An Extension Theorem of Nonlinear Lipschitz Functional and its Application in Banach Spaces

Choe Ryong Gil

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Abstract

In this paper we have obtained a new theorem that a nonlinear Lipschitz (Lip-) functional defined on the closed subset of Banach spaces can be extended to the whole space with Lip-continuity and maintenance of Lip-constant, which would be called an extension theorem (ET). This theorem is a generalization to the Lip-functional of the famous Hahn-Banach theorem on the bounded linear functional. By the ET, we have completely solved the open problem on the relation of the invertibility between the Lip-operator and its Lip-dual operator.

Keywords: Lipschitz operator; Lipschitz functional; Lipschitz dual space; Lipschitz dual operator

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1. Introduction

The Lipschitz (Lip-) operator is one of the most important nonlinear operators with the monotone operator, compact operator and convex function in Banach spaces. Recently, there have been published many research results on the nonlinear Lip-operator [1,2].

In this paper we have obtained a new theorem that a nonlinear Lipschitz (Lip-) functional defined on the closed subset of Banach spaces can be extended to the whole space with Lip-continuity and maintenance of Lip-constant, which would be called an extension theorem (ET). This theorem is a generalization to the Lip-functional of the famous Hahn-Banach theorem on the bounded linear functional. By the ET, we have completely solved the problem in [1] on the relation of the invertibility between the Lip-operator and its Lip-dual operator. This problem was considered also in [2]. The result obtained from [2] is the same with our, but the method is different each other. In the future, the ET will be used constantly in the several sides of the study of the Lip-operator.

First, we recall the concepts on Lip- operator [1].

Let X and Y be real or complex Banach spaces, M and D closed subsets of X , Y respectively. Let $0 \in M$, $0 \in D$ and $T : M \to D$ be an operator. Unless otherwise stated, in this paper we will not repeat above assumptions. If there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L \ \|x - y\| \qquad \text{for all } x, y \in M, \tag{1}$$

then operator T is called a Lip-operator on M . And

$$L_M(T) = \sup_{\substack{x,y \in M \\ x \neq y}} \frac{\|Tx - Ty\|}{\|x - y\|} \tag{2}$$

is called a Lip-constant of T on M . We'll often use the set ([2]):

$$Lip_0(M,D)=\{T:M\to D \mid T(0)=0,\ T \text{ is an Lip-operator on } M\}.$$

If the set D is a linear subspace of Y, then the set $Lip_0(M,D)$ is a vector space and the Lip-constant $L_M(T)$ is a norm of T in $Lip_0(M,D)$. Therefore if D is a closed linear subspace, in short, a closed subspace, then the vector space $Lip_0(M,D)$ is a Banach space by the norm $L_M(T)$. In particular, if $D=K$ (real or complex field), then the space $Lip_0(M,D)$ is called a Lip-dual space of M ([2]). We will denote it by M_L^*. And the element of M_L^* is called a Lip-functional. In the case of $M=X$, we denote by X_l^* the ordinary dual space of Banach space X, which consists of all bounded linear functionals defined on X and would be called a linear dual space of X, in distinction from Lip-dual space X_L^* of X. Then it is clear that X_l^* is a closed subspace of X_L^*. For any $x\in M$, $f\in D_L^*$, an operator defined by

$$\left(T_L^* f\right)(x)=\left(f\circ T\right)(x)=f\left(Tx\right)$$

is called a Lip-dual operator of T and we denote it by T_L^* ([2]). Then it is clear that $T_L^*\in BL(D_L^*,M_L^*)$ and $L_M(T)=\|T_L^*\|$, where $BL(D_L^*,M_L^*)$ is the Banach space consisting of all the bounded linear operators on D_L^* into M_L^* ([2]). Since the space M_L^* is a Banach space and the operator T_L^* is a bounded linear, we can define a linear dual space $M_{Ll}^{**}=(M_L^*)_l^*$ of M_L^* and a linear dual operator $T_{Ll}^{**}=(T_L^*)_l^*$ of T_L^* respectively. Then it is easy to see that $T_{Ll}^{**}\in BL(M_{Ll}^{**},\ D_{Ll}^{**})$ and $L_M(T)=\|T_L^*\|=\|T_{Ll}^{**}\|$.

2. Extension theorem

In the study of Lip-operator, the need to extend the Lip-functional satisfying certain conditions is presented frequently, but it is reduced to the possibility of the extension to whole space of Lip-functional defined at a subset of Banach spaces. In this section we will consider a theorem on the extension to whole space with Lip-continuity and maintenance of Lip-constant of Lip-functional defined at a closed subset of Banach spaces. As will be seen below, we can say that the extension theorem (ET) is a generalization of the Hahn-Banach theorem ([4]) on the extension of the bounded linear functional.

Theorem 1. Let f be a real-valued Lip-functional defined on a closed subset M of a real Banach space X . Then there exists a real-valued Lip-functional F defined on X such that 1) F is an extension of f , i.e., $F(x) = f(x)$ for $x \in M$, and 2) $L_X(F) = L_M(f)$.

Proof. If $X \neq M$, choose $x_0 \in X \setminus M$, and define $M_1 = M \cup \{x_0\}$. The first part of the proof is to extend the functional f from M to M_1 with Lip-continuity and maintenance of Lip-constant. We define a functional f_1 on M_1 by

$$f_1(x) = \begin{cases} f(x), & if\ x \in M \\ C, & if\ x = x_0 \end{cases} \tag{3}$$

Here a constant C is determined concretely as follows.

Since f is Lip-functional on M , we have, for all $x_1, x_2 \in M$, that

$$f(x_1) - f(x_2) \leq L_M(f)\|x_1 - x_2\| \leq L_M(f)\|x_1 - x_0\| + L_M(f)\|x_2 - x_0\|.$$

Hence $f(x_1) - L_M(f)\|x_1 - x_0\| \leq f(x_2) + L_M(f)\|x_2 - x_0\|$.

We now denote constants A and B by

$$A = \sup_{x \in M}\left(f(x) - L_M(f)\|x - x_0\|\right),\ B = \inf_{x \in M}\left(f(x) + L_M(f)\|x - x_0\|\right),$$

then we obtain that $A \leq B$ because $x_1, x_2 \in M$ are arbitrary. Therefore there exists a constant C such that $A \leq C \leq B$. The constant C in (3) is such. We'll show that the functional f_1 defined by (3) is the Lip-functional on M_1 and

$L_{M_1}(f_1)=L_M(f)$. Choose any $x,y\in M_1$. If $x,y\in M$, then it is clear that $\left|f_1(x)-f_1(y)\right|=\left|f(x)-f(y)\right|\le L_M(f)\|x-y\|$. If $x\in M, y=x_0$, then we have

$$f_1(x)-f_1(y)=f(x)-C\le f(x)-A\le L_M(f)\|x-x_0\|=L_M(f)\|x-y\|,$$

$$f_1(y)-f_1(x)=C-f(x)\le B-f(x)\le L_M(f)\|x-x_0\|=L_M(f)\|x-y\| .$$

Hence $\left|f_1(x)-f_1(y)\right|\le L_M(f)\|x-y\|$. The same is true of $y\in M, x=x_0$.

Therefore the functional f_1 is the real-valued Lip-functional on M_1 and $L_{M_1}(F)\le L_M(f)$. On the other hand, since f_1 is the extension of f from M to M_1, we have that $L_{M_1}(f_1)\ge L_M(f)$, hence $L_{M_1}(f_1)=L_M(f)$. Thus there exists a real-valued functional f_1 such that f_1 is the extension of f from M to M_1 with Lip-continuity and maintenance of Lip-constant.

The second part of proof, that is, the existence of a real-valued Lip-functional defined on X satisfying conditions 1) and 2) can be proved by repeating the process of the Hahn-Banach theorem, which was done by Zorn's lemma. So the below part may be omitted.

Let now $\Re$ be a collection of all functionals, which are extensions of f from M with Lip-continuity and maintenance of Lip-constant.

We make this collection $\Re$ into a partially ordered collection by defining $p\succ q$ in the sense that p is an extension of q. Now let $\Im$ be a totally ordered subset of $\Re$ and we define F by

$$\begin{cases} D(F)=\bigcup\limits_{p\in\Im} D(p) \\ x\in D(p),\ p\in\Im \Rightarrow F(x)=p(x). \end{cases} \tag{4}$$

Then it is clear that the functional F is an upper bound of $\Im$. Thus every totally ordered subset $\Im$ of the partially ordered set $\Re$ have the upper bound. Therefore there exists a maximal element in $\Re$ by Zorn's lemma [4]. Let F be a maximal element of $\Re$, then we have that $D(F)=X$. In fact, if it doesn't, then there exists a point x_0 such that $x_0\in X\setminus D(F)$. Hence we can extend the functional F from $D(F)$ to $D(F)\cup\{x_0\}$ as stated above, but this contradicts that F is the maximal element of $\Re$. This completes the proof. □

There are some notes for the theorem 1.

① If X is a separable Banach space, then the proof of the theorem, which does not use Zorn's lemma, can be done as follows.

First, if $\{x_n\}_{n=1}^{\infty}$ is the everywhere dense set of a countable number of $X\setminus M$, then we extend inductively the functional f from M to

$$M_1 = M\cup\{x_1\},\ M_2 = M_1\cup\{x_2\},\cdots,\ M_n = M_{n-1}\cup\{x_n\},\cdots .$$

Next, it is easy to extend F to whole space, which is the closure of the set $M\cup\{x_n\}_{n=1}^{\infty}$, with Lip-continuity and maintenance of Lip-constant.

② In general, because the constant C such that $A\le C\le B$ is not defined uniquely, the functional f_1 given by (3) is not unique.

③ In particular, the theorem is also valid by defining $f_1(x_0)=A$ or B, i.e.,

$$f_1(x_0)=\sup_{x\in M}\left(f(x)-L_M(f)\|x-x_0\|\right) \text{ or } f_1(x_0)=\inf_{x\in M}\left(f(x)+L_M(f)\|x-x_0\|\right).$$

④ As may be seen from process of the proof of the theorem 1, the problem on the extension of given Lip-functional is related not that Banach space X is real or complex, but that the functional is real-valued or complex. In next corollaries we'll show that there exist non-trivial, real-valued Lip-functionals satisfying certain conditions defined on real or complex Banach space X.

From the theorem 1 we could obtain some corollaries, which are similar to one of the Hahn-Banach theorem.

Corollary 1. Let $x_0\ne 0$ be an element of X. Then there exists a real-valued Lip-functional f defined on X such that

$$f(x_0)=\|x_0\| \text{ and } L_X(f)=1 .$$

Proof. We denote by M a closed subset consisting of all points of form αx_o, where α is non-negative real. Define f on M by $f(x)=\|x\|$. Then we have $f(x_0)=\|x_0\|$ and $L_M(f)=1$. In fact, it is clear that $f(x_0)=\|x_0\|$. And, since, for all $x,y\in M$, $|f(x)-f(y)|=|\|x\|-\|y\||\leq\|x-y\|$, we have f is Lip-functional on M and $L_M(f)\leq 1$. On the one hand, since there exist, for all $x,y\in M$, real numbers α_1,α_2 such that $x=\alpha_1 x_0$, $y=\alpha_2 x_0$, we have that

$$|f(x)-f(y)|=|\|\alpha_1 x_0\|-\|\alpha_2 x_0\||=|\alpha_1\|x_0\|-\alpha_2\|x_0\||=$$
$$=|\alpha_1-\alpha_2|\,\|x_0\|=\|(\alpha_1-\alpha_2)x_0\|=\|\alpha_1 x_0-\alpha_2 x_0\|=\|x-y\|,$$

and so $L_M\geq 1$. Therefore we obtain that $L_M(f)=1$. We now extend the functional f from M to X by the theorem 1. □

Corollary 2. For all $x\in X$, we have

$$\|x\|=\sup_{\substack{f\in X_L^*\\ f\neq 0}}\frac{|f(x)|}{L_X(f)}=\sup_{\substack{f\in X_L^*\\ L_X(f)\leq 1}}|f(x)| \tag{5}$$

Proof. Since if $x=0$ then $f(x)=0$, then equality (5) is trivial. So we assume that $x\neq 0$. Since $|f(x)|\leq L_X(f)\|x\|$ for all $f\in X_L^*$, if $L_X(f)\neq 0$ then

$$\sup_{\substack{f\in X_L^*\\ f\neq 0}}\frac{|f(x)|}{L_X(f)}\leq\|x\|. \tag{6}$$

By the corollary 1, there exists a functional $f\in X_L^*$ such that $L_X(f)=1$ and $f(x)=\|x\|$, hence $(|f(x)|/L_X(f))=\|x\|$. Therefore

$$\sup_{\substack{f\in X_L^*\\ f\neq 0}}\frac{|f(x)|}{L_X(f)}\geq\|x\|. \tag{7}$$

Thus the fist part of (5) follows from (6) and (7). The same is also true of the proof of the second part of (5). □

Corollary 3. For any $x_0 \in X \setminus M$, there exists a real-valued Lip-functional f defined on X such that 1) $f(x)=0$ for $x \in M$, 2) $f(x_0)=d$, and 3) $L_X(f)=1$, where $d = \inf_{z\in M} \|z - x_0\| > 0$.

Proof. We denote by M_1 any closed subset of X, which includes given set M and the point x_0, and define a functional f on M_1 by $f(x) = \inf_{z\in M} \|z - x\|$. Then it is not difficult to see that f satisfies condition 1) and 2). We'll show that f satisfies condition 3). To do that, we take any $x, y \in M_1$. Then it is clear that

$$|f(x) - f(y)| = \left| \inf_{z\in M} \|z - x\| - \inf_{z\in M} \|z - y\| \right| \le \sup_{z\in M} \left| \|z - x\| - \|z - y\| \right| \le \|x - y\|.$$

Hence f is Lip- functional on M_1 and $L_{M_1}(f) \le 1$. On the other hand, since there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset M$ such that $\|z_n - x_0\| \to d$ as $n \to \infty$, we have

$$d = |f(z_n) - f(x_0)| \le L_{M_1}(f) \|z_n - x_0\| \to L_{M_1}(f) \cdot d \text{ as } n \to \infty$$

and so $L_{M_1}(f) \ge 1$. Thus we must have that $L_M(f) = 1$. We now extend the functional f from M_1 to X by the theorem 1. □

Note 1. As stated above, these corollaries are very similar to the corollaries of the Hahn-Banach theorem, but there are some essential differences. In fact, first, all the functionals given in corollaries are not linear but real-valued. Actually, these corollaries would be used more often to apply than the ET. Second, the corollary 2 shows that the topology of Banach space X is not only defined on the unit sphere of the linear dual space X_l^*, but also on the unit sphere of Lip-dual space X_L^* of X. In the future, the corollary 2 will be applied to the introduction of a new topology in Banach space, which would be called a L-topology. Finally, the corollary 3 shows that the closed subset M of X and the point $x \in X \setminus M$ may be separated by a real-valued nonlinear Lip-functional in X_L^*. Except these, there are some more corollaries similar to the corollaries of the Hahn-Banach theorem.

Theorem 2. Let f be a complex Lip-functional defined on a closed subset M of a complex Banach space X. Then there exists a complex Lip-functional F defined on X such that 1) $F(x) = f(x)$ for $x \in M$, and 2) $L_X^2(F) \le L_M^2(\operatorname{Re} f) + L_M^2(\operatorname{Im} f)$, where $\operatorname{Re} f$ and $\operatorname{Im} f$ are the real and imaginary parts of f respectively

Proof. If $f(x) = g(x) + i\, h(x)$, where $g(x) = \operatorname{Re} f(x)$ and $h(x) = \operatorname{Im} f(x)$, then g and h are real-valued Lip-functionals defined on M. For all $x, y \in M$, since

$$f(x) - f(y) = (g(x) - g(y)) + i\,(h(x) - h(y)),$$

we have $|g(x) - g(y)| \le |f(x) - f(y)| \le L_M(f)\|x - y\|$ and

$$|h(x) - h(y)| \le |f(x) - f(y)| \le L_M(f)\|x - y\|.$$

At the same time, we have $L_M(g) \le L_M(f)$ and $L_M(h) \le L_M(f)$. On the other hand,

$$\begin{aligned} &|f(x) - f(y)|^2 = |g(x) - g(y)|^2 + |h(x) - h(y)|^2 \le \\ &\le (L_M^2(g) + L_M^2(h))\|x - y\|^2 \end{aligned}$$

implies that $L_M^2(f) \le L_M^2(g) + L_M^2(h)$. Now the theorem may be proved similarly to the theorem 1. If $X \ne M$, choose $x_0 \in X \setminus M$, and define $M_1 = M \cup \{x_0\}$. First, we extend the functional f from M to M_1 with Lip-continuity. We define a functional f_1 on M_1 by

$$f_1(x) = \begin{cases} f(x), & if\ x \in M \\ C + i\,D, & if\ x = x_0. \end{cases} \tag{8}$$

Here constants C and D are as follows. Since $g(x)$ and $h(x)$ are real-valued Lip-functional on M, it is clear that there exist, by repeating the same argument as the theorem 1, constants C and D such that

$$\sup_{x \in M}\big(g(x) - L_M(g)\|x - x_0\|\big) \le C \le \inf_{x \in M}\big(g(x) + L_M(g)\|x - x_0\|\big),$$

$$\sup_{x \in M}\big(h(x) - L_M(h)\|x - x_0\|\big) \le D \le \inf_{x \in M}\big(h(x) + L_M(h)\|x - x_0\|\big).$$

The constant C and D in (8) are such. We'll show that the functional f_1 defined by (8) is the Lip-functional on M_1 and $L_M^2(f_1) \le L_M^2(g) + L_M^2(h)$. Choose any $x, y \in M_1$. If $x, y \in M$, then it is clear that

$$\left| f_1(x) - f_1(y) \right| = \left| f(x) - f(y) \right| \le L_M(f) \|x - y\|.$$

If $x \in M, y = x_0$, then $f_1(x) - f_1(y) = f(x) - (C + iD) = (g(x) - C) + i(h(x) - D)$,

and, since $|g(x) - C| \le L_M(g)\|x - x_0\|,\ |h(x) - D| \le L_M(h)\|x - x_0\|$, we have that

$$\left| f_1(x) - f_1(y) \right|^2 = |g(x) - C|^2 + |h(x) - D|^2 \le \left(L_M^2(g) + L^2(h)\right)\|x - x_0\|^2.$$

The same is also true of $y \in M, x = x_0$. Thus there exists a Lip-functional f_1 such that f_1 is the extension of f from M to M_1 with Lip-continuity. The second part of the proof, in other words, the existence of the complex Lip-functional defined on X satisfying conditions 1) and 2) may be proved by repeating the process of the above theorem 1, which was done by Zorn's lemma. □

Note 2. If $f(x) = g(x) + i\,h(x)$ and we take $g(x) = x^2, h(x) = x - x^2$, $0 \le x \le 1/2$, then we have that $L_M^2(f) \ne L_M^2(g) + L_M^2(h)$. But, in many applications, it is frequently presented the need to extend the complex Lip-functiona with maintenance of not Lip-constant but only Lip-continuity. Moreover, the results of the theorem 2 would be equal to one of the theorem 1 if we define the Lip-constant of the complex Lip-functional $f(x) = g(x) + i\,h(x)$ by $L'_M(f) = \left(L_M^2(g) + L_M^2(h)\right)^{1/2}$. Then we could obtain the following theorem.

Theorem 2$'$. Let f be a complex Lip-functional defined on a closed subset M of a complex Banach space X. Then there exists a complex Lip-functional F defined on X such that 1) $F(x) = f(x)$ for $x \in M$, and 2) $L'_X(F) = L'_M(f)$.

3. Invertibility of Lip-operator and its Lip-dual operator

The dual operator of the linear operator has played very important role in the linear operator theory. In general, to compose a dual operator corresponding to the nonlinear operator is impossible in a measure, but it is possible for Lipschitz (Lip-) operator. A concept of a Lip-dual operator of the nonlinear Lip-operator in Banach spaces was introduced in the paper [2]. That idea was based upon the Lip-dual space of Banach spaces ([1]). In this section we'll consider the problem in [1] on the relation of invertibility between nonlinear Lip-operator and its Lip-dual operator. Invertibility of the Lip-operator is the important problem presented in several sides including its spectrum analysis.

Proposition. Let $T \in Lip_0(M, D)$. Then M is a certain subset of M_{Ll}^{**} in isometric embedding sense. If an operator $J : M \to M_{Ll}^{**}$ is such isometric mapping, then, for all $x, y \in M$, we have

$$\|x-y\| = \|Jx - Jy\| = \sup_{\substack{f \in M_L^* \\ L(f) \le 1}} |f(x) - f(y)|, \tag{9}$$

$$\|Tx - Ty\| = \|T_{Ll}^{**}(Jx) - T_{Ll}^{**}(Jy)\|. \tag{10}$$

Proof. Choose any $x \in M$ and fix it. Consider the functional $J(x)$ defined on M_L^* by $J(x)(f) = f(x)$. Then the functional $J(x)$ is a bounded linear on M_L^* because $|J(x)(f)| = |f(x)| \le L(f)\|x\|$, that is, $J(x) \in M_{Ll}^{**}$. Now put $JM = \{J(x) \in M_{Ll}^{**} \mid x \in M\}$ and we shall show that the operator $J : M \to JM$ is an isometric mapping. If $x, y \in M$, then we obtain that

$$\|J(x)-J(y)\| = \sup_{\substack{f\in M_L^* \\ L(f)\le 1}} \|(J(x)-J(y))(f)\| = \sup_{\substack{f\in M_L^* \\ L(f)\le 1}} |f(x)-f(y)| \le$$
$$\le \sup_{\substack{f\in M_L^* \\ L(f)\le 1}} L(f)\|x-y\| \le \|x-y\| .$$

If $x \ne y$, then, by the Hahn-Banach theorem, there exists a functional $f_0 \in X_l^*$ such that $\|f_0\| = L(f_0) = 1,\ f_0(x-y) = \|x-y\|$. Now we denote again by f_0 a contraction to M of f_0, then it is clear that $f_0 \in M_L^*$. Hence we have that

$$\|x-y\| = |f_0(x-y)| = |f_0(x)-f_0(y)| \le \sup_{\substack{f\in M_L^* \\ L(f)\le 1}} |f(x)-f(y)| =$$
$$= \sup_{\substack{f\in M_L^* \\ L(f)\le 1}} |(J(x)-J(y))(f)| = \|J(x)-J(y)\| .$$

This is proof of the equality (9). We shall show the equality (10).

For all $x \in M$ and all $f \in D_L^*$, since $f\circ T \in M_L^*$ and $J(x) \in M_{Ll}^{**}$, we have that

$$f(Tx) = (f\circ T)(x) = (T_L^* f)(x) = J(x)(T_L^* f) =$$
$$= (J(x)\circ T_L^*)(f) = T_{Ll}^{**}(J(x))(f) .$$

On the other hand, if operator $J^/ : D \to D_{Ll}^{**}$ is such an isometric mapping as J, then $J^/(Tx) \in D_{Ll}^{**}$. Hence $f(Tx) = J^/(Tx)(f)$.

Thus we have $J^/(Tx)(f) = T_{Ll}^{**}(J(x))(f)$ for all $f \in D_L^*$, that is, $J^/(Tx) = T_{Ll}^{**}(J(x))$. Therefore, by the equality (10), we obtain

$$\|Tx-Ty\| = \|J^/(Tx)-J^/(Ty)\| = \|T_{Ll}^{**}(J(x)) - T_{Ll}^{**}(J(y))\| .$$

This completes the proof. □

We’ll here show the **necessary** and **sufficient** conditions for invertibility

First, we recall the concept of invertibility of Lip- operator [5].

Let $T \in Lip_0(M,D)$. The operator T is said to be invertible in $Lip_0(M,D)$, If $R(T) = D$, T^{-1} exists, and $T^{-1} \in Lip_0(D,M)$, where $R(T)$ is a range of T. Similarly, the operator T_L^* is said to be invertible in $BL(D_L^*, M_L^*)$, if $R(T_L^*) = M_L^*$, $(T_L^*)^{-1}$ exists. It is to be noted here that the condition $(T_L^*)^{-1} \in BL(M_L^*, D_L^*)$ follows from that $R(T_L^*) = M_L^*$ and $(T_L^*)^{-1}$ exists by the open mapping theorem

([4]), because T_L^* is the bounded linear operator defined on D_L^*. We'll always assume that $T \in Lip_0(M,D)$ below.

Lemma 1. If $l(T) > 0$, then $R(T)$ is a closed set, T^{-1} exists, and $T^{-1} \in Lip_0(R(T),M)$, where

$$l(T) = \inf_{\substack{x,y\in M\\ x\neq y}} \frac{\|Tx - Ty\|}{\|x - y\|}, \tag{11}$$

which was called a glb-Lipschitz constant of T ([3]).

Proof. For all $x, y \in M$, since $l(T)\|x-y\| \le \|Tx - Ty\|$, if $l(T) > 0$, then $Tx = Ty$ implies $x = y$. Therefore T^{-1} exists.

On the one hand, for any $x_1, y_1 \in R(T)$ there exist $x, y \in M$ such that $Tx = x_1, Ty = y_1$. Hence T^{-1} is a Lip-operator on $R(T)$ into M, i.e. $T^{-1} \in Lip_0(R(T),M)$ and we have $L(T^{-1}) \le l^{-1}(T)$. We now show that $R(T)$ is a closed set. To do that, we take any $\{y_n\}_{n=1}^{\infty}$ such that $\{y_n\}_{n=1}^{\infty} \subset R(T)$ and $y_n \to y_0$ *as* $n \to \infty$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset M$ such that $y_n = Tx_n$, i.e., $x_n = T^{-1}y_n (n=1,2,\cdots)$. Hence

$$\|x_n - x_m\| = \|T^{-1}y_n - T^{-1}y_m\| \le l^{-1}(T)\|y_n - y_m\| \to 0 \text{ as } n,m \to \infty.$$

This shows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence of X. Since X is complete and M is closed, there exists a point $x_0 \in M$ such that $x_n \to x_0$ as $n \to \infty$. By the continuity of T, we have that $y_0 = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tx_0 \in R(T)$. □

It is easy to see that the same result for T_L^* is true.

Lemma 2. If $R(T)$ is a closed set, T^{-1} exists, and $T^{-1} \in Lip_0(R(T), M)$, then we have $R(T_L^*) = M_L^*$.

Proof. For any $y \in R(T)$ and any $f \in M_L^*$, set $g(y) = f(T^{-1}y)$. Since both T^{-1} and f are Lip-continuous on $R(T)$ and M respectively, the functional g is also

Lip-continuous on $R(T)$. By the theorem 2, we can extend g from $R(T)$ to D with Lip-continuity. We denote again by g the extended functional, then it is clear that $g \in D_L^*$. Therefore, for all $x \in M$,

$$\left(T_L^* g\right)(x) = g(Tx) = g(y) = f\left(T^{-1}y\right) = f\left(T^{-1}Tx\right) = f(x).$$

Thus $T_L^* g = f$. This shows that $R\left(T_L^*\right) = M_L^*$. □

Lemma 3. If $R(T)$ is a closed set and $\left(T_L^*\right)^{-1}$ exists, then $R(T) = D$.

Proof. Assume that there exists a point $y_0 \in D \setminus R(T)$. Because $R(T)$ is the closed set, we have that $d = \inf_{y \in R(T)} \|y - y_0\| > 0$. By the corollary 3 of theorem 1, there exists a functional $f_0 \in D_L^*$ such that $f_0(y) = 0$ for $y \in R(T)$, $f_0(y_0) = d$ and $L(f_0) = 1$. On the one hand, since $Tx \in R(T)$ for all $x \in M$, we have that $f_0(Tx) = \left(T_L^* f_0\right)(x) = 0$. So $T_L^* f_0 = 0$. Since $\left(T_L^*\right)^{-1}$ exists, we have $f_0 = 0$, but this contradicts that $L(f_0) = 1$.

It is easy to see that the following lemma is valid, but we shall check it for the sake of the proof of the next main theorem. We need not only the lemma 4 but also the remark of it. □

Lemma 4. The operator T is invertible in $Lip_0(M,D)$ if and only if $l(T) > 0, R(T) = D$. Then we have $l(T) = L^{-1}\left(T^{-1}\right)$.

Proof. First, we show that if T is invertible in $Lip_0(M,D)$ then $l(T) = L^{-1}\left(T^{-1}\right)$. If it be so, then it is clear that T is one-to-one and $0 < L(T) < +\infty$. Hence, for any $x, y \in M$ ($x \neq y$), there exist $x_1, y_1 \in D$ ($x_1 \neq y_1$) such that $T^{-1}x_1 = x,\ T^{-1}y_1 = y$. Conversely, for any $x_1, y_1 \in D$ ($x_1 \neq y_1$), there exist $x, y \in M$ ($x \neq y$) such that $Tx = x_1, Ty = y_1$. Therefore

$$l(T)=\inf_{\substack{x,y\in M\\ x\neq y}}\|Tx-Ty\|\cdot\|x-y\|^{-1}=\inf_{\substack{x_1,\,y_1\in D\\ x_1\neq y_1}}\left(\ \|x_1-y_1\|^{-1}\cdot\left\|T^{-1}x_1-T^{-1}y_1\right\|\right)^{-1}=$$
$$=\left(\sup_{\substack{x_1,\,y_1\in D\\ x_1\neq y_1}}\left\|T^{-1}x_1-T^{-1}y_1\right\|\cdot\|x_1-y_1\|^{-1}\right)^{-1}=L^{-1}\left(T^{-1}\right)\ . \tag{12}$$

Next, we show the fore part of the theorem. If T is invertible in $Lip_0(M,D)$, then $L(T^{-1})>0$ and hence we have $l(T)>0$ and $R(T)=D$. Conversely, if $l(T)>0$ and $R(T)=D$, then it is clear that T is invertible in $Lip_0(M,D)$ by the lemma 1. □

Note 3. It is easy to see that the same result for T_L^* is true. In other words, we can obtain following result for T_L^* by repeating the same argument as the theorem 1. The operator T_L^* is invertible in $BL(D_L^*,M_L^*)$ if and only if $l(T_L^*)>0$ and $R(T_L^*)=M_L^*$.

The following theorem is the main result on the relation of invertibility between Lip-operator and its Lip-dual operator. This theorem gives us the complete answer to the question in [1].

Theorem 3. The operator T is invertible in $Lip_0(M,D)$ if and only if the operator T_L^* is invertible in $BL(D_L^*,M_L^*)$. Then we have

$$\left(T_L^*\right)^{-1}=\left(T^{-1}\right)_L^*.$$

Proof. If T is invertible in $Lip_0(M,D)$, then the lemma 4 implies $l(T)>0$. Hence we have that $R(T_L^*)=M_L^*$ by the lemma 1 and lemma 2. On the one hand, since T^{-1} belongs in the space $Lip_0(D,M)$, Lip-dual operator $\left(T^{-1}\right)_L^*$ of T^{-1} is defined on M_L^* and $(T^{-1})_L^*\in BL(M_L^*,D_L^*)$. And, because $I_M=T^{-1}\circ T,\ I_D=T\circ T^{-1}$ are identity operators on M, D respectively, we have

$$\left(I_M\right)_L^*=(T^{-1}\circ T)_L^*=T_L^*\circ(T^{-1})_L^*, \tag{13}$$

$$\left(I_D\right)_L^*=(T\circ T^{-1})_L^*=(T^{-1})_L^*\circ T_L^*, \tag{14}$$

where $(I_M)_L^*, (I_D)_L^*$ are identity operators on M_L^*, D_L^* respectively ([2]). Therefore, By (13) and (14), we have that $(T^{-1})_L^*$ exists and $(T_L^*)^{-1} = (T^{-1})_L^*$. Hence $(T_L^*)^{-1} \in BL(M_L^*, D_L^*)$, thus T_L^* is invertible in $BL(D_L^*, M_L^*)$.

Conversely, assume that T_L^* is invertible in $BL(D_L^*, M_L^*)$. We have to prove $l(T) > 0, R(T) = D$. If T_L^* is invertible, then it is true that the linear dual operator T_{Ll}^{**} of bounded linear operator T_L^* defined on Banach space D_L^* is also invertible and $\|T_L^*\| = \|T_{Ll}^{**}\|$. Therefore, by lemma 4, we have that $l(T_L^*) = L^{-1}((T_L^*)^{-1})$. Hence $L((T_L^*)^{-1}) = L((T_{Ll}^{**})^{-1})$ and $l(T_L^*) = L^{-1}((T_L^*)^{-1}) = l(T_{Ll}^{**})$ follows from $(T_{Ll}^{**})^{-1} = ((T_L^*)^{-1})_l^*$ and $\|(T_L^*)^{-1}\| = \|(T_{Ll}^{**})^{-1}\|$. And by the lemma 4, the equality (9) and (10), we obtain that

$$0 < l(T_L^*) = l(T_{Ll}^{**}) = \inf_{\bar{x}, \bar{y} \in M_{Ll}^{**}} \|T_{Ll}^{**}\bar{x} - T_{Ll}^{**}\bar{y}\| \cdot \|\bar{x} - \bar{y}\|^{-1} \leq$$

$$\leq \inf_{\substack{Jx, Jy \in JM \\ Jx \neq Jy}} \|T_{Ll}^{**}(Jx) - T_{Ll}^{**}(Jy)\| \cdot \|Jx - Jy\|^{-1} =$$

$$= \inf_{\substack{x, y \in M \\ x \neq y}} \|Tx - Ty\| \cdot \|x - y\|^{-1} = l(T).$$

Next, since $(T_L^*)^{-1}$ exists and $R(T)$ is closed, by the lemma 3, we have that $R(T) = D$. Therefore T is invertible in $Lip_0(M, D)$. Finally, using $L(T^{-1}) = \|(T_L^*)^{-1}\|$ and the equality (13), we have that $l(T) = L^{-1}(T^{-1}) = L^{-1}((T^{-1})_L^*) = L^{-1}((T_L^*)^{-1}) = l(T_L^*)$.

This completes the proof. □

Corollary. The following statements are equivalent to each other:

1) The operator T is invertible in $Lip_0(M, D)$,

2) $l(T) > 0$ and $(T_L^*)^{-1}$ exists,

3) $l(T) \cdot l(T_L^*) > 0$.

Proof. If T is invertible, then, by the lemma 4 and theorem 3, $l(T) > 0$ and $\left(T_L^*\right)^{-1}$ exists. And, since T_L^* is invertible, we also have that $l(T_L^*) > 0$. Thus $l(T) \cdot l(T_L^*) > 0$. On the other hand, if $l(T) > 0$ and $\left(T_L^*\right)^{-1}$ exists, then, by the lemma 1 and lemma 3, T is invertible in $Lip_0(M, D)$. □

The results in this paper can be used effectively at the study of the topology of Banach space by the Lip-functional, and at the study of the operator equation with Lip-operator. In fact, the extension theorem 1 is important at the introduction and the consideration of a new topology in Banach space, and the theorem 3 is significant at the resolution of nonlinear Lip-operator equation by the linear Lip-dual operator equation.

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Choe Ryong Gil:
Department of Mathematics,
University of Sciences,
Unjong District, Gwahak 1-dong,
Pyongyang,
D.P.R.Korea,
E-mail: ryonggilchoe@163.com