Calculation of the Area of a Sampled Boundary Using Fourier Components

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Abstract. A reformulation of the area of a planar two-dimensional object in the frequency domain allows for the computation of the true area of a band-limited boundary to be calculated.

Introduction

Represent the boundary of a planar object as an ordered set of points on the complex plane, $\{u[j]\}$, with j an integer in the range (0,N-1) as discussed in [1]. Applying the discrete Fourier transform to these points gives an equivalent set of points $U[n]$, with n in the range (0,N-1). The area swept out by traversing such a boundary in an anti-clockwise direction, is the sum of all of the elementary trapezoids between adjacent boundary points and the x-axis, such as those identified by the four points $x[j], u[j], u[j+1], x[j+1]$ in Fig. 1.

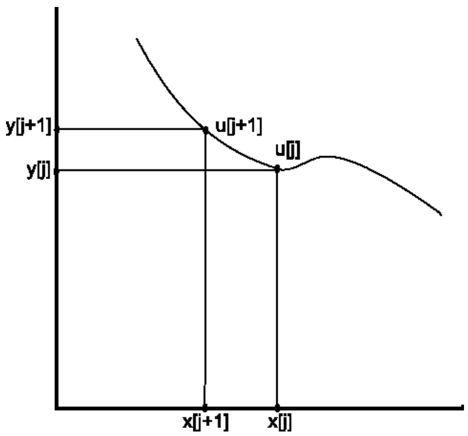

Figure 1

The area of the object is the sum of these elements for all j:

$$A = \sum_{j=0}^{N-1} \frac{1}{2}(x[j] - x[j+1])(y[j] + y[j+1])$$

$$\therefore A = \frac{1}{2}\sum_{j=0}^{N-1}(x[j]y[j+1] - x[j+1]y[j])$$

Expressing this equation in terms of points on the complex plane gives:

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$$A = \frac{1}{4i}\sum_{j=0}^{N-1}\left(u^*[j]u[j+1] - u[j]u^*[j+1]\right)$$

Transforming to the Frequency Domain

Transforming to the frequency domain using the general form of Parseval's theorem, we get:

$$A = \frac{N}{4i}\sum_{n=0}^{N-1}\left(U^*[n]U[-n]e^{\frac{-i2\pi n}{N}} - U[n]U^*[n]e^{\frac{-i2\pi n}{N}}\right)$$

$$\therefore A = \frac{N}{4i}\sum_{n=0}^{N-1}\left(|U[-n]|^2 e^{\frac{-i2\pi n}{N}} - |U[n]|^2 e^{\frac{-i2\pi n}{N}}\right)$$

$$\therefore A = \frac{N}{4i}\sum_{n=0}^{N-1}|U[n]|^2\left(e^{\frac{i2\pi n}{N}} - e^{\frac{-i2\pi n}{N}}\right)$$

$$\therefore A = \frac{N}{2}\sum_{n=0}^{N-1}|U[n]|^2 sin\left(\frac{2\pi n}{N}\right)$$

The sums involved remain true over any N consecutive points because of the cyclic continuation of the boundary and of its' discrete Fourier transform; in particular, over the N points symmetrically disposed about zero.

$$\text{Denote this sum by the symbol} \sum_n \; where\ for\ odd\ N, \sum_n \; replaces \sum_{n=\frac{-N-1}{2}}^{\frac{N-1}{2}}$$

If N is even, add an additional element with zero intensity at the midpoint so that the total number of elements becomes an odd number.

We now use this alternative form

$$A = \frac{N}{2}\sum_n |U[n]|^2 sin\left(\frac{2\pi n}{N}\right)$$

This is the area defined by an N point polygonal. If the boundary is band limited in the frequency domain, or is made so by some pre-processing operation, we can increase the number of points represented by extending the spectrum with additional zero power elements as explained in an earlier paper[1]. In the limit, we can calculate the true area of a band limited boundary by increasing the number of points to infinity:

$$A = \lim_{N\to\infty}\left\{\frac{N}{2}\sum_n |U[n]|^2 sin\left(\frac{2\pi n}{N}\right)\right\}$$

$$\therefore A = \lim_{N\to\infty}\left\{\frac{N}{2}\sum_n |U[n]|^2\left(\frac{2\pi n}{N} + \frac{1}{3!}\left(\frac{2\pi n}{N}\right)^3 + \cdots\right)\right\}$$

$$\therefore A = \pi\sum_n n|U[n]|^2$$

[1] D. Proffitt, “Normalization of discrete planar objects,” *Pattern Recognition*, vol. 15, no. 3, pp. 137-143, 1982.