On Building 4-critical Plane and Projective Plane Multiwheels from Odd Wheels. Extended Abstract

Dainis Zeps

Institute of Mathematics and Computer Science,
University of Latvia,
Rainis blvd., Riga, Latvia
dainize@mii.lu.lv
http://www.ltn.lv/~dainize/

Abstract. We build unbounded classes of plane and projective plane multiwheels that are 4-critical that are received summing odd wheels as edge sums modulo two. These classes can be considered as ascending from single common graph that can be received as edge sum modulo two of the octahedron graph O and the minimal wheel W_3.

Keywords: graph coloring, chromatic critical graphs, wheels, planar graphs, projective planar graphs, Grötzsch graph, Mycielski's construction

1 Introduction

This is extended abstract without proofs of the full version of paper [5].

We consider cube graph with cut off a corner, see in fig.1 [5] left, that in [3] is denoted G_3, but in the article [5] gets several denotations due to its particular features, w_{111} or w_{1_3}, g_{111} or g_{1_3}, see lower. In the introduction we consider some simple features of this graph w_{1_3} called *base graph* in frames of the article, but further we generalize these features to two, plane and projective plane, unbounded classes of graphs.

Simple observation gives that the base graph w_{1_3} turns into the wheel graph W_3 after contracting three edges incident to three and four degree vertices. Graph w_{1_3} is chromatic 4-critical [3] and smallest non-trivial one of this quality.

The base graph w_{1_3} can be considered as edge sum modulo two of three graphs W_3 in the way that each pair overlap just in one edge, and all three wheels have one vertex in common, see fig.2 [5].

Further, the base graph w_{1_3} may be embedded on projective plane, quadrangulating it, see a) in fig.8 [5]. w_{1_3} is selfdual, if considered plane, but not such on projective plane.

2 Defining 4-critical plane multiwheels

Let us assume that the wheel W_k is built from k *simple sections* where as simple section we take triangle C_3 with one vertex common from each triangle for the wheel's hub, and opposite edge of triangle as for forming wheel's rim.

Let us do similarly summation of edge sets of some wheels. Let us replace each section in a simple odd wheel with another arbitrary odd wheel in a way that edge sets of sections are summed modulo two. This new aggregation of wheels M may be expressed as $\sum W_{k_i}$ where summation is modulo two over index i numbering sections that were replaced by wheels of order k_i in each case. Evidently the summation itself is indeterministic because result of it depends on how wheels overlap each other by summation. Now we ask: under which conditions the resulting graph is 4-critical. It turns out that the answer directly is connected with the number of edges wheels intersect by summation. Theorem 5 below says that this number must be equal to two.

But first we are to prove some lemmas about intersecting wheels without edge losses by summation modulo two. For example, $W_5 + W_5$ may be formed with all five rim vertices of both wheels common and forming subgraph K_5, but resulting 6-chromatic graph is in no way chromatic critical.

Lemma 1 *Let k odd wheels by summation possibly intersect in vertices but not in edges. The sum of these wheels can't give 4-critical graph except in case $k = 1$.*

Let us configure two odd wheels so that they have common two adjacent vertices, and sum their edges modulo two. It is easy to see that the resulting graph is 3-chromatic. It suffices to notice that losing of an edge in both odd wheels allow to color them in 3-colors so that lost edge's ends receive the same color. Further, we may easily apply the use of this fact to *unclosed sequence of wheels* $W_{q_1},...,W_{q_i}$ where two proximal wheels overlap in two adjacent vertices, but next two possibly only in one. Let us formulate it as a lemma.

Lemma 2 *Let summation of edges modulo two is applied to unclosed sequence of wheels. The resulting graph is 3-chromatic.*

We need one more crucial feature of 4-critical graphs. Let graph H is 4-critical and let $H' = H \odot w$ be graph H with vertex w split into two new vertices and edges incident to w be connected either to one or other vertex. We ask whether graph H' can remain to be 4-critical. Of course, it is expectable that H' becomes 3-chromatic.

Lemma 3 *Let graph H be 4-critical and $w \in V(H)$. Then graph $H \odot w$ is always 3-chromatic.*

We are going to use lemma 3 in the following way. By summation of edge sets of wheels modulo two in order to build new 4-critical graphs we may ignore cases where wheels intersect only in vertices without incident edges, knowing that this can't lead to new 4-critical graphs. Suppose we received 4-critical graph in this way. Then splitting all vertices that were merged by summation backwards we should receive 3-chromatic graph but it might not be true. Let us formulate this fact as lemma.

Lemma 4 *Let by summation of edge sets of wheels some wheels intersect in vertices without incident edges. Then resulting graph can't be 4-critical.*

Now we may go over to the main theorem of this chapter.

Theorem 5 *Let $2k+1$ $(k>0)$ arbitrary odd wheels be summed in a way that edges of wheels are summed modulo two and all wheels have one overlapping vertex. The resulting graph M is 4-critical if and only if each wheel by summation modulo two looses just two of its edges and resulting graph is planar.*

The proof the theorem see in [5].

3 Grötzsch graph, Mycielski's Construction and 4-critical projective plane multiwheels

We start with observation that Grötzsch graph [1] may be considered as edge sum modulo two of five wheels w_1 and one wheel w_2, see fig.6 [5].

Grötzsch graph is 4-critical and it quadrangulates projective plane. Indeed, it has 11 vertices and 20 edges, i.e., it belongs to $2n-2$-edges-class of graphs, and fig.7 [5] shows how this embedding on projective plane is performed.

If in place of Grötzsch graph formed as $5w_1 + w_2$ we take only three plus one wheel, we get graph that is isomorphic to the base graph w_{1_3}. Taking this fact into account, we designate this graph g_{111} or g_{1_3} and traditional Grötzsch graph as g_{11111} or g_{1_5}. First, let us notice that both graphs are 4-critical, both quadrangulate projective plane, first being planar, bet second - projective planar. We might ask - are all graphs $qw_1 + w_q$ belonging summed according multiwheel summation pattern 4-critical? The answer is quite obviously positive, and we express the fact in the lemma what follows. We say that sum qw_1+w_q, $k=2q+1$, $q>0$, modulo two is got according multiwheel pattern if k wheels w_1 each looses two edges and w_q looses k edges. This class of graphs we call Grötzsch class.

Lemma 6 *For $k = 2q+1$, $q>0$, resulting graphs from $kw_1 + w_q$ summed according multiwheel pattern are 4-critical and quadrangulate projective plane.*

This class $kw_1 + w_q$, extending the base graph and the Grötzsch graph may be got as the Mycielski's Construction [1]. For that we are to take 3-critical graph, i.e., arbitrary odd cycle C_k, and apply Mycielski's Construction.

Further we are going to build more multiwheels, but the previous class should be the only that were quadrangulating projective plane.

Further we generalize projective planar multiwheels similarly as in case plane multiwheels, i.e., sections of w_1 may be replaced with arbitrary odd wheels. Fig.8 [5] shows simplest properly projective plane multiwheel q_{112}.

Construction 7 *Let us take odd in number ($k = 2q + 1$) odd wheels and one wheel w_q. Let us take in each of first wheels two proximal spikes and rim edge so that they do not form triangle, and middle spike edge match with central wheel w_q, and other two chosen edges (spike and rim edge) match in cyclical sequence of wheels.*

The resulting graph built according construction 7 belongs to $2n - 2$ edges class and is 4-critical. We call the resulting graph *multiwheel* similarly to those planar ones.

Multiwheels built according construction 7 are 4-critical. Multiwheel quadrangulates projective plane only if it belongs to Grötzsch class. Proofs see in [5].

4 Conclusions

In Hajoś construction cycle is summed with k-critical graphs modulo two where each section looses an edge with matching lost edge in the cycle [4]. In our construction of plane and projective plane multiwheels we sum odd wheels so that sections loose edge per conjunction with other section. We ask whether this can't be general paradigm for building of all k-critical graphs, i.e., lost edge representing other part of graph outside separating vertices.

References

1. Bondy J. A., Murty U.S.R., *Graph Theory*, Springer, 2008, 657pp.
2. Jensen Tommy R., Toft Bjarne. *Graph Coloring Problems*, John Willey and Sons, 1995, 295pp.
3. Zeps D. *4-critical wheel graphs of higher order*, `arXiv:1106.1336v1`, 2007, 4 pp, `congreso.us.es/eurocomb07/program_thursday.html`
4. Zeps D. *Application of the Free Minor Closed Classes in the Context of the Four Color Theorem*, `hal-00408145`, 2008, 19pp.
5. Zeps D. *On building 4-critical plane and projective plane multiwheels from odd wheels*, `arXiv:1202.4862v1`, 2012, 12pp.