Inelastic spherical caps with defects

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Abstract : Limit analysis and minimum weight design of stepped spherical shells is studied. The caps have piece wise constant thickness and are subjected to the uniform external pressure. The shells are made of an inelastic material obeying an approximation of the Tresca yield surface. The aim of the paper is to develop a procedure for minimum weight design for given limit load. Necessary optimality conditions are derived with the aid of variational methods of the theory of optimal control. Numerical results are presented for a simply supported spherical cap.

Keywords: optimization, spherical shell, inelastic material, crack.

1. Introduction

Various approaches to the limit analysis of axisymmetric plates and shells and solutions of problems of load carrying capacity of spherical caps can be find in books by Hodge [3], [4]; Chakrabarty [2].

Inelastic spherical cap with a central hole was studied by Lellep and Tungel [7] assuming that the thickness was piece wise constant and the material obeyed generalised square yield condition. Minimum weight designs for shallow shells are obtained by Lellep and Hein [5]. The shell under consideration is pierced with a central hole and it is subjected to the initial impact loading. Optimal designs of shells of piece wise constant thickness are established under the condition that the maximal residual deflection attains the minimum value for given total weight. Spherical shells of Mises material were studied by Lellep and Tungel [8] whereas conical shells were considered by Lellep and Puman [6].

In [8] an optimization procedure is developed for spherical shells of piece wise constant thickness made of an inelastic material obeying the Mises material and associated flow law. The designs of spherical shells corresponding to maximal load carrying capacity are established for given material volume or weight of the shell.

In the present paper stepped spherical caps with cracks at re-entrant corners of steps are considered making use of an approximation of the Tresca yield condition. The aim of the paper is to establish minimum weight designs of the shell for given load carrying capacity.

2. Formulation of the problem

Let us consider a spherical cap of radius A simply supported at the edge with central angle

$\varphi = \beta$ (Fig. 1). The shell is subjected to the uniform external pressure of intensity P. The pressure loading is assumed to be quasi-static, inertial effects will be neglected.
Let the thickness of the shell be piece wise constant, e.g. $h = h_j$, for $\varphi \in (\alpha_j; \alpha_{j+1})$ where $j = 0,...,n$ and $\alpha_{n+1} = \beta$. Thicknesses $h_j (j = 0,...,n)$ and angles $\alpha_j (j = 1,...,n)$ will be treated as design parameters to be defined so that a cost function attains its minimal value. It is wellknown that sharp corners in structures generate stress concentration which entails cracks. It is assumed herein that at $\varphi = \alpha_j, (j = 1,...,n)$ circular cracks are located.

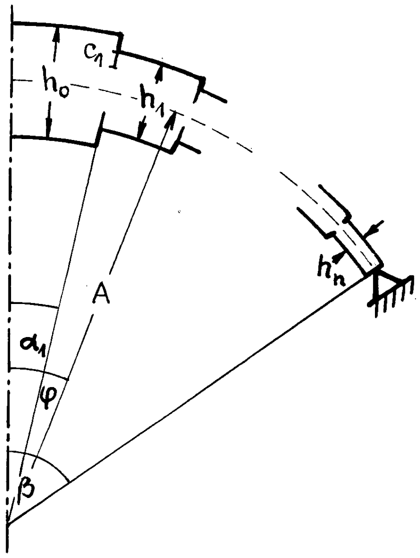

Fig. 1: Geometry of the shell.

We are looking for the minimum weight design of the spherical cap for the fixed limit load. The other problem we are dealing with consists in the maximization of the limit load for given material consumption.
Since the middle surface of the cap is a sphere of radius A the cost criterion for the problem of minimum weight can be presented as

$$V = \sum_{j=0}^{n} h_j (\cos\alpha_j - \cos\alpha_{j+1}), \qquad (1)$$

where $V = \bar{V} / 2\pi A^2$.
If the problem is to maximize the ultimate load to be sustained by the cap then the volume V is considered as a given constant.

3. Basic equations

In the case of rotational symmetry the equilibrium equations of a shell element can be presented as (see Chakrabarty [2], Lellep and Tungel [7], [8])

$$\begin{aligned} &(N_\varphi \sin\varphi)' - N_\theta \cos\varphi = S \sin\varphi, \\ &(N_\varphi + N_\theta + PA)\sin\varphi = -(S \sin\varphi)', \qquad (2) \\ &(M_\varphi \sin\varphi)' - M_\theta \cos\varphi = AS \sin\varphi \end{aligned}$$

In (2) N_φ, N_θ stand for membrane forces and M_φ, M_θ for bending moments in the two principal directions, respectively, and S is the shear force. Here and henceforth prims denote the differentiation with respect to φ.
We shall use a simple approximation of the exact yield surface is obtained assuming that the stress state of the shell corresponds to the ridge

$$\frac{M_\theta}{M_{0j}} = \pm(1 - (\frac{N_\theta}{N_{0j}})^2) \qquad (3)$$

of the exact yield surface (Hodge, [3], [4]). whereas bending moments M_φ, M_θ satisfy the yield condition (hexagon) on the plane of moments. Here M_{0j} and N_{0j} stand for the limit moment and limit force for a portion of the shell with thickness h_j, e.g.

$$M_{0j} = \sigma_0 h_j^2 / 4, \;\; N_{0j} = \sigma_0 h_j, \;\; \sigma_0 \qquad (4)$$

being the yield stress of the material.
In the limit analysis as well as dynamic plasticity of axisymmetric shells it is usual that the bending moment M_θ attains its limit values. Thus it is reasonable to expect that
$M_\theta = M_{0j}$
(5)

and according to (3) $N_\theta = 0$ for $\varphi \in \lfloor \alpha_j, \alpha_{j+1} \rfloor; \; j = 0,...,n$. The latter means that $N_\theta = 0$ throughout the shell.

Let h_* be the thickness of a reference shell of constant thickness and N_*, M_* - yield force and yield moment for the reference shell. It seems to be reasonable to introduce following non-dimensional quantities

$$\gamma_j = \frac{h_j}{h_*}, \quad n_{1,2} = \frac{N_{\varphi,\theta}}{N_*}, \quad m_{1,2} = \frac{M_{\varphi,\theta}}{M_*}$$
$$k = \frac{M_*}{AN_*}, \quad p = \frac{PA}{N_*}, \quad s = \frac{S}{N_*} \tag{6}$$

Variables (6) admit to present the equilibrium equations (2) with (5) as

$$\begin{aligned} n_1' &= -n_1 \cot\varphi + s \\ s' &= -n_1 - s\cot\varphi - p \\ m_1' &= -m_1 \cot\varphi + \gamma_j^2 \cot\varphi + s/k \end{aligned} \tag{7}$$

for $\varphi \in \lfloor \alpha_j, \alpha_{j+1} \rfloor; \; j = 0,...,n$.

Boundary conditions for equations (7) are

$$\begin{aligned} &m_1(0) = m_2(0) = \gamma_0^2, \quad s(0) = 0, \\ &n_1(0) = n_2(0) = 0, \quad m_1(\beta) = 0 \end{aligned} \tag{8}$$

In the case of simply supported spherical caps it is expected that the optimal shape of the shell is such that

$$\gamma_j > \gamma_{j+1} \tag{9}$$

for each $j = 0,...,n-1$.

Since we are looking for statically admissible solution of the problem we have to check if

$$0 \le m_1(\varphi) \le \gamma_j^2$$
(10)

for $\varphi \in (\alpha_j, \alpha_{j+1}); \; j = 0,...,n$ and if

$$|n_1(\varphi)| \le \gamma_j$$
(11)

It can be shown that the moment m_1 and membrane force n_1 are monotonic functions of the angle φ. Thus the admissible values of m_1 and n_1 are exceeded at boundary points φ_j of intervals $\lfloor \alpha_j; \alpha_{j+1} \rfloor$, if any.

This means that we have to check the admissibility of stress components at $\varphi = \alpha_j (j = 0,...,n)$. We must bear in mind that the sections $\varphi = \alpha_j$ of the cap are weakened by cracks of depth c_j. These sections are able to sustain bending moments with maximal value $M_{\varphi m} = \frac{\sigma_0}{4}(h_j - c_j)^2$ at $\varphi = \alpha_j$.

Similarily, the maximal admissible value of the membrane force N_φ is $N_{\varphi m} = \sigma_0 (h_j - c_j)$ for $\varphi = \alpha_j$.

Therefore, the constraints (10) and (11) can be replaced by equalities

$$m_1(\alpha_j) - \nu_j^2 \gamma_j^2 + \theta_{j1}^2 = 0,$$
(12)

and

$$|n_1(\alpha_j)| - \nu_j \gamma_j + \theta_{j2}^2 = 0$$
(13)

provided (9) holds good and $j = 1,...,n$. In (12), (13) θ_{j1}, θ_{j2} stand for so-called slack variables and

$$\nu_j = 1 - \frac{c_j}{h_j}$$
(14)

for $j = 1,...,n$.

The problem posed above will be treated as a problem of the theory of optimal control (see Bryson, [1]).

4. Optimality conditions

In order to derive necessary conditions of optimality for the problem with cost function (1) and state equations (7) with boundary conditions (8) and constraints (10)-(12) we compile an extended functional (see Bryson and Ho, [1]; Lellep and Hein, [5]; Lellep and Puman [6]; Lellep and Tungel, [7], [8])

$$J_* = \sum_{j=0}^{n} \{\gamma_j(\cos\alpha_j - \cos\alpha_{j+1}) +$$
$$+ \int_{\alpha_j}^{\alpha_{j+1}} [\psi_1(n_1' + n_1\cot\varphi - s) +$$
$$+ \psi_2(s' + n_1 + s\cot\varphi + p) +$$
$$\psi_3(m_1' + m_1\cot\varphi - \frac{s}{k} - \gamma_j^2\cot\varphi)]d\varphi\} + \qquad (15)$$
$$+ \sum_{j=1}^{n} \{\varphi_{j1}(m_1(\alpha_j) - \nu_j^2\gamma_j^2 + \theta_{j1}^2) +$$
$$+ \varphi_{2j}(\left|n_1(\alpha_j)\right| - \nu_j\gamma_j + \theta_{j2}^2)\} + \mu_0(m_1(0) -$$
$$- \gamma_0^2) + \mu_1 n_1(0) + \mu_2 s(0) + \mu_3 m_1(\beta)$$

In (15) ψ_1, ψ_2, ψ_3 stand for co-state (conjugate) variables and $\varphi_{j1}, \varphi_{j2} (j = 1,...,n); \mu_0, \mu_1, \mu_2, \mu_3$ are unknown Langrange'ian multipliers. It is worthwhile to emphasize that the co-state variables ψ_1, ψ_2, ψ_3 are certain functions of φ whereas Lagrange'ian multipliers are treated as unknown constants.

When calculating the total variation of the extended functional (15) one has to take into account the distinctions between ordinary (weak) variations of state variables and total variations at boundary points of intervals $(\alpha_j; \alpha_{j+1})$. We call $\Delta z(\alpha_j \pm)$ the total variation of a variable z at $\varphi = \alpha_j$ and $\delta z(\alpha_j \pm)$ the value of the ordinary variation δz at $\varphi = \alpha_j$.

It is known that (see Bryson and Ho [1], Lellep and Puman [6], Lellep and Tungel [7,8])

$$\Delta z(\alpha_j \pm 0) = \delta z(\alpha_j \pm 0) + z'(\alpha_j \pm 0) \cdot \Delta\alpha_j$$

It means that at the boundary points of intervals $(\alpha_j; \alpha_{j+1})$ one has

$$\Delta n_1(\alpha_j -) = \Delta n_1(\alpha_j +)$$
$$\Delta m_1(\alpha_j -) = \Delta m_1(\alpha_j +)$$
$$\Delta s(\alpha_j -) = \Delta s(\alpha_j +)$$

Note that when deducing the last relations the continuity of state variables $n_1, s, m_1, e.g.$

$$n_1(\alpha_j +) = n_1(\alpha_j -)$$
$$s_1(\alpha_j +) = s_1(\alpha_j -)$$
$$m_1(\alpha_j +) = m_1(\alpha_j -)$$

is taken into account.

Guiding by the considerations given above it can be rechecked that the equation $\Delta J_* = 0$ yields the co-state system

$$\psi_1' = \psi_1\cot\varphi + \psi_2$$
$$\psi_2' = -\psi_1 + \psi_2\cot\varphi - \frac{\psi_3}{k}$$
$$\psi_3' = \psi_3\cot\varphi$$

(16)

also equations for determination of parameters

$$\cos\alpha_j - \cos\alpha_{j+1} - 2\varphi_{j1}\nu_j^2\gamma_j - \varphi_{j2}\nu_j -$$
$$-2\gamma_j \int_{\alpha_j}^{\alpha_{j+1}} \psi_3 \cot\varphi d\varphi = 0$$

(17)

for $j = 1,...,n$ and

$$\cos\alpha_0 - \cos\alpha_1 - 2\mu_0\gamma_0 - 2\gamma_0 \int_0^{\alpha_1} \psi_3 \cot\varphi d\varphi = 0$$

(18)

for $j = 0$ and

$$\varphi_{j1}\theta_{j1} = 0, \quad \varphi_{j2}\theta_{j2} = 0$$

(19)

for $j = 1,...,n$.

The transversality conditions at boundary points are

$$\psi_1(\beta) = \psi_2(\beta) = 0, \quad \psi_3(\beta) = -\mu_3$$

(20)

and

$$\psi_1(0) = \mu_1, \quad \psi_2(0) = \mu_2, \quad \psi_3(0) = \mu_0$$

(21)

whereas jump conditions for co-state variables have the form

$$\psi_1(\alpha_j-)-\psi_1(\alpha_j+)=\pm\varphi_{j2}$$
$$\psi_2(\alpha_j-)-\psi_2(\alpha_j+)=0$$
$$\psi_3(\alpha_j-)-\psi_3(\alpha_j+)=\pm\varphi_{j1}$$
(22)

where $j=1,...,n$.
Finally, variation of (15) yields
$$\sin\alpha_j(-\gamma_j+\gamma_{j-1})+\psi_1(\alpha_j+)n_1'(\alpha_j+)+$$
$$+\psi_2(\alpha_j+)s'(\alpha_j+)+\psi_3(\alpha_j+)m_1'(\alpha_j+)-$$
$$-\psi_1(\alpha_j-)n_1'(\alpha_j-)-\psi_2(\alpha_j-)s'(\alpha_j-)-$$
$$-\psi_3(\alpha_j-)m_1'(\alpha_j-)=0$$
(23)

for each $j=1,...,n$. Making use of (7), (22) the equations (23) can be put into the form

$$\sin\alpha_j(-\gamma_j+\gamma_{j-1})\pm\varphi_{j2}(s(\alpha_j)-n_1(\alpha_j)\cot\alpha_j)+$$
$$[\psi_3(\alpha_j)](\frac{s(\alpha_j)}{k}-m_1(\alpha_j)\cot\alpha_j)+$$
$$+\cot\alpha_j(\gamma_j^2\psi_3(\alpha_j+)-\gamma_{j-1}^2\psi_3(\alpha_j-))=0$$

(24)
for $j=1,...,n$. In (24) square brackets denote finite jumps, $[\psi(\alpha_j)]=\psi(\alpha_j+0)-\psi(\alpha_j-0)$.
It is easy to recheck that the general solution of the system (16) is

$$\psi_1=(A_j\sin\varphi+B_j\cos\varphi-\frac{C_j}{k})\sin\varphi$$
$$\psi_2=(A_j\cos\varphi-B_j\sin\varphi)\sin\varphi$$
$$\psi_3=C_j\sin\varphi$$
(25)

for $\varphi\in(\alpha_j;\alpha_{j+1})$; $j=0,...,n$.
In order to solve the posed problem up to the end one has to integrate (7) making use of (8) and to solve equations (17)-(24) making use of (25).
It appears that equations (7) can be integrated in each region $(\alpha_j;\alpha_{j+1})$; $j=0,...,n$. The result is

$$n_1=\frac{p}{2}\varphi\cot\varphi+D_j\cot\varphi+E_j$$
$$s=\frac{p}{2}(\cot\varphi-\varphi)-D_j+E_j\cot\varphi$$
$$m_1=\gamma_j^2+\frac{1}{k}(\frac{p}{2}\varphi\cot\varphi+D_j\cot\varphi+E_j)+\frac{F_j}{\sin\varphi}$$

(26)
for $\varphi\in(\alpha_j,\alpha_{j+1})$; $j=1,...,n$, where D_j,E_j,F_j are arbitrary constants.
Unknown constants D_j,E_j,F_j in (26) can be defined from the continuity of state variables n_1,s,m_1 at $\varphi=\alpha_j$, making use of boundary conditions (8).

5. Numerical results and discussion

The detailed analysis shows that finally we obtain $3n+3$ equations for determination of arbitrary constants A_j,B_j,C_j $(j=0,...,n)$.

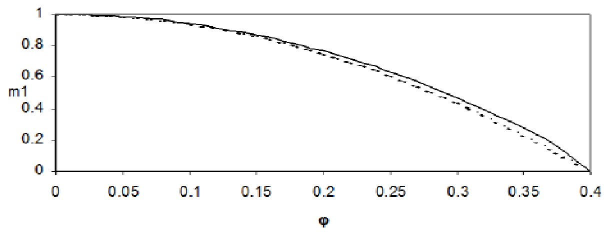

Fig. 2: Bending moment m_1.

This set of equations is solved numerically. Results of calculations are presented in Fig. 2 and Table 1 for the spherical cap with single step. In Fig. 2 the distributions of the bending moment m_1 are presented.

Solid line corresponds to the optimized stepped shell whereas dashed line is associated with the reference shell of constant thickness. It can be seen from Fig. 2 that bending moment m_1 monotonically decreases from its limit value γ_0^2 at the pole until zero at the supported edge. It is somewhat surprising that the distributions of the bending moment m_1 corresponding to the optimized shell and to the reference shell of constant thickness, respectively, are quite close to each other.

Calculations showed that the stress state of the shell is statically admissible.

In Table 1 optimal values of parameters α_1 and γ_1 are presented for different values of the intensity of the external pressure. Here V stands for the the optimal value of the material volume whereas V_0 is the material volume of the reference shell of constant thickness and P_0 is the collapse pressure of the shell of constant thickness. Table 1 corresponds to shells with $k = 0.001; \nu = 0.9$.

Note that the material volume of the reference shell with thickness h_* can be expressed as

$V_0 = h_*(\cos\beta - 1)$.

For the sake of simplicity in the Table 1 are accommodated the data corresponding to the case when $h_* = h_0$.

Table 1: Optimal design for $\beta = 0.2$.

p	α	γ_1	$e = V / V_0$
$0.99P_0$	0.1979	0.1950	0.9832
$0.98P_0$	0.1956	0.2782	0.9687
$0.97P_0$	0.1931	0.3440	0.9557
$0.96P_0$	0.1903	0.4017	0.9438
$0.95P_0$	0.1872	0.4550	0.9328
$0.91P_0$	0.1666	0.6752	0.9007

Calculations carried out showed that the efficiency of the design $e = V / V_0$ depends on the shell parameters β and k, on the loading p and on the crack length ν. For instance, if $\beta = 0.2; k = 0.001$ and $\nu = 0.9; p = 0.91P_0$ one has $\alpha_1 = 0.1666$, γ_1=0.6752 and $e = 0.9007$. It means that in this case the material saving is 9.93%.

6. Concluding remarks

The behaviour of inelastic spherical caps under uniformly distributed external pressure loading was studied. The material of the cap obeys an approximation of the Tresca yield surface in the space of membrane forces and bending moments. Necessary optimality conditions for the posed problem are derived with the aid of variational methods of the theory of optimal control. Numerical results are presented for a simply supported cap with unique step.

In the study it was assumed that the stepped caps had circular cracks at re-entrant corners of steps. It was established that the depth of a crack had relatively weak influence on the optimal design of the shell.

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