A L-Topology of Banach space and Separability of Lipschitz dual space

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Abstract

In this paper we have introduced a new topology and a convergence in Banach space, which would be called a L-topology and a L-convergence. It is similar to the weak topology and weak convergence, but there are some essential differences. For example, the L-topology is stronger than weak topology, but weaker than the strong one. On the basis of the notion, we have considered the problem on the separability and reflexibility of Lipschitz (Lip-) dual space. Furthermore, we have introduced a new topology of Lip-dual space, which is similar to the weak***** (W*-) topology of linear dual of Banach space and would be called an L*-topology, and we have considered the problems on the metrizability of L*-topology and on the L*-separability of Lip-dual space, too.

Keywords: Lipschitz functional; Lipschitz operator; Lipschitz dual space; Lipschitz dual operator

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1. Introduction

There have been published many research results on the nonlinear Lipschitz (Lip-) operator in Banach space [1,2,6,7,8]. The Lip-operator is one of the most important nonlinear operators with the monotone operator, compact operator and convex function, but its properties are well known recently.

In this paper we shall introduce a L-topology and a L-convergence of a sequence in Banach space, and, on the basis of it, we'll consider the problem in [1] on the separability and refilexibility of Lip-dual space of Banach space. And we'll introduce a new topology of Lip-dual space and we will consider the problem in [1] on L*-separability of Lip-dual space.

First, we'll recall the concepts on Lip- operator [7,8].

Let X and Y be real or complex Banach spaces, M and D closed subsets of X , Y respectively. Let $0 \in M$, $0 \in D$ and $T : M \to D$ be an operator. Unless otherwise noted, in this paper we shall not repeat above assumptions. If there exists a constant $L \geq 0$ such that, for all $x, y \in M$, $\|Tx - Ty\| \leq L\|x - y\|$, then operator T is called a Lip-operator on M . And $L_M(T) = \sup_{x \neq y} \|Tx - Ty\| / \|x - y\|$ is called a Lip-constant of T on M . We'll often use a following set:

$$Lip_0(M, D) = \{T : M \to D |\ T(0) = 0,\ T \text{ is an Lip-operator on } M\}.$$

If the set D is a linear subspace of Y , then the set $Lip_0(M, D)$ is a normed linear space and the Lip-constant $L_M(T)$ is a norm of T in $Lip_0(M, D)$. And if D is a closed linear subspace, in short, a closed subspace, then the normed linear space $Lip_0(M, D)$ is a Banach space by the norm $L_M(T)$. In particular, if $D = K$ (real or complex field), then the space $Lip_0(M, D)$ is called a Lip-dual space of M . We denote it by M_L^*. And the element of M_L^* is called a Lip-functional. In the case of $M = X$, we denote by X_l^* the ordinary dual space of Banach space X , which consists of all bounded linear functionals defined on X and would be called a linear dual space of X, in distinction from Lip-dual space X_L^* of X. Then it is clear that X_l^* is a closed subspace of X_L^*. For any $x \in M$, $f \in D_L^*$, an operator defined by $(T_L^* f)(x) = (f \circ T)(x) = f(Tx)$ is called a Lip-dual operator of T and we denote it by T_L^*. Then it is clear that $T_L^* \in BL(D_L^*, M_L^*)$ and $L_M(T) = \|T_L^*\|$, where $BL(D_L^*, M_L^*)$ is the Banach space consisting of all the bounded linear operators on D_L^* into M_L^* ([2]). Since the space M_L^* is a Banach space and the operator T_L^* is a bounded linear, it is defined a linear dual space $M_{Ll}^{**} = (M_L^*)_l^*$ of M_L^* and a linear dual operator $T_{Ll}^{**} = (T_L^*)_l^*$ of T_L^* respectively. Then it is easy to see that $T_{Ll}^{**} \in BL(M_{Ll}^{**}, D_{Ll}^{**})$ and $L_M(T) = \|T_L^*\| = \|T_{Ll}^{**}\|$. In the study of Lip-operator, the need to extend the Lip-functional satisfying certain conditions is presented frequently, but that is reduced to the possibility of the extension to whole space with Lip-continuity and maintenance of Lip-constant of Lip-functional defined at a subset of Banach spaces. The following theorem gives us a sure guarantee for such possibility ([7]).

Theorem 1[7]. Let f be a real-valued Lip-functional defined on a closed subset M of a real Banach space X. Then there exists a real-valued Lip-functional F defined on X such that 1) F is an extension of f, i.e., $F(x) = f(x)$ for $x \in M$, and 2) $L_X(F) = L_M(f)$.

Theorem 1$'$[7] Let f be a complex Lip-functional defined on a closed subset M of a complex Banach space X. Then there exists a complex Lip-functional F defined on X such that 1) $F(x) = f(x)$ for $x \in M$, and 2) $L'_X(F) = L'_M(f)$, where $L'_M(f) = \left(L_M^2(g) + L_M^2(h)\right)^{1/2}$ and, g and h are the real and imaginary parts of f respectively.

As will be seen from these theorems, we can say that the extension theorem is a generalization to nonlinear Lip-functional of the Hahn-Banach theorem

([4]) on the extension of the bounded linear functional. Some corollaries follow from the extension theorem.

Corollary 1[7]. For all $x \in M$, we have $\|x\| = \sup_{\substack{f \in M_L^* \\ L(f) \neq 0}} |f(x)| / L(f)$

Corollary 2[7]. For any $x_0 \in X \setminus M$, there exists a real-valued Lip-functional f defined on X such that 1) $f(x) = 0$ for $x \in M$, 2) $f(x_0) = d$, and 3) $L_X(f) = 1$, where $d = \inf_{z \in M} \|z - x_0\| > 0$.

Proposition 1[7]. Let $T \in Lip_0(M, D)$. Then M is a certain subset of M_{Ll}^{**} in isometric embedding sense. If an operator $J : M \to M_{Ll}^{**}$ is such isometric mapping, then we have, for all $x, y \in M$,

$$\|x - y\| = \|Jx - Jy\| = \sup_{f \in M_L^*, L(f) \leq 1} |f(x) - f(y)| \text{ and } \|Tx - Ty\| = \|T_{Ll}^{**}(Jx) - T_{Ll}^{**}(Jy)\|.$$

Here, for any $x \in M$, a functional $J(x)$ defined on M_L^* by $J(x)(f) = f(x)$ is a bounded linear and an operator $J : M \to JM = \; = \{ J(x) \in M_{Ll}^{**} | x \in M \}$ is an isometric mapping satisfying the conditions of the theorem.

2. A L-topology and A L-convergence

In the section we will introduce a L-topology and a L-convergence in Banach space and we will consider its some properties. First we need some information. Now let A be a finite subset of M_L^* and let $\varepsilon > 0$ be a given positive number. Take any $x_0 \in M$ and fix it. The set

$$U(x_0; A; \varepsilon) = \left\{ x \in M \,\middle|\, \sup_{f \in A} |f(x) - f(x_0)| < \varepsilon \right\}$$

would be called a ε-neighbourhood of the point x_0 with respect to A. And we denote by $\vartheta(x_0)$ a set consisting of all $U(x_0; A; \varepsilon)$'s, when $\varepsilon > 0$ and the finite set A are arbitrary. Then $\vartheta(x_0)$ is a family of neighbourhoods of x_0. The following statement is valid.

Proposition 2. The family $\vartheta(x_0)$ satisfies; **i)** $\vartheta(x_0) \neq \varnothing$, **ii)** $U \in \vartheta(x_0)$ implies $x_0 \in U$, **iii)** $U, V \in \vartheta(x_0)$ then there exists a $W \in \vartheta(x_0)$ such that $W \subset U \cap V$, **iv)** $U \in \vartheta(x_0)$ then, for any $y \in U$, there exists a $V \in \vartheta(y_0)$ such that $V \subset U$, and **v)** $x_0 \neq y_0$ then there exists a $U \in \vartheta(x_0)$ and $V \in \vartheta(y_0)$ such that $U \cap V = \varnothing$.

It is not difficult to check that this proposition is true. Next we denote by $\mathfrak{I}$ a set consisting of all $\vartheta(x_0)$'s, when $x_0 \in M$ is arbitrary. Then $\mathfrak{I}$ is a family of all neighbourhoods of all $x_0 \in M$. And $\mathfrak{I}$ satisfies following property.

Proposition 3. Let $\tau_L = \{G = \cup_\alpha U_\alpha | U_\alpha \in \mathfrak{I},\ \alpha \in \Lambda$ (index set)$\}$. Then (M, τ_L) is a topological space.

Proof. To show that a system τ_L is a topology of M, it would be sufficient to show that the family $\mathfrak{I}$ satisfies; i) for any $x \in M$, there exists $U \in \mathfrak{I}$ such that $x \in U$, and ii) for any $U, V \in \mathfrak{I}$ and any $z \in U \cap V$, there exists $W \in \mathfrak{I}$ such that $z \in W \subset U \cap V$. Then $\mathfrak{I}$ becomes a basis of the topology of (M, τ_L).

Here i) is clear. And it is not difficult to see that ii) is true.

Definition 1. The system τ_L would be called a L-topology of M. Since τ_L satisfies the property **v)** of the proposition 1, the topological space (M, τ_L) satisfies Hausdorff's axiom of separation. Unless otherwise stated, we shall often omit τ_L and refer to M as a topological space.

The following theorem shows a relation of L-topology with weak and strong topology.

Theorem 1. The L-topology of M is stronger than the weak topology, but weaker than the strong one of M, where the weak or strong topology of M means the relative topology of the subset M of X as the weak or strong topological space, respectively.

Proof. To explain shortly and clearly the process of the proof, we'll do as follows. We shall change the finite set $A \subset M_L^*$ in the definition of the neighbourhood $U(x_0; A; \varepsilon)$ of the point $x_0 \in M$ into arbitrary bounded subset of Lip-dual space M_L^* or into arbitrary finite subset $A \subset \tilde{M}_L^*$, and denote by τ_S or τ_W the topologies generated from those, respectively. Here $\tilde{M}_L^*$ is the set consisting of all $\tilde{f}$'s, i.e., $\tilde{M}_L^* = \left\{ \tilde{f} = f|_M \,;\, f \in X_l^* \right\}$ and $\tilde{f} = f|_M$ is the contraction of f to M for any $f \in X_l^*$ (the linear dual space). Then clearly both systems τ_S and τ_W satisfy the five and two conditions of the proposition 2 and 3. Therefore (M, τ_S) and (M, τ_W) are two topological spaces, which satisfy Hausdorff's axiom of separation. And, by the definition, we see easily that the L-topology τ_L is weaker than τ_S, but stronger than τ_W. To prove the theorem, we'll show that the topology τ_S is equivalent to the relative topology of M of the strong one by the norm of X. In fact, a ε-neighbourhood of the point $x_0 \in M$ in the relative strong topology of M of the strong one by the norm of X is the set $B(x_0; \varepsilon) = \left\{ x \in M \,\middle|\, \|x - x_0\| < \varepsilon \right\}$. Now let A be any bounded subset of M_L^*, that is, there exists a non-negative constant c such that $\sup_{f \in A} L(f) \le c < +\infty$. Then, since

$$\sup_{\substack{f \in A \\ A \subset M_L^*}} |f(x) - f(x_0)| \le \sup_{\substack{L(f) \le c \\ f \in M_L^*}} |f(x) - f(x_0)| \le c \cdot \|x - x_0\|,$$

the topology τ_S is weaker than the relative topology of M of the strong one by the norm of X. On the other hand, by the corollary 1 of the extension theorem and by the proposition 1, we have $\|x - x_0\| = \sup_{\substack{L(f) \le 1 \\ f \in M_L^*}} |f(x) - f(x_0)|$. Now let S^L be the unit sphere of M_L^*, then we have $B(x_0;\,\varepsilon) = U(x_0; S^L; \varepsilon)$. This means that the topology τ_S is stronger than the relative topology of M of the strong one by the norm of X. Therefore the tow topology is equivalent to each other. It is clear that the topology τ_W is equivalent to relative topology of M of the weak one of X, too. This shows that the relation of the three topologies is true.

Remark. From the definition of each topology**,** it is clear that the converse relations are not true in general. And $x_n,\, x_0 \in M$, then it is easy to see that the sequence $\{x_n\}$ converges to x_0 in the L-topology of M if and only if, for any $f \in M_L^*$, $f(x_n) \to f(x_0)$. In connection with this, we may introduce a following notion.

Definition 2. A sequence $\{x_n\}$ in M would be said to be L-convergent if a finite $\lim_{n\to\infty} f(x_n)$ exists for each $f \in M_L^*$; $\{x_n\}$ would be said to L-converge to an element $x_0 \in M$ if $f(x_n) \to f(x_0)$ for all $f \in M_L^*$. In the letter case, x_0 is uniquely determined by Hahn-Banach theorem; we shall write $L - \lim_{n\to\infty} x_n = x_0$ or, in short, $x_n \xrightarrow{L} x_0$. And M would be said to be L-complete if every L-convergent sequence of M L-converges to an element of M.
The following theorems are valid.

Theorem 2. The following conditions are equivalent to each other.
 i) $x_n \xrightarrow{L} x_0$, ii) $x_n \xrightarrow{S} x_0$, and iii) $\{\|x_n\|\}$ is bounded, and $f(x_n) \to f(x_0)$ for any f in the strongly dense subset of M_L^*.
The proof of the equality of i) and iii) follows from the definition 2, and it is similar to one of weak convergence. We'll show only i) implies ii). If it does

not, the point x_0 is not in the strong closure of $\{x_n\}$. We denote by M_0 the strong closure of $\{x_n\}$. Then we have $d = \inf_{z \in M_0} \|z - x_0\| > 0$. Therefore, by the corollary 2 of the extension theorem, there exists a functional $f_0 \in M_L^*$ such that $f_0(z) = 0\ (z \in M_0')$, $f_0(x_0) = d$ and $L(f_0) = 1$. This is contrary to that $\{x_n\}$ L-converges to x_0.

Remark. In general, the following statement is true: " if, for any $f \in D_L^*$, the sequence $\{f(y_n)\}$ converges to $f(y_0)$, then $\{y_n\}$ converges strongly to y_0." Without using the extension theorem, this is easily proved as follows. We define a functional f_0 on D by $f_0(y) = \|y - y_0\| - \|y_0\|$ for $y \in D$. Then it is clear that $f_0 \in D_L^*$ and $L_D(f_0) = 1$. Therefore we have

$$\|y_n - y_0\| = \|\|y_n - y_0\| - \|y_0\| + \|y_0\|\| = |f(y_n) - f(y_0)| \to 0 \ .$$

In the future, we shall denote by L- (S- or W-) all the notions and representations with respect to L- (strong or weak) topology. Thus the representation $x_n \xrightarrow{S} x_0$ in the theorem 2 means the sequence $\{x_n\}$ converges strongly to x_0.

We'll now discuss shortly the L-completeness of M. From the definition 2, we see that a sequence $\{x_n\}$ in M is L-convergent then it is weakly convergent. Therefore, the weak completeness of M is an important condition for L-completeness of M. Of course, all finite dimensional spaces are L-complete. But there is an infinite dimensional Banach space, too. For example, let $X = l_1$ or $X = L_1[0,1]$, then, as is well known, these spaces two are weakly complete, and both weak and strong convergence of the sequences in these are equivalent to each other, by the Schur's theorem ([5]). Therefore spaces $X = l_1$ or $X = L_1[0,1]$ are surely L-complete. In addition, we know that the strong and weak topologies of the Banach space are equivalent to each other if and only if the dimension of the space is finite ([5]). Thus the

Schur's theorem shows that the weak convergence of the sequence is of insufficient in the study of the weak topology. This is also true for L-topology and L-convergence. The following theorem is valid.

Theorem 3. Let X be a uniformly convex Banach space and satisfy the condition such that $x_n \xrightarrow{W} x_0$ and $\lim_{n\to\infty} \|x_n(t)\| = a$ (a real) imply $a = \|x_0\|$. Then X is L-complete.

Proof. Let a sequence $\{x_n\}$ be arbitrary L-convergent from X. Then, for any $f \in X_l^*$ and for the Lip-functional $f(x) = \|x\|$ defined on X, a finite $\lim_{n\to\infty} f(x_n)$ exists. Therefore $\{x_n\}$ is W-convergent. Since X is W-complete, there exists an element $x_0 \in X$ such that $x_n \xrightarrow{W} x_0$. By the condition, we have $\lim_{n\to\infty} \|x_n(t)\| = \|x_0\|$. Again by the uniform convexity of X, we have $x_n \xrightarrow{S} x_0$ ([3]). Thus X is L-complete.

Remark. ① In general, not every uniformly convex Banach space satisfies the condition in the theorem 3. For example, Let $X = l_p \quad (1 < p < +\infty)$ and $e_n = (\overbrace{0,0,\ldots 0}^{n-1},1,0,\ldots)\ (n = 1,2,\ldots)$, $e_0 = (0,0,\ldots)$. Then $e_n \xrightarrow{W} e_0$ and $\lim_{n\to\infty} \|e_n\| = 1$, but $\|e_0\| \neq 1$.

② If X is not uniformly convex, then L-convergence of the sequence in X doesn't follows in general even if the condition in the theorem 3 is satisfied. The following example shows; Let $X = C[0,1]$. Take a sequence $\{x_n(t)\}$ from X by $x_n(t) = 1 - nt$ for $0 \le t \le 1/n$, $x_n(t) = nt - 1$ for $1/n \le t \le 2/n$, and $x_n(t) = 1$ for $2/n \le t \le 1$. Then $x_n(t) \xrightarrow{W} x_0(t) \equiv 1$, $\lim_{n\to\infty} \|x_n(t)\| = \|x_0(t)\|$, but we have $x_n(t) \overset{L}{\nrightarrow} x_0(t)$.

From the theorem 2, we see that a linear normed space, which is L-complete, is also strongly complete, that is, a Banach space. Therefore the linear normed space, which is not Banach space, is also not L-complete. We have to show an example of Banach space that is not L-complete.

3. The separability and reflexivity of the Lip-dual spaces

The separability and reflexivity of a Banach space, as is well known, are very important concepts for the research of the operator defined in the space. Those two are closely connected each other and deeply depended on the algebraic and topological structures of original Banach space. So those of the linear dual space are not complicated so much. But, since the Lip-dual space is dependent only on topological structure of the Banach space and is independent of its algebraic one, those for Lip-dual are not simple. However, similarly to that those for linear dual are closely connected with W-completeness of the Banach space, it is clear that those for Lip-dual are also connected with L-completeness of the Banach space.
We can prove

Theorem 4. Let M be L-complete. If M_L^* is separable, then each bounded subset of M is relatively strong compact.
Proof. Let M_0 be arbitrary bounded subset of M and $\{f_n\}$ a sequence of a countable number of Lip-functionals, which is strongly dense in M_L^*. Take arbitrary sequence $\{x_n\}$ from M_0. Since $\{x_n\}$ is norm bounded, the sequence $\{f_1(x_n)\}$ is bounded. Thus there exists a subsequence $\{x_{n_1}\}$ for which the sequence $\{f_1(x_{n_1})\}$ is convergent. Since the sequence $\{f_2(x_{n_1})\}$ is bounded, there exists a subsequence $\{x_{n_2}\}$ of $\{x_{n_1}\}$ such that $\{f_2(x_{n_2})\}$ is convergent. Proceeding in this way, we can choose a subsequence $\{x_{n_{i+1}}\}$ of the sequence $\{x_{n_i}\}$ such that the sequence of numbers $\{f_j(x_{n_{i+1}})\}$ converges for $j = 1, 2, \ldots, i+1$. Hence the diagonal subsequence $\{x_{n_n}\}$ of the original sequence $\{x_n\}$ satisfies the condition that the sequence $\{f_j(x_{n_n})\}$ converges for $j = 1, 2, \ldots$. Let an

arbitrary $f \in M_L^*$ and $\varepsilon > 0$ be given. Since the sequence $\{f_n\}$ is strongly dense in M_L^*, there exists a functional $f_{n_0} \in \{f_n\}$ such that $L(f - f_{n_0}) < \varepsilon$. Therefore, for n, m, we have

$$\begin{aligned}
&\left|f(x_{n_n}) - f(x_{m_m})\right| \le \left|f(x_{n_n}) - f_0(x_{n_n})\right| + \left|f_0(x_{n_n}) - f_0(x_{m_m})\right| + \\
&+\left|f_0(x_{m_m}) - f(x_{m_m})\right| \le L(f - f_0)\left\|x_{n_n}\right\| + \left|f_0(x_{n_n}) - f_0(x_{m_m})\right| + \\
&+L(f - f_0)\left\|x_{m_m}\right\| < 2\left(\sup_n \left\|x_n\right\|\right)\cdot \varepsilon + \left|f_0(x_{n_n}) - f_0(x_{m_m})\right|
\end{aligned}$$

Thus, for any $f \in M_L^*$, the sequence of numbers $\left\{f(x_{n_n})\right\}$ is a Cauchy sequence, that is, the sequence $\left\{x_{n_n}\right\}$ is L-convergent. By the assumption, there exists an element $x_0 \in M$ such that $\left\{x_{n_n}\right\}$ L-converges to x_0. By the theorem 2, $\left\{x_{n_n}\right\}$ converges strongly to x_0.

From this theorem we see easily

Corollary. Let M be L-complete. If M_L^* is separable, then both weak and strong convergences of the sequence in M are equivalent to each other. In the case of $M = X$, in particular, the Banach space X is of finite dimensional.

Remark. We could obtain the following results.

If M satisfies one of the below conditions, then M_L^* is not separable.

i) M is not separable, **ii)** M is separable, but $\tilde{M}_L^*$ is not one, **iii)** M is L-complete, but there exists a sequence $\{x_n\}$ in M such that $x_n \xrightarrow{W} x_0$ and $\|x_n\| \not\to \|x_0\|$, and **iv)** The Banach space $M = X$ is weakly complete, but not reflexive.

Let us show now the reflexivity of Lip-dual space.

Theorem 5. The space M_L^* is reflexive if and only if the strong closure of linear subspace generated by JM in M_{Ll}^{**} is equal to M_{Ll}^{**}, that is,

$[Lin\{JM\}] = M_{Ll}^{**}$, where JM is the set in the proposition 1 and “equal to” means that two spaces are linear isomorphic and isometric.

Proof. Put $\tilde{X} = [Lin\{JM\}]$. Then, by the theorem 1 in the paper [1], it is true that $\tilde{X}_l^* = M_L^*$. Thus the reflexivity of M_L^* is equivalent to one of X_l^*. On the other hand, in general, the reflexivity of Banach space is also equivalent to one of its linear dual. Therefore, if M_L^* is reflexive then, since $\tilde{X}_l^*$ is also so, we have $M_{Ll}^{**} = \left(M_L^*\right)_l^* = \left(\tilde{X}_l^*\right)_l^* = \tilde{X}_{ll}^{**} = \tilde{X}$. Conversely, if $M_{Ll}^{**} = \tilde{X}$ then we have $\left(M_L^*\right)_{ll}^{**} = \left(M_{Ll}^{**}\right)_l^* = \tilde{X}_l^* = M_L^*$. Thus M_L^* is reflexive.

Corollary. If M is separable and M_L^* is reflexive, then M_L^* is separable.

Proof. It is clear that M is separable then $\tilde{X} = [Lin\{JM\}]$ is so, and M_L^* is reflexive then $\tilde{X}$ is also such. Therefore $\tilde{X}$ is separable and reflexive. By the theorem 4, M_L^* is separable.

Theorem 6. Let G be an open convex subset of X and let $0 \in G$, $M = [G]$ strong closure of G. Then M_L^* is not reflexive. In particular, M is a real interval $[0, 1]$ then M_L^* is not reflexive.

The proof is easily. In fact, it would be sufficient to show a closed subspace of that, which is not reflexive. For example, we could take the space $C_0^1[0, 1]$ consisting of all continuous differentiable real functions with a norm $\|x\| = \sup_{0 \le t \le 1} |x'(t)|$.

Remark. We could obtain the following results.

If M satisfies one of the below conditions, then M_L^* is not reflexive.

i) M is separable, but $\tilde{M}_L^*$ is not one, **ii)** M is separable and L-complete, but there exists a sequence $\{x_n\}$ in M such that $x_n \xrightarrow{W} x_0$ and $\|x_n\| \not\to \|x_0\|$, and **iii)** The Banach space $M = X$ is not reflexive.

4. The L*-separability of Lip-dual space

In this section we shall introduce an L*-topology and an L*-convergence of sequence of Lip-functionals in Lip-dual space, which are similar to W*-topology and W*-convergence in linear dual space, and consider the separability of Lip-dual space in the sense of L*-convergence. As may be seen from the results, we can say that the L*-separability of Lip-dual space is simply obtained in some measure.

Let A be a finite subset containing at least two distinct points of M and let $\varepsilon > 0$ be given. Take arbitrary $f_0 \in M_L^*$ and define

$$L_A(f - f_0) = \sup_{\substack{x, y \in A \\ x \neq y}} \|x - y\|^{-1} \cdot \left|(f(x) - f(y)) - (f_0(x) - f_0(y))\right|.$$

Then a set $U(f_0; A; \varepsilon) = \{ f \in M_L^* \mid L_A(f - f_0) < \varepsilon \}$ would be called a ε-neighbourhood of the point $f_0 \in M_L^*$ with respect to A. And we denote by $\mathfrak{A}(f_0)$ a set consisting of all $U(f_0; A; \varepsilon)$'s, when $\varepsilon > 0$ and A are arbitrary. Then $\mathfrak{A}(f_0)$ is a family of neighbourhoods of f_0 and satisfies the conditions of the proposition 2. The topology of M_L^* generated by $\mathfrak{A}(f_0)$ would be called a L*-topology and we shall denote it by τ_{L^*}. It is clear that the strong topology of M_L^* is defined by $U(f_0; A; \varepsilon)$, when A is bounded. So L*-topology of M_L^* is weaker than the strong one. On the other hand, we see easily that the topology defined by the neighbourhoods $U(f_0; A; \varepsilon) = \{ f \in M_L^* \mid L_A^W(f - f_0) < \varepsilon \}$, where $L_A^W(f-f_0)=\sup_{\substack{\tilde{x},\tilde{y}\in A\\ \tilde{x}\neq\tilde{y}}}\|\tilde{x}-\tilde{y}\|^{-1}\cdot\left|\left(\tilde{x}(f)-\tilde{y}(f)\right)-\left(\tilde{x}(f_0)-\tilde{y}(f_0)\right)\right|$

and when A is arbitrary finite subset of M_{Ll}^{**}, is equivalent to the weak topology of M_L^*. Therefore L*-topology of M_L^* is weaker than the weak one, too. From the definition, it follows that the converse relation of weak and L*-topology is not true. The following theorem shows the metrizabil- ity of L*-topology.

Theorem 7. The L*-topology of the closed unit sphere of M_L^* is metri- zability if and only if M is separable.

Proof. Let us denote by S^L the closed unit sphere of M_L^*. First, it is to be noted here that L*-topology of S^L is relative one of S^L as subset of M_L^*. Let $\{x_n\}$ be a strong dense subset of a countable number of elements of M. For $f,g\in S^L$, define

$$\rho(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{|f(x_n)-g(x_n)|}{1+|f(x_n)-g(x_n)|}.$$

Then it is clear that $\rho(f,g)$ is a distance defined on S^L. We denote by τ_ρ the topology of S^L defined by the distance $\rho(f,g)$. It is not difficult to see that the topology τ_ρ is weaker than τ_{L^*} (see the theorem V. 5. 1 of [5]). On the other hand, the metric space (S^L,ρ) is surely Hausdorff space and, by the theorem 8 below, the topological space (S^L,τ_{L^*}) is compact. Thus identity mapping I on (S^L,τ_{L^*}) onto (S^L,ρ) is continuous because the topology τ_ρ is weaker than τ_{L^*}. Since one-to-one continuous mapping on the compact topological space onto Hausdorff space is topological homeomorphism, the topology τ_ρ is equivalent to τ_{L^*} (see the lemma I. 5. 8 of [5]).

Conversely, let the topology τ_{L^*} of S^L be defined by a certain distance. Then there exist a family $U_n^*=U(f_0;A_n;\varepsilon_n)$ of neighbourhoods of $f_0=0$ such that

$\bigcap_{n=1}^{\infty} U_n^* = \{f_0\}$. Since each A_n is countable, the union set $A = \bigcup_{n=1}^{\infty} A_n$ is also countable. For any $x \in A$, $f(x) = 0$ ($f \in M_L^*$) then it is clear that $f = f_0$. Denote by M_0 the strong closure of A then M_0 is clearly separable. And it is true that $M_0 = M$. If it doesn't, then there exists an element $x_0 \in M$, which is not in M_0, such that $d = \inf_{x \in M} \|x - x_0\| > 0$. By the corollary 2 of the extension theorem, there exists a functional $f_0 \in M_L^*$ such that $f_0(x) = 0 (x \in M_0)$, $f_0 \neq 0$. This completes the proof.

Let f_n, $f_0 \in M_L^*$. Then it is easy to see that the sequence $\{f_n\}$ converges to f_0 in the L*-topology of M_L^* if and only if, for any $x \in M$, $f_n(x) \to f_0(x)$. In connection with this, we may introduce a following notion ([8]).

Definition 3[8]. A sequence $\{f_n\}$ in Lip-dual space M_L^* would be said to be L*-convergent if a finite $\lim_{n\to\infty} f_n(x)$ exists for each $x \in M$; $\{f_n\}$ would be said to L*-converges to an element $f_0 \in M_L^*$ if $\lim_{n\to\infty} f_n(x) = f_0(x)$ all $x \in M$. In the letter case, we'll write $f_0 = L^* - \lim_{n\to\infty} f_n$ or, in short, $f_n \xrightarrow{L^*} f_0$.

Remark. Set $f_n(x) = \sin nx / \sqrt{n}$ for $x \in M = [0,1] \subset R^1$ (real number field), then it is clear that $\{f_n\} \subset M_L^*$ and $f_n \xrightarrow{L^*} f_0 = 0$, but $\sup_n L(f_n) = +\infty$.

The linear dual space of Banach space was always complete in the sense of W*-convergence of the sequence of bounded linear functionals. But it follows from this example that the Lip-dual space may not be complete in the sense of L*-convergence of the sequence of Lip-functionals. And, in general, the Banach-Steinhaus theorem - the resonance theorem ([4]) is not valid for the sequence of Lip-functionals. In other words, it is not true that L^*-convergence of the sequence $\{f_n\}$ implies $\sup_n L(f_n) < +\infty$.

The following properties and their proofs for L*-convergence are very similar to one of the W*-convergence of the sequence of the bounded linear functionals.

Proposition 4 [8]. i) If $\{f_n\}$ is strongly convergent to f_0, that is $\|f_n - f_0\| \to 0$, then $f_0 = L^* - \lim_{n\to\infty} f_n$, but not conversely. **ii)** If $\sup_n L(f_n) < +\infty$ and $f_0 = L^* - \lim_{n\to\infty} f_n$, then f_0 is unique and $L(f_0) \le \sup_n L(f_n)$. **iii)** Suppose that $\sup_n L(f_n) < +\infty$. Then a sequence $\{f_n\}$ L*-converges to an element $f_0 \in M_L^*$ if and only if $\lim_{n\to\infty} f_n(x) = f_0(x)$ on a strongly dense subset of M.

Definition 4 [8]. Let $A, M_0^{/} \subset M_L^*$. The set $M_0^{/}$ would be said to be separable in the sense of L*-convergence, or, in short, L*-separable if, for any $f \in M_0^{/}$ and $\varepsilon > 0$, there exists a functional $g \in A$ such that $|f(x) - g(x)| < \varepsilon$ for $x \in M$ and A is a set of countable elements. And $M_0^{/}$ would be called a relatively compact in the sense of L*-convergence, or, in short, L*-relative compact if every sequence $\{f_n\}$ in $M_0^{/}$ contains a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ L*-converges to an element $f_0 \in M_L^*$.

We can obtain the following statement.

Theorem 8. i) If $M_0^{/}$ is L*-relative compact then it is L*-separable. **ii)** If M is separable, then each bounded subset of M_L^* is a L*-relatively compact. **iii)** If M is separable then M_L^* is L*-separable.

Proof. The proof of i) and ii) is done by the above properties ii), iii) and the diagonal method. That is similar to one that if the Banach space X is separable then the bounded subset of the linear dual space X_l^* of X is W*-relatively compact ([4]). On the other hand, iii) is proved easily as follows.

It is true that $M_L^* = \bigcup_{n=1}^{\infty} M_n^{\prime}$, where $M_n^{\prime} = \left\{ f \in M_L^* \middle| L(f) \le n \right\}$, and each $M_n^{\prime}$ is L*-separable. Thus, by i) and ii), M_L^* is L*-separable.

Remark. It is well known that any bounded subset of X_l^* is W*-relatively compact without the separability of X (the theorem 2 in Chapter V, 4 of [5]). But, for Lip-functional, the assumption that M is separable is essential.

5. A L(f_0)-topology and A L(f_0)-convergence

In the section we shall introduce a new $\mathrm{L}\left(f_0\right)$-topology and a $\mathrm{L}\left(f_0\right)$-convergence of a sequence in Banach space and consider its some properties. First we need some information. For any $f \in X_l^*$, we denote by $\tilde{f}$ the contraction of f to M, i.e., $\tilde{f} = f|_M$. Then it is clear that $\tilde{f} \in M_L^*$. Let us denote by $\tilde{M}_L^*$ the set consisting of all $\tilde{f}$'s, that is, $\tilde{M}_L^* = \left\{ \tilde{f} = f|_M ;\ f \in X_l^* \right\}$. On the other hand, we put $f_0(x) = \|x\|$ for any $x \in M$, then f_0 belongs surely to M_L^* and $L(f_0) = 1$. We'll denote by ${}^0M_L^*$ the set consisting of $\tilde{M}_L^*$ and f_0, that is, ${}^0M_L^* = \tilde{M}_L^* \cup \left\{ f_0 \right\}$. Then ${}^0M_L^*$ is a proper subset of M_L^*. Now let A be a finite subset of ${}^0M_L^*$ and let $\varepsilon > 0$ be given. Take any $x_0 \in M$ and fix it. The set

$$U(x_0; A; \varepsilon) = \left\{ x \in M \,\middle|\, \sup_{f \in A} \left| f(x) - f(x_0) \right| < \varepsilon \right\}$$

would be called a ε-neighbourhood of the point x_0 with respect to A. And we denote by $\vartheta(x_0)$ a set consisting of all $U(x_0; A; \varepsilon)$'s, when $\varepsilon > 0$ and the finite set A are arbitrary. Then $\vartheta(x_0)$ is a family of neighbourhoods of x_0. The following statement is valid.

Proposition 5. $\vartheta(x_0)$ satisfies the following conditions.

i) $\vartheta(x_0) \neq \varnothing$, **ii)** $U \in \vartheta(x_0)$ implies $x_0 \in U$, **iii)** $U, V \in \vartheta(x_0)$ then there exists a $W \in \vartheta(x_0)$ such that $W \subset U \cap V$, **iv)** $U \in \vartheta(x_0)$ then, for any $y \in U$, there exists a $V \in \vartheta(y_0)$ such that $V \subset U$, and **v)** $x_0 \neq y_0$ then there exists a $U \in \vartheta(x_0)$ and $V \in \vartheta(y_0)$ such that $U \cap V = \varnothing$.

It is not difficult to check that this proposition is true. Next we denote by $\Im$ a set consisting of all $\vartheta(x_0)$'s, when $x_0 \in M$ is arbitrary. Then $\Im$ is a family of all neighbourhoods of all $x_0 \in M$. And $\Im$ satisfies following property.

Proposition 6. Let $\tau_L = \{G = \cup_\alpha U_\alpha \mid U_\alpha \in \Im, \alpha \in \Lambda$(a set of parameters)$\}$,

Then (M, τ_L) is a topological space.

Proof. To show that a system τ_L is a topology of M, it would be sufficient to show that the family $\Im$ satisfies; i) for any $x \in M$, there exists a $U \in \Im$ such that $x \in U$, and ii) for any $U, V \in \Im$ and any $z \in U \cap V$, there exists a $W \in \Im$ such that $z \in W \subset U \cap V$. Then $\Im$ becomes a base of the topology of (M, τ_L). i) is clear. And it is not difficult to see that ii) is true.

Definition 5. The system τ_L would be called L(f_0)-topology of M. Since τ_L satisfies the property v) of the proposition 1, the topological space (M, τ_L) satisfies Hausdorff's axiom of separation. Unless otherwise stated, we shall often omit τ_L and refer to M as a topological space.

The following theorem shows a relation of L(f_0)-topology with weak or strong topology.

Theorem 9. The L(f_0)-topology of M is stronger than the weak topology, but weaker than the strong one of M, where the weak or strong topology of

M means the relative topology of the subset M of X as the weak or strong topological space, respectively. The converse is not true in general.

Proof. We shall change the finite set $A \subset {}^0M_L^*$ in the neighbourhood $U(x_0; A; \varepsilon)$ in the definition of L(f_0)-topology into any bounded subset of Lip-dual space M_L^* or into any finite subset $A \subset \tilde{M}_L^*$, and denote by τ_S or τ_W a topology generated from these respectively. Then clearly both systems τ_S and τ_W satisfy the five and tow conditions of the proposition 2 and 3. Therefore (M, τ_S) and (M, τ_W) are tow topological spaces, which satisfy Hausdorff's axiom of separation. And, by the definition, we see easily that the L(f_0)-topology τ_L is weaker than τ_S, but stronger than τ_W. To prove the theorem, we'll show that the topology τ_S is equivalent to the relative topology of M of the strong one by the norm of X. In fact, a ε-neighbourhood of the point $x_0 \in M$ in the relative topology of M of the strong one by the norm of X is the set $B(x_0;\, \varepsilon) = \left\{\, x \in M \,\middle|\, \|x - x_0\| < \varepsilon \,\right\}$. Now let A be any bounded subset of M_L^*, that is, there exists a non-negative constant c such that $\sup_{f \in A} L(f) \le c < +\infty$. Then, since

$$\sup_{\substack{f \in A \\ A \subset M_L^*}} \left| f(x) - f(x_0) \right| \le \sup_{\substack{L(f) \le c \\ f \in M_L^*}} \left| f(x) - f(x_0) \right| \le c \cdot \|x - x_0\|,$$

the topology τ_S is weaker than the relative topology of M of the strong one by the norm of X. On the other hand, by the corollary 1 of the extension theorem and by the proposition 1, we have $\|x - x_0\| = \sup_{\substack{L(f) \le 1 \\ f \in M_L^*}} \left| f(x) - f(x_0) \right|$. Now let S^L be the unit sphere of M_L^*, then we have $B(x_0;\, \varepsilon) = U(x_0; S^L; \varepsilon)$. This means that topology τ_S is stronger than the relative topology of M of the strong one by the norm of X. Therefore the tow topology is equal to each other. It is clear that the topology τ_W is equal to relative topology of M of the weak one of X. This shows that the relation of the topologies in the theorem is true. Finally, we'll give a short and clear explanation that the converse relations are not true. In general, as it is well known, the topologies of the topological spaces determine the convergences of the sequence in the spaces. And the convergences by the tow topologies equal to each other are also equal. In the below example, we'll show that there exists a sequence $\{x_n\}$ such that $\{x_n\}$ is convergent in the topology τ_L but not strongly, or $\{x_n\}$ is weakly but not in τ_L. This completes the proof.

Let $x_n, x_0 \in M$. Then it is easy to see that the sequence $\{x_n\}$ converges to x_0 in the L(f_0)--topology of M if and only if, for any $f \in {}^0M_L^*$, $f(x_n) \to f(x_0)$. In connection with this, we may introduce a following notion.

Definition 6. A sequence $\{x_n\}$ in M would be said to be L(f_0)-convergent if a finite $\lim_{n\to\infty} f(x_n)$ exists for each $f \in {}^0M_L^*$; $\{x_n\}$ would be said to L(f_0)-converge to an element $x_0 \in M$ if $f(x_n) \to f(x_0)$ for all $f \in {}^0M_L^*$. In the letter case, x_0 is uniquely determined by Hahn-Banach theorem; we shall write $L(f_0) - \lim_{n\to\infty} x_n = x_0$ or, in short, $x_n \xrightarrow{L(f_0)} x_0$. And M would be said to be L(f_0)-complete if every L(f_0)-convergent sequence of M L(f_0)-converges to an element of M.

The following theorems are valid.

Theorem 10. The following conditions are equivalent to each other.

i) $x_n \xrightarrow{L(f_0)} x_0$, ii) $\{\|x_n\|\}$ is bounded, and $f(x_n) \to f(x_0)$ for any f in the strongly dense subset of ${}^0M_L^*$, and iii) $x_n \xrightarrow{W} x_0$, $\|x_n\| \to \|x_0\|$.

The proof follows from the definition 2 and it is similar to one of weak convergence. In the future, we shall denote by L(f_0)- (S- or W-) all the notions and representations with respect to L(f_0)- (strong or weak) topology.

In this instance, the representation $x_n \overset{W}{\to} x_0$ in the theorem 2 means the sequence $\{x_n\}$ converges weakly to x_0.

Theorem 11. Let $x_n,\, x_0 \in M$. Then $x_n \overset{S}{\to} x_0$ implies $x_n \overset{L(f_0)}{\to} x_0$ and $x_n \overset{L(f_0)}{\to} x_0$ implies $x_n \overset{W}{\to} x_0$, but not conversely in general.

Proof. Since, for any $f \in M_L^*$, $|f(x_n) - f(x_0)| \le L(f)\|x_n - x_0\|$, first part is clear. The second part is surely valid from the definition 2. The following examples show that the converse relations are not true in general. Let $X = C[0,1]$. Take both sequence $\{x_n(t)\}$ and $\{x_n'(t)\}$ from X by

$$x_n(t) = \begin{cases} nt, & 0 \le t \le 1/n \\ 2 - nt, & 1/n \le t \le 2/n \\ 0, & 2/n \le t \le 1 \end{cases} \qquad x_n'(t) = \begin{cases} 1 - nt, & 0 \le t \le 1/n \\ nt - 1, & 1/n \le t \le 2/n \\ 1, & 2/n \le t \le 1 \end{cases}.$$

Then $x_n(t) \overset{W}{\to} x_0(t) \equiv 0$, but $x_n(t) \overset{L(f_0)}{\not\to} x_0(t)$ by iii) of the theorem 2 because $\|x_n(t)\| = 1 \not\to \|x_0(t)\| = 0$. And $x_n'(t) \overset{L(f_0)}{\to} x_0'(t) \equiv 1$, but $x_n'(t) \overset{S}{\not\to} x_0'(t)$ by $\|x_n'(t) - x_0'(t)\| = 1 \not\to 0$.

Remark. The examples in the theorem 2 show that the converse relations with respect to the several topologies in the theorem 1are not true in general. We'll now discuss the L(f_0)-completeness of M . From the definition 2, we see that a sequence $\{x_n\}$ in M is L(f_0)-convergent then it is weakly convergent. Therefore, the weak completeness of M is an important condition for L(f_0)-completeness of M . For example, the sequence $\{x_n(t)\}$ in the theorem 3 is L(f_0)-convergent because $x_n(t) \overset{W}{\to} x_0(t) \equiv 0$ and a finite $\lim_{n\to\infty}\|x_n\|$ exists, but doesn't L(f_0)-converges to any element of $X = C[0,1]$ at all. This example shows also that the space $C[0,1]$ is not L(f_0)-complete. If M is weakly complete and satisfies the condition such that $x_n \overset{W}{\to} x_0$ and $\|x_n(t)\| \to a$ (a real) implies $a = \|x_0\|$, then it is clearly L(f_0)-complete. For example, let $X = l_1$

or $X = L_1[0,1]$, then, as is well known, these spaces two are weakly complete, and both weak and strong convergence of the sequences in these are equivalent to each other, by the Schur's theorem ([3]). Therefore spaces $X = l_1$ or $X = L_1[0,1]$ are surely L(f_0)-complete. In addition, we know that the strong and weak topologies of the Banach space are equivalent to each other if and only if the dimension of the space is finite ([4]). Thus the Schur's theorem shows that the weak convergence of the sequence is of insufficient to study the weak topology. This is also true for L(f_0)-convergence. The following concept is an important one for L(f_0)-completeness.

Definition 7. A subset M_0 of M would be called a relatively L(f_0)-compact if every sequence $\{x_n\}$ in M_0 contains a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ L(f_0)-converges to an element $x_0 \in M$.

We can prove

Theorem 12. Let any bounded subset of M be a relatively L(f_0)-compact. Then M is L(f_0)-complete.

Proof. Let $\{x_n\}$ be an arbitrary L(f_0)-convergent sequence from M . Clearly, $\{x_n\}$ is a bounded as a subset of M . By the assumption, there exists a subsequence $\{x_{n_k}\}$ and an element $x_0 \in M$ such that $\{x_{n_k}\}$ L(f_0)-converges to x_0. Thus $\{x_n\}$ L(f_0)--converges to x_0.

References

[1] J. G. Peng, Z. B. Xu. A Novel Dual Notion of a Banach Space: Lipschitz Dual Space, Acta Mathematica Sinica, 42(1), 1999, 61-70, in Chinese

[2] J. G. Peng, Z. B. Xu, A Novel Dual Notion of a Nonlinear Lipschitz

Operator: Lipschitz Dual Operator, Acta Mathematica Sinica, 45(3), 2002, 469-480, in Chinese
[3] R. H. Martin, Nonlinear operator and differential equations in Banach spaces, New York, J. Wiley sons, 1976
[4] Kosaku Yosida, Functional Analysis, sixth edition, Berlin, 1980
[5] N. Dunford, J. Schwartz, Linear Operator, Vol. I, 1958
[6] K. Y. Chen, A. N. Fang, Bi-Lipschitz Maps in Q-regular Loewner Spaces, Acta Math. Sinica, English. Series, Sep. 2008. Vol. 24, No.9, 1555-1568
[7] R. G. Choe, An Extension Theorem of Nonlinear Lipschitz Functional and its Application in Banach Spaces, February 2012, viXra.org e-Print archive, viXra1202.0060, http://vixra.org/abs/1202.0060
[8] R. G. Choe, A L*-Convergence of Sequence of Nonlinear Lipschitz Functionals and its Applications in Banach Spaces, February 2012, viXra.org e-Print archive, viXra1202.0069, http://vixra.org/abs/1202.0069

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