimeter $C_1B_1 + B_1A_1 + A_1C_1$ of orthic triangle $A_1B_1C_1$ is the minimum of all triangles in ABC.

I. Perspectivity :

In Projective geometry, (*Desargues` theorem*), two triangles are in perspective *axially*, if and only if they are in perspective *centrally*. Show that, Projective geometry is an **Extrema** in Euclidean geometry.





Perspective triangles. Corresponding sides of the triangles, when extended, meet at points on a line called the axis of perspectivity. The lines which run through corresponding vertices on the triangles meet at a point called the center of perspectivity. Desargues' theorem guarantees that the truth of the first condition is necessary and sufficient for the truth of the second. 3

F11.(1) F11.(2) F11.(3)

Two points P , P′ on circumcircle of triangle ABC , form **Extrema** on line PP′ . Symmetrical axis for the two points is the mid-perpendicular of PP′ which passes through the centre O of the circle , *therefore* , Properties of axis PP′ are transferred on the Symmetrical axis in rapport with the center O (central symmetry) , *i.e.* the *three points of intersection A_1 ,B_1 ,C_1 are Symmetrically placed as A′ , B′ , C′ on this Parallel axis .* F11.(1)

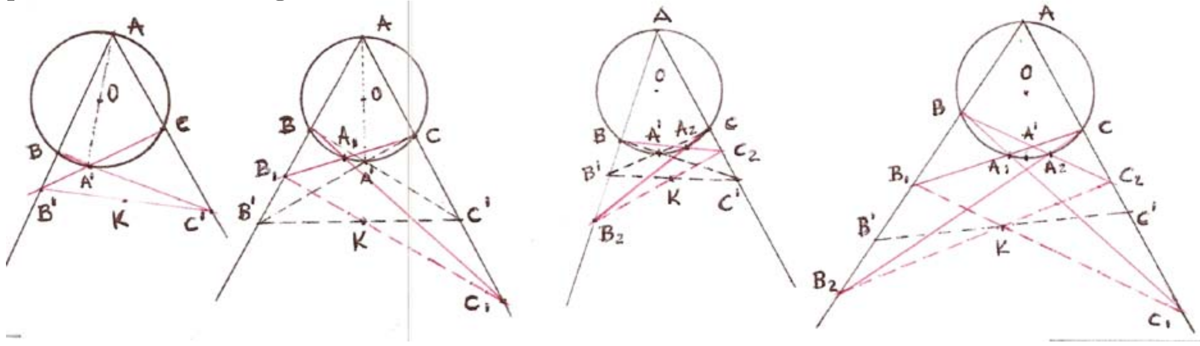
a. In case points P , P′ are on **any diameter** of the circumcircle F11.(2) , then line PP′ coincides with the parallel axis , the points A′ , B′ , C′ are Symmetric in rapport with center O , and *the Perspective lines AA′ , BB′ , CC′ are concurrent in a point O′ situated on the circle .*
When a pair of lines of the two triangles (ABC , abc) are parallel F11.(3) , where the point of intersection recedes to infinity , axis PP′ passes through the circumcenters of the two triangles , (Maxima) and is not needed “ ***to complete*** *“ the Euclidean plane to a projective plane .i.e.*
Perspective lines of two Symmetric triangles in a circle , on the diameters and through the vertices of corresponding triangles , are concurrent in a point on the circle .

b. *When all pairs of lines of two triangles are parallel ,* equal triangles *, then points of intersection recede to infinity , and axis PP` passes through the circumcenters of the two triangles* (*Extrema*).

c. *When second triangle is a point* ***P*** *then axis PP′ passes through the circumcenter of triangle .*

Now is shown that *Perspectivity* exists between a triangle ABC , a line PP′ and any point P where then exists Extrema , i.e. ***Perspectivity in a Plane is transferred on line and from line to Point This is a compact logic in Euclidean geometry which holds in Extrema Points .***

J. A point A_1 and triangle ABC in a circle of diameter AA′ .



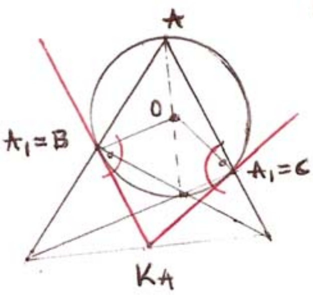
F12.(1) F12.(2) F12.(3) F12.(4)

In F12 . Lines CA′, BA′ produced intersect lines AB , AC at points B′, C′ respectively .
A_1 is any point on the circle between points B , A′.
CA_1 , BA_1 produced intersect lines AB , AC at points B_1 , C_1 respectively .

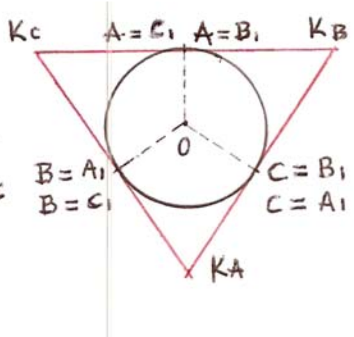
Show that lines B_1C_1 are concurrent at the circumcenter K of triangles CC′B′ , BB′C′ .

Proof :
Angle C′CA′= C′ CB′ = 90° therefore circumcenter of triangle CC′B′ is point K , the middle point of B′C′
Angle B′BA′= B′BC′ = 90° therefore circumcenter of triangle BB′C′ is point K , the middle point of B′C′
Considering angle < C′CA′= 90° as constant then all circles passing through points C , A′, C′ have their center on KC .
Considering angle B′BA′= 90° as constant then all circles passing through points B , A′, B′ have their center on KB .
Considering both angles < C′CA′= B′BA′= 90° then lines BA′, CA′ produced meet lines AB′, AC′ at points B′, C′ such that line B′C′ passes through point K (*common to KC , KB*)
Extrema for points A_1 ,A_2 are points C, B and in case of points B ,C coincide with point A
i.e. line KA is Extrema line for lines BA , CA .

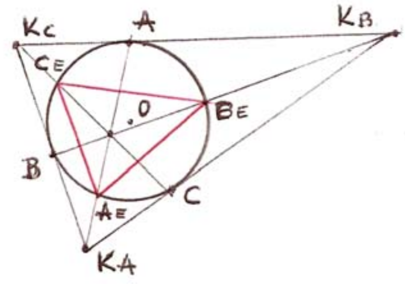
On the contrary , In any angle < BAC of triangle ABC exists a constant point K_A such that all lines passing through this point intersect sides AB , AC at points C_1 , B_1 so that internal lines CB_1 , BC_1 concurrence on the circumcircle of triangle ABC and in case this point is A , then concurrence on line AK_A . F12 . (4) , F13 .



F13.(1)



F13.(2)



F13.(3)

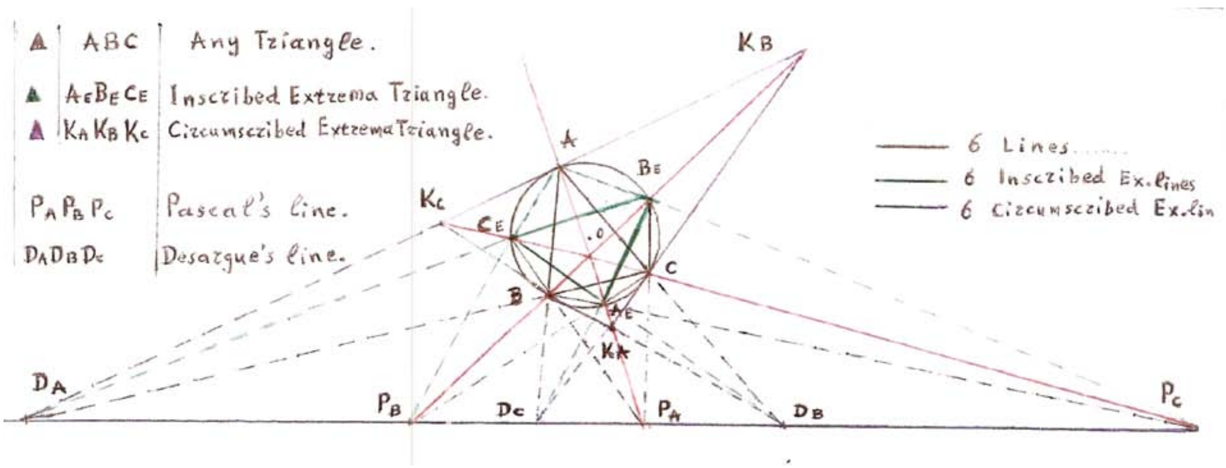
a) When point A_1 is on point B (Superposition of points A_1 , B) then line BA_1 is the tangent at point B , where then angle < OBK_A = 90° . When point A_1 is on point C (Superposition of points A_1 , C) then line CA_1 is the tangent at point C , where then angle < OCK_A = 90° . F13.(1) Following the above for the three angles BAC , ABC , ACB F13.(2) then ,

K_A B , K_AC are tangents at points B and C and angle < OBK_A = OCK_A = 90 °.
K_B C , K_BA are tangents at points C and A and angle < OCK_B = OAK_B = 90° .
K_C A , K_CB are tangents at points A and B and angle < OAK_C = OBK_C = 90° .

Since at points A , B , C of the circumcircle exists only one tangent then ,
The sum of angles OCK_A + OCK_B = 180° therefore points K_A , C , K_B are on line K_AK_B .
The sum of angles OAK_B + OAK_C = 180° therefore points K_B , A , K_C are on line K_BK_C .
The sum of angles OBK_C + OBK_A = 180° therefore points K_C , B , K_A are on line K_AK_C . i.e.

Circle (O , OA = OB = OC) is inscribed in triangle $K_AK_BK_C$ and circumscribed on triangle ABC .

b) Theorem : *On any triangle* **ABC** *and the circumcircle exists one inscribed triangle* **$A_EB_EC_E$** *and another one circumscribed* ***Extrema*** **triangle** **$K_AK_BK_C$** *such that the* ***Six*** *points of intersection of the six pairs of* ***triple lines*** *are collinear* .→ (3+3) . 3 = 18



F.14. ***The Six , Triple Points , Line*** → ***The Six , Triple Concurrency Points , Line***
Proof :
F13. (1 – 2 – 3) , F14
Let A_E ,B_E ,C_E are the points of intersection of circumcircle and lines AK_A ,BK_B ,CK_C respectively. Since circle (O ,OA) is incircle of triangle $K_AK_BK_C$ therefore lines AK_A ,BK_B ,CK_C concurrence.

1. When points A_1 , A coincide , then internal lines CB_1 , BC_1 coincide with sides CA , BA , so line K_AA is constant . Since point A_E is on ***Extrema line*** AK_A then lines C_EB ,B_EC concurrent on line AK_A . The same for tangent lines K_AK_B , K_AK_C of angle < $K_BK_AK_C$.
2. When points A_1 , B coincide , then internal lines CA_1 , AC_1 coincide with sides CB , AB , so line K_BB is constant . Since point B_E is on ***Extrema line*** BK_A then lines A_EC ,C_EA concurrent on line BK_B . The same for tangent lines K_BK_C , K_BK_A of angle < $K_CK_BK_A$.
3. When points A_1 , C coincide , then internal lines AB_1 , BA_1 coincide with sides AC , BC , so line K_CC is constant . Since point C_E is on ***Extrema line*** CK_C then lines B_EA ,A_EB concurrent on line CK_C .The same for tangent lines K_CK_A , K_CK_B of angle < $K_AK_CK_B$, ***i.e.***

Since Perspective lines , on Extrema triangles $A_EB_EC_E$, $K_AK_BK_C$ ***concurrent , and since also Vertices A , B , C of triangle ABC lie on sides of triangle*** $K_AK_BK_C$ ***, therefore all corresponding lines of the three triangles when extended concurrent , and so the three triangles ABC ,*** $A_EB_EC_E$, $K_AK_BK_C$ ***are Perspective between them .***

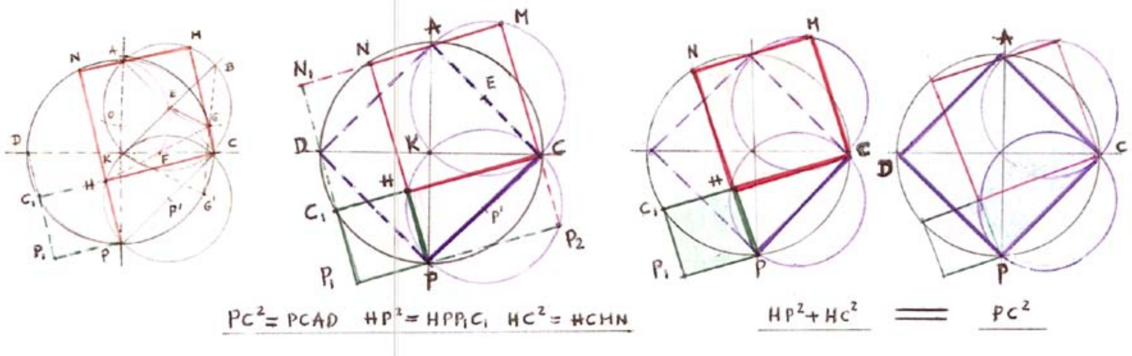
In triangles ABC , $A_EB_EC_E$ all corresponding Perspective lines concurrent on AK_A , BK_B , CK_C lines , and concurrency points P_A , P_B , P_C collinear . The pairs of Perspective lines [AA_E , CB_E , BC_E] , [BB_E , CA_E , AC_E] , [CC_E , AB_E , AB_E] concurrent in points P_A , P_B , P_C respectively and lie on a line .
In triangles ABC , $K_AK_BK_C$ vertices A ,B ,C lie on triangle`s $K_AK_BK_C$ sides , therefore all corresponding Perspective lines concurrent in points D_A ,D_B ,D_C respectively (sides of triangle $K_AK_BK_C$ are tangents on circumcircle) . The pairs of Perspective lines [K_BA , CB , C_EB_E] , [K_AB , AC , A_EC_E] , [K_BC, BA , B_EA_E] , concurrent in points D_A , D_B , D_C respectively .

In triangles $A_EB_EC_E$, $K_AK_BK_C$, and ABC corresponding Perspective lines concurrent in the same points D_A , D_B , D_C because triangles` $K_AK_BK_C$ sides are tangent on circumcircle , so **Extrema** lines K_BK_C , K_CK_A , K_AK_B for both sets of sides lines [BC , B_EC_E , K_BK_C] , [AC , A_EC_E ,K_AK_C] [AB , A_EB_E ,K_AK_B] concurrent in points of the same line . i.e.

This compact logic of the points **[A , B , C] , [** A_E **,** B_E **,** C_E **] , [** K_A **,** K_B **,** K_C **]** ***when is applied on the three lines*** K_AK_B **,** K_AK_C **,** K_BK_C ***, THEN THE SIX pairs of the corresponding lines which extended are concurrent at points*** P_A **,** P_B **,** P_C ***for the triple pairs of lines*** **[** AA_E **,** BC_E **,** CB_E **] [** BB_E **,**CA_E **,**AC_E**] , [** CC_E **,**AB_E **,**BA_E **]** ***and at points*** D_A **,** D_B **,** D_C ***for the other Triple pairs of lines*** **[CB ,** K_BA **,** B_EC_E **] , [AC ,** K_AB **,** C_EA_E **] , [BA ,** K_BC **,** A_EB_E **] ,** ***lie on a straight line .***

Remarks :
1. Since Inscribed Extrema Triangle [A_E,B_E,C_E] defines , axis AA_E ,BB_E ,CC_E of Projective geometry and Circumscribed Extrema Triangle [K_A,K_B,K_C] , axis K_BK_C , K_AK_C , K_AK_B of Perspective geometry therefore , Projective and Perspective geometry are Extrema in Euclidean geometry i.e. an Extension of the two fundamental branches of geometry .

c) . Pythagoras`s Theorem and the Mould (*Machine*) of changeable Squares . 25 / 2 / 2012



F15.(1) F15.(2) F15.(3) F15.(4)

It has been proved [5] , F15.(1) that any two equal and Perpendicular first dimentional Units CA ,CP and the hypotenuse AP , formulate the two equal circles with diameters CA ,CP and the circumscribed circle (K ,KA) of triangle CAP , (*Machine* $CA \perp CP = CA$) , which forms on the three circles ,{ *for any point* ***H*** *on circle (P′ ,P′C)* }, **the rectangle HNMC which is Square , (** HNMC = HC^2 **) and the Square** $HPP_1C_1 = HP^2$**.**

Show that : $PC^2 = HP^2 + HC^2$ or → Rectangle CADP = HCMN + HPP_1C_1

Proof :
Complete Rectangles $PNMP_2$, $PHCP_2$ on the circle (P′, P′ P) and HNN_1C_1 , PNN_1P_1 on the circle (K , KC = KD) and the squares HPP_1C_1 , HNMC in the three circles . F15.(2)
Rightangled triangles CMA , CP_2P , DP_1P, DN_1A , are equal between them because these have equal their two sides , (CA = CP = DP = DA *and* CM = PP2 = DP1 = AN1) , therefore other sides $AM = CP_2 = PP_1 = DN_1$, and areas of triangles CAM , CPP_2 , DPP_1 , DAN_1 are also equal .

1. Since HC = HN then Rectangles HCP_2P , HNN_1C_1 are equal , and both equal to 2. CHP.
2. In square $MP_2P_1N_1$ → Square CADP = Square $MP_2P_1N_1$ – 4. Triangles CHP .
3. In square $MP_2P_1N_1$ → Square HCMN + Square HPP_1C_1 = Square $MP_2P_1N_1$ – [2. Rectangles $PHCP_2$ = 4. Triangles CHP] = Square $MP_2P_1N_1$ – 4. Triangles CHP … i.e.

Square CADP = Square HCMN + Square HPP_1C_1 or → $PC^2 = HC^2 + HP^2$
which is Pythagoras`s theorem .

Remarks :

a. In machine $CA \perp CP$, CA = CP of the two equal circles (E , EA) ,(P′, P′C) and the circumcircle (K , KA) of the triangle ACP , are drawn two Changeable Squares HNMC , HPP_1C_1 , which Sum is equal to the inscribed one CADP , or $PC^2 = HC^2 + HP^2$.

b.**Extrema** exists at edge points **P , C** where then point **H** coincides with points P and C , and $PC^2 = PP^2 + PC^2 = PC^2$ and $PC^2 = PC^2 + CC^2 = PC^2$, and when coincides with K , then the Sum of the two Changeable Squares HNMC, HPP_1C_1 is $KC^2 + KP^2 = 2.\Delta$ (KCA)+2.Δ (KCP) = 2.Δ(KCA+ KCP) = CADP = PC^2 .

c.The Position of the three Points only , defines Quantities (sides CP , HP , HC) , and Qualities (triangle PCH , Squares PDAC , HPP_1C_1 , HCMN and any classification of numbers is included).

d. It has been proved [5] that the two equal and perpendicular Units CA , CP, (in plane ACP) construct the Isosceles rightangled triangle ACP and the three circles on the sides as diameters . From any point **H** on the first circle are conscructed Squares HNMC , HPP_1C_1 with vertices on the three circles . This Plane Geometrical Formation is a , **mooving Machine** , and is called < **Plane Formation of Constructing Squares** >. *The same analogues exist in Space and is called,* **< Space Formation of Constructing Cubes ,** and for Cubes $HO^3 = HP^3 + HC^3 + HJ^3$.

The Plane System of triangle ACP with the three circles on the sides as diameters consists ***the Steady Formulation*** **,** and squares HNMC , HPP_1C_1 , ***is the moving Changeable Formulation of this twin Supplementary System*** **,** (*The Sum of Areas of the two Changeable Squares is constant and equal to that of Hypotenuse*) . i.e. ***The Total Area is conserved on the Changable System*** .

e. Pythagoras theorem defines the relation between the three sides of the rightangled triangle HCP which is an equality of Squares . The Sum of the Quantities, *Moving Squares HNMC , HPP_1C_1* , is constant and equal to that of PCAD , which is the Conservation of Areas in motion .i.e.

In Mechanics , the fundamental laws of Conservation of Energy is not Energy`s Property , but of Geometry`s (Since the Total Area , *quantity* ***, Conserves on the Changeable System).***

The Position of Points only , defines Properties and Laws of Universe (Quantities and Qualities). Conservation laws are the most fundamental principle of Mechanics . The Conserved Quantities (Energy , Momentum , Angular Momentum etc.) follow the Pythagoras theorem , applied on the moving machine [CA ⊥ CP , CA = CP] . The parallel transformations are ,

Kinetic Energy (KE) + Potential Energy (PE) = Total Energy .

Changeable Square HNMC + Changeable Square HPP_1C_1 = Total Area of Square CADP .

f. In applying the method on the three dimension Space (*Four points = Three equal and perpendicular First dimensional Units* OA , OB , OC *construct an Isosceles Orthogonal System with four Spheres , three on the Units and the fourth on the Resultant as diameters*) , exists a Changeable Space Formation such that the Sum of the Volumes of Spheres to be equal to that of the Resultant , meaning that , conservation of Volumes exists on the three Perpendicular Units. Energy (*Motion*) follows this Euclidean Mould , because this Proposition (*Principle*) belongs to geometry`s , *and not to Energy`s which is only motion .*

Open Problem ? Which logic exists on this moving machine such that the area of the changeable Square *or changeable Cube* , to be equal to that of the circle , *or Cube* ?

References :

[1] The great text of J . L .Heiberg (1883-1886) and the English translation by Richard Fitzpatrick .
[2] ELEMENTS BOOK 1
[3] WIKIPEDIA , THE FREE ENCYCLOPEDIA .
[4] GREEK MATHEMATICS by, Sir Thomas L .Heath - Dover Publications, Inc , New York. 63-3571.
[5] A SIMPLIFIED APPROACH OF SQUARING THE CIRCLE . (**MELAN.doc**)
[6] THE PARALLEL POSTULATE IS DEPENDED ON THE OTHER AXIOMS (**EUCLID.doc**)
[7] MEASURING REGULAR POLYGONS AND HEPTAGON IN CIRCLE **(REGULAR.doc**)
[8] THE TRISECTION OF ANY ANGLE . (**TRISECTION.doc**)
[9] THE EUCLIDEAN PHILOSOPHY OF UNIVERSE . (**EUCLID .Spaces.doc**) .
[10] UNIVERSE ORIGINATED **not** WITH BIG BANG(**EUCLID .not.BIG BANG .doc**)
[11] COMPLEX NUMBERS , QUANTUM MECHANICS FROM EUCLIDEAN UNIVERSE . (**ComplexQuantumEuclid.doc**) .
[12] TIME REGISTRATION IN PRIMARY NEUTRAL SPACE , ANTI - SPACE **(Time-Spaces.doc**)
[13] ZENOS PARADOX , NATURE OF POINTS IN QUANTIZED EUCLIDEAN GEOMETRY **(ZENO - PARADOX. doc**)
[14] THE DECREASING TUNNEL BY PR.FLORENDIN SMARANDASHE (**ORTHIC.doc**)
[15] THE SIX –TRIPLE CONCURENCY POINTS – LINE (Six.TriplePoi.Line**.doc)**
[16] ENERGY LAWS FOLLOW EUCLIDEAN MOULDS (Pythagoras**.doc)**

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