A L*-Convergence of Sequence of Nonlinear Lipschitz Functionals and its Applications in Banach Spaces

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February 20, 2012

Abstract:

In this paper we have introduced a new concept on the convergence of a sequence of the nonlinear Lipschitz (Lip-) functionals, which would be called an L*-convergence, and we have considered its applications in Banach spaces. This convergence is very similar to the weak* (W*-) convergence of the sequence of the bounded linear functionals, but there are some differences. By the L*-convergence, we have considered the problem on the relations of the compactness between the Lip-operator and its Lip-dual operator, and we have obtained the mean ergodic theorems for the Lip-operator.

Keywords: Lipschitz functional; Lipschitz operator; Lipschitz dual space; Lipschitz dual operator

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2010 Mathematics Subject Classification: 46B07, 47H10;

1. Introduction

The monotone operator, compact operator and convex function are the typical nonlinear operators in Banach spaces. The Lipschitz (Lip-) operator is one of the most important nonlinear operators, but its properties are well known recently. There have been published many research results on the nonlinear Lip-operator [1,2,3,4,7].

In this paper we will introduce a L*-convergence of a sequence of the nonlinear Lip-functionals and, on the basis of it, we'll consider the problem in [1] on the relations of the compactness of Lip-operator and its Lip-dual operator, and obtain the mean ergodic theorems for the Lip-operator.

First, we'll recall the concepts on Lip- operator [8].

Let X and Y be real or complex Banach spaces, M and D closed subsets of X , Y respectively. Let $0 \in M$, $0 \in D$ and $T : M \to D$ be an operator. Unless otherwise noted, in this paper we shall not repeat above assumptions. If there exists a constant $L \geq 0$ such that, for all $x, y \in M$, $\|Tx - Ty\| \leq L\|x - y\|$, then operator T is called a Lip-operator on M .

And $L_M(T) = \sup_{x \neq y} \|Tx - Ty\| / \|x - y\|$ is called a Lip-constant of T on M .

We'll often use a set $Lip_0(M,D)$, that is,

$$Lip_0(M,D) = \{T : M \to D \mid T(0) = 0,\ T \text{ is an Lip-operator on } M\}.$$

If the set D is a linear subspace of Y, then the set $Lip_0(M,D)$ is a normed linear space and the Lip-constant $L_M(T)$ is a norm of T in $Lip_0(M,D)$. And if D is a closed linear subspace, in short, a closed subspace, then the normed linear space $Lip_0(M,D)$ is a Banach space by the norm $L_M(T)$. In particular, if $D = K$ (real or complex field), then the space $Lip_0(M,D)$ is called a Lip-dual space of M . We denote it by M_L^*. And the element of M_L^* is called a Lip-functional. In the case of $M = X$, we denote by X_l^* the ordinary dual space of

Banach space X, which consists of all bounded linear functionals defined on X and would be called a linear dual space of X, in distinction from Lip-dual space X_L^* of X. Then it is clear that X_l^* is a closed subspace of X_L^*. For any $x \in M$, $f \in D_L^*$, an operator defined by $(T_L^* f)(x) = (f \circ T)(x) = f(Tx)$ is called a Lip-dual operator of T and we denote it by T_L^*. Then it is clear that $T_L^* \in BL(D_L^*, M_L^*)$ and $L_M(T) = \|T_L^*\|$, where $BL(D_L^*, M_L^*)$ is the Banach space consisting of all the bounded linear operators on D_L^* into M_L^*([2]). Since the space M_L^* is a Banach space and the operator T_L^* is a bounded linear, it is defined a linear dual space $M_{Ll}^{**} = (M_L^*)_l^*$ of M_L^* and a linear dual operator $T_{Ll}^{**} = (T_L^*)_l^*$ of T_L^* respectively. Then it is easy to see that $T_{Ll}^{**} \in BL(M_{Ll}^{**}, D_{Ll}^{**})$ and $L_M(T) = \|T_L^*\| = \|T_{Ll}^{**}\|$.

In the study of Lip-operator, the need to extend the Lip-functional satisfying certain conditions is presented frequently, but that is reduced to the possibility of the extension to whole space with Lip-continuity and maintenance of Lip-constant of Lip-functional defined at a subset of Banach spaces. The following theorem gives us a sure guarantee for such possibility.

Theorem 1[8]. Let f be a real-valued Lip-functional defined on a closed subset M of a real Banach space X. Then there exists a real-valued Lip-functional F defined on X such that 1) F is an extension of f, i.e., $F(x) = f(x)$ for $x \in M$, and 2) $L_X(F) = L_M(f)$.

Theorem $1'$[8]. Let f be a complex Lip-functional defined on a closed subset M of a complex Banach space X. Then there exists a complex Lip-functional F defined on X such that 1) $F(x) = f(x)$ for $x \in M$, and 2) $L'_X(F) = L'_M(f)$, where $L'_M(f) = \left(L_M^2(g) + L_M^2(h)\right)^{1/2}$ and, g and h are the real and imaginary parts of f respectively.

As will be seen from these theorems, we can say that the extension theorem is a generalization to nonlinear Lip-functional of the Hahn-Banach theorem ([5]) on the extension of the bounded linear functional. Some corollaries follow from the extension theorem. The following corollary is one of those.

Corollary [8]. For any $x_0 \in X \setminus M$, there exists a real-valued Lip-functional f defined on X such that 1) $f(x)=0$ for $x \in M$, 2) $f(x_0)=d$, and 3) $L_X(f)=1$, where $d = \inf_{z \in M} \|z - x_0\| > 0$.

Proposition 1 [8]. Let $T \in Lip_0(M,D)$. Then M is a certain subset of M_{Ll}^{**} in isometric embedding sense. If an operator $J : M \to M_{Ll}^{**}$ is such isometric mapping, then we have, for all $x, y \in M$,

$$\|x-y\| = \|Jx-Jy\| = \sup_{f \in M_L^*, L(f) \le 1} |f(x)-f(y)| \text{ and } \|Tx-Ty\| = \|T_{Ll}^{**}(Jx) - T_{Ll}^{**}(Jy)\|.$$

Here, for any $x \in M$, a functional $J(x)$ defined on M_L^* by $J(x)(f) = f(x)$ is a bounded linear and an operator $J : M \to JM = \{J(x) \in M_{Ll}^{**} | x \in M\}$ is an isometric mapping satisfying the conditions of the theorem.

2. A L*-convergence of sequence of Lip-functionals

We shall introduce a new convergence of a sequence of Lip-functionals in Lip-dual space. This convergence is similar to W*-convergence of the sequence of bounded linear functionals in the linear dual space of Banach space, but there are some differences.

Definition 1. A sequence $\{f_n\}$ in Lip-dual space M_L^* would be said to be L*-convergent if a finite $\lim_{n\to\infty} f_n(x)$ exists for each $x \in M$; $\{f_n\}$ would be said to L*-converges to an element $f_0 \in M_L^*$ if $\lim_{n\to\infty} f_n(x) = f_0(x)$ all $x \in M$. In the later case, we'll write $f_0 = L^* - \lim_{n\to\infty} f_n$ or , in short, $f_n \xrightarrow{L^*} f_0$.

Remark. In general, the Banach-Steinhaus theorem - the resonance theorem ([5]) is not valid for the sequence of Lip-functionals. In other words, it is not true that L^* -convergence of the sequence $\{f_n\}$ implies $\sup_n L(f_n) < +\infty$. For example, set $f_n(x) = \sin nx/\sqrt{n}$ for $x \in M = [0,1] \subset R^1$ (the set of real numbers), then it is clear that $\{f_n\} \subset M_L^*$ and $f_n \xrightarrow{L^*} f_0 = 0$, but $\sup_n L(f_n) = +\infty$. The linear dual space of Banach space was always complete in the sense of W*-convergence of the sequence of bounded linear functionals. But it follows from this example that the Lip-dual space may not be complete in the sense of L*-convergence of the sequence of Lip-functionals. (Private talk; we have discussed this example in our work in 2003. And we have seen this example at the paper [3], too. In [3], taken f_n by $f_n(x) = \sin n^2 x/n$ and the example have been discussed not in the sense of L*-convergence of a sequence of Lip-functionals, but W*-convergence. However, the ideas in the papers [3] and our work look equal to each other. Of course, there would be some differences in viewpoint of its discussion. We were surprised that our research method and idea for Lip-operator are coinciding with Prof. Peng Jigen'. This gave us a mind to study the Lip-operator theory with great confidence.)

The following properties and their proofs for L*-convergence are very similar to one of the W*-convergence of the sequence of the bounded linear functionals.

Proposition 2. i) If $\{f_n\}$ is strongly convergent to f_0, that is $\|f_n - f_0\| \to 0$, then $f_0 = L^* - \lim_{n\to\infty} f_n$, but not conversely. **ii)** If $\sup_n L(f_n) < +\infty$ and $f_0 = L^* - \lim_{n\to\infty} f_n$, then f_0 is unique and $L(f_0) \le \sup_n L(f_n)$. **iii)** Suppose that $\sup_n L(f_n) < +\infty$. Then a sequence

$\{f_n\}$ L*-converges to an element $f_0 \in M_L^*$ if and only if $\lim_{n\to\infty} f_n(x) = f_0(x)$ on a strongly dense subset of M.

Definition 2. A subset M_0' of M_L^* would be called a L*-relatively compact if every sequence $\{f_n\}$ in M_0' contains a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ L*-converges to an element $f_0 \in M_L^*$.

We can obtain the following statement.

Proposition 3. If M is separable, then each bounded subset of M_L^* is a L*-relatively compact.

The proof is done by the above properties ii), iii) and the diagonal method. That is similar to one that if the Banach space X is separable then the bounded subset of the linear dual space X_l^* of X is W*-relatively compact ([5]). On the other hand, It is well known that any bounded subset of X_l^* is W*-relatively compact without the separability of X (the theorem 2 in Chapter V, 4 of [6]). But, for Lip-functional, the assumption that M is separable is essential.

3. The relations of the compactness between Lip-operator and its Lip-dual operator

We'll recall the concepts of the weak or strong compact operator.

Let M_0 be the bounded subset of M. An operator $T \in Lip_0(M,D)$ is said to be weakly (or strongly) compact (W.C. or S.C.) if the image $T(M)$ is relatively weakly (or strongly) compact in D [4, 6].

We'll always assume that $T \in Lip_0(M,D)$ below.

The following theorems for the compactness of Lip-operator are valid.

Theorem 2. If T is S.C., then T_L^* is L*-compact in the sense that the image $T_L^*(D_0^/)$ by T_L^* of any bounded subset $D_0^/$ of D_L^* is L*-relatively compact of M_L^*.

Proof. Let $\{f_n\}$ be a sequence from $T_L^*(D_0^/)$. Then there exists $\{g_n\}$ in the subset $D_0^/$ such that $f_n = T_L^* g_n$. We denote by $\tilde{g}_n$ the contraction to the subset $T(M)$ of g_n, i.e., $\tilde{g}_n = g_n |_{T(M)}$. Then $\{\tilde{g}_n\}$ is a clearly bounded set. On the other hand, it is well known that if T is S.C. then the range $T(M)$ of T is strong separable. Therefore, by the proposition 3, the set $\{\tilde{g}_n\}$ of Lip-functionals defined on $T(M)$ is L*-relatively compact. Hence there exist a subsequence $\{\tilde{g}_{n_k}\}$ of $\{\tilde{g}_n\}$ and a functional $\tilde{g}_0$ defined on $T(M)$ such that $\tilde{g}_{n_k}(y)$ converges $\tilde{g}_0(y)$ for any $y \in T(M)$. Since $\sup_k L(\tilde{g}_{n_k}) < +\infty$, we have, for any $y_1, y_2 \in T(M)$, $\left|\tilde{g}_{n_k}(y_1) - \tilde{g}_{n_k}(y_2)\right| \le \sup_k L(\tilde{g}_{n_k}) \|y_1 - y_2\|$. Here, by letting $k \to \infty$, we have $\left|\tilde{g}_0(y_1) - \tilde{g}_0(y_2)\right| \le \sup_k L(\tilde{g}_{n_k}) \|y_1 - y_2\|$. Thus $\tilde{g}_0$ is a Lip-functional defined on $T(M)$. By the extension theorem, we can extend $\tilde{g}_0$ from $T(M)$ to D and we denote it by g_0. Then it is clear that $g_0 \in D_L^*$. Put $f_0 = T_L^* g_0$, then we have $f_0 \in M_L^*$ and $f_{n_k} \xrightarrow{L^*} f_0$. In fact, for any $x \in M$,

$$\begin{aligned} &f_{n_k}(x) = T_L^* g_{n_k}(x) = g_{n_k}(Tx) = \tilde{g}_{n_k}(y) \to \\ &\to \tilde{g}_0(y) = \tilde{g}_0(Tx) = g_0(Tx) = T_L^* g_0(x) = f_0(x). \end{aligned}$$

This completes the proof. □

Theorem 3. If T_L^* is S.C. then T is S.C.

Proof. Since T_L^* is a bounded linear operator on D_L^*, if T_L^* is S.C. then its linear dual operator T_{Ll}^{**} is also S.C. Therefore the image $T_{Ll}^{**}(M_0^*)$ by T_{Ll}^{**} of any bounded subset M_0^* of M_{Ll}^{**} is relatively S.C. in D_{Ll}^{**}. Let M_0 be any bounded subset of M. We have to prove that the image $T(M_0)$ is relatively S.C. in D. Take any sequence $\{y_n\}$ from $T(M_0)$. As was stated in the above proposition 1, we'll denote by J_M and by J_D the isometric mappings from

M and D into M_{Ll}^{**} and D_{Ll}^{**} respectively. Then $J_M(M_0)$ is a bounded subset in M_{Ll}^{**} and so $T_{Ll}^{**}(J_M(M_0))$ is relatively S.C. in D_{Ll}^{**}. On the other hand, since $J_D(T(M_0)) = T_{Ll}^{**}(J_M(M_0))$, the set $J_D(T(M_0))$ is also relatively S.C. in D_{Ll}^{**}. Hence, by putting $\tilde{y}_n = J_D(y_n)$, then $\tilde{y}_n \in J_D(T(M_0)) = \ = T_{Ll}^{**}(J_M(M_0))$ and there exist a subsequence $\{\tilde{y}_{n_k}\}$ of $\{\tilde{y}_n\}$ such that $\{\tilde{y}_{n_k}\}$ converges strongly in $T_{Ll}^{**}(J_M(M_0))$. Thus $\{\tilde{y}_{n_k}\}$ is strong Cauchy sequence. On the other hand, since the mapping J_D is isometric, we have $\|y_{n_k} - y_{n_l}\| = \|J_D(y_{n_k}) - J_D(y_{n_l})\| = \|\tilde{y}_{n_k} - \tilde{y}_{n_l}\|$ and so $\{y_{n_k}\}$ is strong Cauchy sequence in D. Therefore $T(M_0)$ is relatively S.C. in D. This completes the proof of the theorem. □

Theorem 4. If T_L^* is W.C. and $J_D(D)$ is closed in the sense of the W*-convergence, then T is S.C.

Proof. As the above theorem 3, Let M_0 be any bounded subset of M and $\{y_n\}$ any sequence from $T(M_0)$. Since T_{Ll}^{**} is also W.C., the set $J_D(T(M_0)) = T_{Ll}^{**}(J_M(M_0))$ is relatively W.C. in D_{Ll}^{**}. Therefore the sequence $\{\tilde{y}_n = J_D(y_n)\}$ contains a subsequence $\{\tilde{y}_{n_k}\}$ such that $\{\tilde{y}_{n_k}\}$ converges weakly in $T_{Ll}^{**}(J_M(M_0))$. Hence $\{\tilde{y}_{n_k}\}$ is W*-convergent. Since D_{Ll}^{**} is a linear dual space of the Banach space D_L^*, it is complete in the sense of the W*-convergence. Thus there exists an element $\tilde{y}_0 \in D_{Ll}^{**}$ such that $\{\tilde{y}_{n_k}\}$ is W*-converges to $\tilde{y}_0$. On the other hand, by the assumption, $\tilde{y}_0$ must belong to $J_D(D)$. Consequently, for any $f \in D_L^*$, the sequence $\{\tilde{y}_{n_k}(f)\}$ converges to $\tilde{y}_0(f)$. Now Let $y_0 \in D$ and $\tilde{y}_0 = J_D(y_0)$. Then, as was stated in the above proposition 1, we have $\tilde{y}_{n_k}(f) = f(y_{n_k}),\ \tilde{y}_0(f) = f(y_0)$ for any $f \in D_L^*$. Since the sequence $\{\tilde{y}_{n_k}(f)\}$ converges to $\tilde{y}_0(f)$, we have, for any $f \in D_L^*$, $f(y_{n_k}) \to f(y_0)$. We now shall show that the sequence $\{y_{n_k}\}$ is strongly convergent to y_0. If it does not, the point y_0 is not in the strong closure of $\{y_{n_k}\}$. We denote by D_0 the strong closure of $\{y_{n_k}\}$. Then we have $d = \inf_{z\in D_0} \|z - y_0\| > 0$. Therefore, by the corollary of the extension theorem, there exists a functional $f_0 \in D_L^*$ such that $f_0(z) = 0\ (z \in D_0)$, $f_0(y_0) = d$ and $L(f_0) = 1$. This is contrary to the above. Therefore $T(M_0)$ is relatively S.C. in D. This completes the proof of the theorem. □

Remark. In general, the following statement is true: " if, for any $f \in D_L^*$, the sequence $\{f(y_n)\}$ converges to $f(y_0)$, then $\{y_n\}$ converges strongly to y_0." Without using the extension theorem, this is easily proved as follows. We define a functional f_0 on D by $f_0(y) = \|y - y_0\| - \|y_0\|$ for $y \in D$. Then it is clear that $f_0 \in D_L^*$ and $L_D(f_0) = 1$. Therefore we have

$$\|y_n - y_0\| = \big|\|y_n - y_0\| - \|y_0\| + \|y_0\|\big| = |f(y_n) - f(y_0)| \to 0 \ .$$

The linear dual operator of the bounded linear operator maps a W*-convergent sequence of bounded linear functionals to the W*-convergent one. Similarly, we see easily that the Lip-dual of Lip-operator maps a L*-convergent sequence of Lip-functionals to the L*-convergent one of Lip-functionals. In this connection, we may introduce a following concept.

The operator T_L^* would be said to satisfy L*-W (or L*-S) property if it maps a L*-convergent bounded sequence of Lip-functionals to the weakly (or strongly) convergent one of Lip-functionals. It is easy to see that if T_L^* is W.C. (or S.C.) then it satisfies L*-W (or L*-S) property, but not conversely. However, the following theorem is valid.

Theorem 5. Let D be separable and T_L^* satisfy the L*-W (or L*-S) property. Then T_L^* is W.C. (or S.C.)

Proof. Take any bounded subset D_0' of D_L^* and a sequence $\{f_n\}$ in $T_L^*(D_0')$. Then there exists $\{g_n\}$ in the subset D_0' such that $f_n = T_L^* g_n$. Since D is separable and D_0' is bounded, by the proposition 3, the subset D_0' is L*-relatively compact, that is, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and an element $g_0 \in D_L^*$ such that $g_{n_k} \xrightarrow{L^*} g_0$. By the hypothesis of the theorem, $f_{n_k} = T_L^* g_{n_k}$ converges weakly (or strongly) to $f_0 = T_L^* g_0$. Therefore T_L^* is W.C. (or S.C.). □

Theorem 6. Let D_L^* be separable and D satisfy following condition:
If $\{y_n\}$ is a sequence of elements in D and a finite $\lim_{n\to\infty} g(y_n)$ exists for each $g \in D_L^*$, then there exists an element $y_0 \in D$ such that $g(y_n) \to g(y_0)$. Then T is S.C.

Proof. As in the theorem 3, let M_0 be any bounded subset of M and take any sequence $\{y_n\}$ from $T(M_0)$. Let $\{g_n\}$ be a strong dense set of a countable number of elements in D_L^*. Then, since $\{y_n\}$ is bounded, we can choose, by the diagonal method, a subsequence $\{y_{n_k}\}$ such that a finite $\lim_{k\to\infty} g_n(y_{n_k})$ exists for every fixed n. And, for any $g \in D_L^*$ and $\varepsilon > 0$, there exists $g_{n_0} \in \{g_n\}$ such that $L(g - g_{n_0}) < \varepsilon$. We have

$$\begin{aligned} &\left|g(y_{n_i}) - g(y_{n_j})\right| \le \left|g(y_{n_i}) - g_{n_0}(y_{n_i})\right| + \left|g_{n_0}(y_{n_i}) - g_{n_0}(y_{n_j})\right| + \left|g_{n_0}(y_{n_j}) - g(y_{n_j})\right| \le \\ &\le L(g - g_{n_0})\left\|y_{n_i}\right\| + \left|g_{n_0}(y_{n_i}) - g_{n_0}(y_{n_j})\right| + L(g - g_{n_0})\left\|y_{n_j}\right\| \end{aligned}$$

and so $g(y_{n_k})$ is Cauchy sequence for each $g \in D_L^*$. By the assumption of the theorem, there exists an element $y_0 \in D$ such that $g(y_{n_k}) \to g(y_0)$. Therefore, by the corollary of the extension theorem, we have that $\{y_{n_k}\}$ converges strongly to y_0. This shows that T is S.C. □

The following theorem shows the properties of the set of the W.C. (or S.C.) operators.

Theorem 7. (i) A linear combination of W.C. (or S.C.) operators is W.C. (or S.C.) respectively. **(ii)** Let a sequence $\{T_n\}$ of W.C. (or S.C.) operators in $Lip_0(M,D)$ converge to a Lip-operator T in the sense that $\lim_{n\to\infty} L(T_n - T) = 0$. Then T is also W.C. (or S.C.). **(iii)** The product of a Lip-operator by a S.C. operator is S.C. operator. In other words, the set of S.C. operators has the like property with closed two-sided ideal in the Banach space $Lip_0(M,M)$. **(iv)** The product of a Lip-operator by a W.C. operator on the left is W.C. operator, but not on the right.

Remark. The proof of this theorem is similar to the case of the bounded linear operator. We show a counter example with respect to the letter part of (iv). To do that, it would be sufficient to show that there exist a Lip-operator in $Lip_0(M,M)$ and a weakly convergent sequence $\{x_n\}$ in M such that the image $\{Tx_n\}$ by T of $\{x_n\}$ doesn't converges weakly.

Let $X = C[0,1]$. Here $C[0,1]$ is the space consisting of all continuous functions defined on the real interval $[0,1]$. Take both sequence $\{x_n\}$ and $\{\tilde{x}_n\}$ by $x_n(t) = nt$ for $0 \le t \le 1/n$, $x_n(t) = 2 - nt$ for $1/n \le t \le 2/n$, $x_n(t) = 0$ for $2/n \le t \le 1$, and $\tilde{x}_n(t) = 1 - nt$ for $0 \le t \le 1/n$, $\tilde{x}_n(t) = 0$ for $1/n \le t \le 1$. And we put $\tilde{x}_0(t) = 1$ for $t = 0$, $\tilde{x}_0(t) = 1$ for $0 < t \le 1$. Then $\{x_n\}$ converges weakly to $x_0(t) \equiv 0$, but $\{\tilde{x}_n\}$ doesn't converges weakly to $\tilde{x}_0$ because $\tilde{x}_0$ isn't in $C[0,1]$ and so $\{\tilde{x}_n\}$ can't contain any weakly convergent subsequence. We define an operator T by $Tx_n = \tilde{x}_n$, then T is a Lip-operator defined on $M = \{x_n\}$.

4. The mean ergodic theorems for Lip-operator

The fixed point of an operator is the solution of the equation containing that operator. The existence of the solution of an operator equation is just existence of the fixed point of the operator. Recently, there have been published many research results on the fixed point of Lip-operator, in particular, the nonexpansive operator [4,7]. An important task in the fixed-point theory of the operator is not only to evidence the existence of the fixed point, but also to compose the approximate sequence converging to one. The mean ergodic theorems are of the important in the construction of such approximate sequence. In the paper [1], there have been considered, by Lip- dual operator of Lip- operator, the mean ergodic theorem of the nonexpansive operator in uniformly convex Banach spaces with Frechet differentiable norm

In this section we will consider a mean ergodic theorem of Lip- operator defined on the closed convex subset of weakly complete Banach space. Our purpose is to obtain the results for Lip-operator by Lip-dual operator as the bounded linear operator.

We put $S_n = \frac{1}{n}\sum_{m=1}^{n} T^m$ and $S_n^L = \frac{1}{n}\sum_{m=1}^{n} (T_L^*)^m$. If $X = Y$ and $M = D$, then we'll denote $Lip_0(M,D)$ by $Lip_0(M,M)$. It is clear that $S_n \in Lip_0(M,M)$ and $S_n^L \in BL\left(M_L^*, M_L^*\right)$ for each n if M is a convex set of X. Unless otherwise noted, in the below theorems we shall always assume following condition, that is, X is a weakly complete Banach space, M is a closed convex subset of X and $T \in Lip_0(M,M)$.

Theorem 8. Suppose that $\sup_{n\geq 1} L(S_n^L) < +\infty$ and $\sup_{n\geq 1} L(S_n) < +\infty$. If, for any $f \in M_L^*$, a sequence $\{S_n^L f\}$ is weakly convergent, then there exists a fixed point x_0 of T in M, and, for any $x \in M$, the sequence $\{S_n x\}$ converges weakly to x_0.

Proof. Since a finite $\lim_{n\to\infty}(S_n^L f)(x)$ exists for any $f \in M_L^*$ and $x \in M$, we shall put $g(x) = \lim_{n\to\infty}(S_n^L f)(x)$. Then it is not difficult to see that $g \in M_L^*$ and $L(g) \leq \sup_n L(S_n^L f)$. In fact, we have, for any $x_1, x_2 \in M$, $\left|(S_n^L f)(x_1) - (S_n^L f)(x_2)\right| \leq \sup_{n\geq 1} L(S_n^L f)\|x_1 - x_2\|$ and**,** by letting $n \to \infty$, we have $\left|g(x_1) - g(x_2)\right| \leq \sup_{n\geq 1} L(S_n^L f)\|x_1 - x_2\|$. Therefore g must belong to M_L^* and we see that $L(g) \leq \sup_n L(S_n^L f)$. And, for any $f \in M_L^*$, the sequence $\{S_n^L f\}$ is L*-convergent to the element $g \in M_L^*$, that is, $S_n^L f \xrightarrow{L^*} g$. We now define an operator $T_0^/$ by $T_0^/ : f \to g$, that is, $T_0^/ f = g$. Then it is clear that $T_0^/ \in BL(M_L^*, M_L^*)$. For, since $(T_0^/ f)(x) = \lim_{n\to\infty}(S_n^L f)(x)$ and each S_n^L is clearly in $BL(M_L^*, M_L^*)$, it is clear that the operator $T_0^/$ defined on M_L^* is also linear. And, by $L(T_0^/ f) \leq \sup_n L(S_n^L f)$ for any $f \in M_L^*$, we have $L(T_0^/) \leq \sup_n L(S_n^L) < +\infty$. On the other hand, we take any bounded linear functional f defined on X and denote by $\tilde{f}$ the contraction to the set M of f, i.e., $\tilde{f} = f|_M$. Then $\tilde{f}$ is a clearly in M_L^*. We define by $\tilde{M}_L^*$ the set consisting of all $\tilde{f}$'s. then we see easily that $\tilde{M}_L^*$ is a subset of M_L^*. We have, for any $x \in M$ and $\tilde{f} \in \tilde{M}_L^*$,

$$
\begin{aligned}
(S_n^L \tilde{f})(x) &= \left(\frac{1}{n}\sum_{m=1}^{n}(T_L^*)^m \tilde{f}\right)(x) = \left(\frac{1}{n}\sum_{m=1}^{n}((T_L^*)^m \tilde{f})(x)\right) = \\
&= \frac{1}{n}\sum_{m=1}^{n}\tilde{f}(T^m x) = \tilde{f}\left(\frac{1}{n}\sum_{m=1}^{n}T^m x\right) = \tilde{f}(S_n x)
\end{aligned}
$$

and so $(S_n^L \tilde{f})(x) = \tilde{f}(S_n x) \to g(x) = (T_0^/ \tilde{f})(x)$. Therefore, a sequence $\{S_n x\}$ is weakly convergent. By the assumption that X is weakly complete and M is closed convex, there exists an element $x_0 \in M$ such that $x_0 = W - \lim_{n\to\infty} S_n x$. We define again an operator T_0 by $T_0 : x \to x_0$, that is, $T_0 x = x_0$. Then it is easy to see that $T_0 \in Lip_0(M,M)$. In fact, we have, for any $x_1, x_2 \in M$,

$$\left|\tilde{f}(S_n x_1) - \tilde{f}(S_n x_2)\right| \le \sup_{n\ge1} L(S_n)\left\|\tilde{f}\right\|\left\|x_1 - x_2\right\|$$

and so $\left|\tilde{f}(T_0 x_1) - \tilde{f}(T_0 x_2)\right| \le \sup_{n\ge1} L(S_n)\left\|\tilde{f}\right\|\left\|x_1 - x_2\right\|$. Thus we must obtain $\left\|T_0 x_1 - T_0 x_2\right\| \le \sup_{n\ge1} L(S_n)\left\|x_1 - x_2\right\|$, so T_0 belongs to $Lip_0(M,M)$. Therefore it is defined Lip-dual operator $(T_0)_L^*$ of T and we have $(S_n^L \tilde{f})(x) = \tilde{f}(S_n x) \to g(x) = (T_0^{/} \tilde{f})(x) = \tilde{f}(T_0 x) = ((T_0)_L^* \tilde{f})(x)$ and $(S_n^L \tilde{f})(Tx) = \tilde{f}(S_n Tx) \to ((T_0)_L^* \tilde{f})(Tx) = ((T_L^*)(T_0)_L^* \tilde{f})(x)$ for any $x \in M$ and $\tilde{f} \in \tilde{M}_L^*$. On the other hand, by the hypothesis such that $\sup_{n\ge1} L(S_n^L) < +\infty$, we have $n^{-1}\left(T_L^*\right)^n \tilde{f} = (n+1)\cdot n^{-1}\left(S_{n+1}^L \tilde{f}\right) - S_n^L \tilde{f} \xrightarrow{L^*} 0$, and, since $(T_0)_L^*$ is com- mutative with each S_n^L, we obtain

$$\begin{aligned}(S_n^L \tilde{f})(Tx) &= \left(T_L^* S_n^L \tilde{f}\right)(x) = \left(S_n^L T_L^* \tilde{f}\right)(x) = \\ &= \left(S_n^L \tilde{f}\right)(x) + \frac{1}{n}\left(\left(\left(T_L^*\right)^n - I_L^*\right)\tilde{f}\right)(x) \quad \to \left(\left(T_0\right)_L^* \tilde{f}\right)(x).\end{aligned}$$

Therefore we have $\left((T_L^*)(T_0)_L^* \tilde{f}\right)(x) = \left((T_0)_L^* \tilde{f}\right)(x) = \tilde{f}(T_0 \cdot Tx) = \tilde{f}(T_0 x)$.

By the arbitrariness of $\tilde{f} \in \tilde{M}_L^*$ and the Hahn-Banach theorem ([5]), it is true that $T_0 \cdot Tx = T_0 x$. Thus we have $\left((T_L^*)(T_0)_L^* f\right)(x) = \left((T_0)_L^* f\right)(x)$, for any $f \in M_L^*$. Again by the arbitrariness of $x \in M$, we must have $(T_L^*)(T_0)_L^* f = (T_0)_L^* f$. This proves that, for any $f \in M_L^*$, the element $(T_0)_L^* f$ is a fixed point of T_L^*. On the other hand, since $\sup_{n\ge1} L(S_n^L) < +\infty$ and the sequence $\{S_n^L f\}$ is weakly convergent, the sequence $\{S_n^L f\}$, for any $f \in M_L^*$, converge strongly to the fixed point $(T_0)_L^* f$ of T_L^* by the theorem 2 in Chapter VIII, 3 of [5], which is the mean ergodic theorem for the bounded linear operator and is proved on the basis of the Mazur' theorem. Therefore we have, for any $f \in M_L^*$, $S_n^L f \xrightarrow{L^*} (T_0)_L^* f$ and so $S_n^L T_L^* f \xrightarrow{L^*} (T_0)_L^* T_L^* f$. By repeating the same argument as above, we have $S_n^L T_L^* f \xrightarrow{L^*} (T_0)_L^* f$ and $(T_0)_L^* T_L^* f = (T_0)_L^* f$. Thus we have $f(T \cdot T_0 x) = f(T_0 x)$. By the arbitrariness of $f \in M_L^*$, we must have $T \cdot T_0 x = T_0 x = x_0$. Therefore, for any $x \in M$, the element $T_0 x = x_0$ is a fixed point of T. This completes the proof. □

Corollary 1. Suppose that there exists a constant $K \geq 0$ such that $L(T^n) \leq K$ for any n or that the operator T is nonexpansive, that is, $L(T) \leq 1$. If, for any $f \in M_L^*$, a sequence $\{S_n^L f\}$ is weakly compact, then there exists a fixed point x_0 of T in M and, for any $x \in M$, the sequence $\{S_n x\}$ converges weakly to x_0.

Proof. Since $L(T) = \|T_L^*\|$ and $(T^n)_L^* = (T_L^*)^n$ for each n, by the assumption, we have that $\sup_{n \geq 1} L(S_n^L) < +\infty$ and $\sup_{n \geq 1} L(S_n) < +\infty$. On the other hand, since, for any $f \in M_L^*$, a sequence $\{S_n^L f\}$ is weakly compact, there exist a subsequence $\{S_{n_k}^L f\}$ such that $\{S_{n_k}^L f\}$ is weakly convergent. Then, by the theorem 2 in Chapter VIII, 3 of [5], for any $f \in M_L^*$, a sequence $\{S_n^L f\}$ is weakly convergent. Therefore the result of the corollary follows from the above theorem 8.

Remark. The Lip-operator T is said to be uniform if there exists a constant $K \geq 0$ such that $L(T^n) \leq K$ for any n ([1]). As would be seen from the mean ergidic theorem, in the discussion of the convergence of the mean sequence for Lip-operator the condition, whether the Lip-operator is uniform or nonexpansive, doesn't have the essential difference.

Corollary 2. Suppose that M is separable and $L(T) \leq 1$. If T_L^* satisfies the L*-W property, then there exists a fixed point x_0 of T in M and, for any $x \in M$, the sequence $\{S_n x\}$ converges weakly to x_0.

Proof. It is clear that $\sup_{n\geq 1} L(S_n^L) < +\infty$ and $\sup_{n\geq 1} L(S_n) < +\infty$. Since M is separable, by the proposition 3, for any $f \in M_L^*$, the sequence $\{S_n^L f\}$ is L*-compact. Therefore there exist a subsequence $\{S_{n_k}^L f\}$ such that $\{S_{n_k}^L f\}$ is L*-convergent. By the assumption, the sequence $\{T_L^* S_{n_k}^L f\}$ is weakly convergent. On the other hand, since $\sup_{n\geq 1} L(S_n^L) < +\infty$, we have $\frac{1}{n}\left(T_L^*\right)^n f = \frac{n+1}{n}\left(S_{n+1}^L f\right) - S_n^L f \xrightarrow{W} 0$. By $S_n^L f = T_L^* S_n^L f - \frac{1}{n}\left(\left(T_L^*\right)^n - I_L^*\right) f$, the sequence $\{S_{n_k}^L f\}$ is weakly convergent. Consequently, the result of the corollary follows from the above theorem 8.

□

Corollary 3. Let X be a reflexive Banach space, M a bounded closed convex subset of X and let $L(T) \leq 1$. Then there exist an operator $T_0 \in Lip_0(M, M)$ such that, for any $\tilde{f} \in \tilde{M}_L^*$, $(T_0)_L^* \tilde{f}$ is a fixed point of T_L^*, and a subsequence $\{S_{n_k}^L \tilde{f}\}$ such that $\{S_{n_k}^L \tilde{f}\}$ L*-converges to $(T_0)_L^* \tilde{f}$. If either $\{S_{n_k}^L T_L^* \tilde{f}\}$ L*-converges to $(T_0)_L^* T_L^* \tilde{f}$, or T_L^* is commutative with $(T_0)_L^*$, then T has a fixed point in M.

Proof. It is clear that $\sup_{n\geq 1} L(S_n^L) < +\infty$ and $\sup_{n\geq 1} L(S_n) < +\infty$. Therefore, for any $x \in M$, the sequence $\{S_n x\}$ is a bounded subset in M. Hence, by the Eberlein-Shmulyan theorem ([5]), the set $\{S_n x\}$ is weakly compact and so there exist a subsequence $\{S_{n_k} x\}$ such that $\{S_{n_k} x\}$ is weakly convergent. Since the reflexive Banach space X is weakly complete and M is closed convex, there exists an element $x_0 \in M$ such that $\{S_{n_k} x\}$ converges weakly to x_0, that is, $S_{n_k} x \xrightarrow{W} x_0$. We now define an operator $T_0; M \to M$ by $T_0 x = x_0$. Then, by repeating the same argument as the theorem 8, we see easily that T_0 is a Lip-operator defined on M. And we have $\tilde{f}\left(S_{n_k} x\right) \to \tilde{f}(T_0 x) = ((T_0)_L^* \tilde{f})(x)$ and

$\tilde{f}\left(S_{n_k}Tx\right) = (S_{n_k}^L \tilde{f})(Tx) \to ((T_0)_L^* \tilde{f})(Tx) = ((T_L^*)(T_0)_L^* \tilde{f})(x)$ for $\tilde{f} \in \tilde{M}_L^*, x \in M$. Thus we have $\left((T_L^*)(T_0)_L^* \tilde{f}\right)(x) = \left((T_0)_L^* \tilde{f}\right)(x) = \tilde{f}(T_0 \cdot Tx) = \tilde{f}(T_0 x)$. Therefore, for any $\tilde{f} \in \tilde{M}_L^*$, $(T_0)_L^* \tilde{f}$ is a fixed point of T_L^*. If T_L^* satisfies the assumptions, then it follows from the process of the theorem 8 that, for any $x \in M$, $T_0 x$ is a fixed point of T. □

Remark. There is noted in [4] that the problem, whether there exists a fixed point of the nonexpansive operator defined on the bounded closed convex subset into itself of a reflexive Banach space or not, is unsolved. Until the present, there have been published many research results on this problem, but we couldn't find the paper completely solving one. We know that the problem was considered very much under the several assumptions on the subset M. The corollary 3 shows that the result is original obtained from the relation of the nonexpansive operator and its Lip-dual.

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