Negative masses in general relativity and the Dirac equation

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It is shown that Einstein's gravitational field equations imply the existence of negative masses. With the inclusion of negative masses, Dirac spinors can be composed of a large positive and likewise large negative mass. The gravitational interaction of these positive-negative masses is positive and leads to a helical motion and not to an unstable vacuum as claimed by Cavalleri and Tonni.

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1. Bondi's Solution

Assuming the existence of negative besides positive masses, the two body problem of two masses equal in magnitude by opposite in sign, was studied in the approximation of the linearized Einstein equations by Bondi [1]. As expected from the equivalence principle, where a positive mass attracts positive or negative masses, and where a negative mass repels all masses regardless of their sign, such a two-body configuration is self-accelerating and would make unstable a vacuum composed of an equal number of positive and negative masses. However, if the mass of the gravitational field set up between the two masses is included the result is very different. This field mass is not included in Bondi's analysis, because it comes from the nonlinear terms in Einstein's gravitational field equations.

2. Negative Masses in Einstein's Gravitational Field Equation

While Bondi simply assumed that negative masses might exist, Eintein's gravitational field equations imply the existence of negative masses [2]. For the proof it is sufficient to consider the gravitational field outside a spherical mass distribution, where one has Schwarzchild's solution. With the line element in spherical coordinates

$$ds^2 = f^2c^2dt^2 - h^2dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1}$$

expressing the components of the metric tensor g_{ik} in space-time by two functions h(r) and f(r), and inserting the thusly given components of the metric tensor in Einstein's vacuum field equation,

$$R_{ik} = 0 \tag{2}$$

one obtains

$$\left.\begin{aligned} &(hf)' = 0 \\ &h(f'' + \frac{2}{r}f') - h'f' = 0 \end{aligned}\right\} \tag{3}$$

For the gravitational field *F*, if measured with the eigen-time fdt, and eigen-length hdr, one obtains for the acceleration and hence the radial force *F*:

$$F = \frac{d}{fdt}\left(\frac{h}{f}\frac{dr}{dt}\right) = -\frac{c^2f'}{hf} \tag{4}$$

With *F* given by (4) one can write for the second equation (3):

$$\frac{1}{hr^2}(r^2F)' - \frac{F^2}{c^2} = 0 \tag{5}$$

The first term is identical to the definition of the divergence of a radial vector $\mathbf{F} = F \cdot \mathbf{r} / r$ which is $\pi r^2 F$ of the flux of **F** through a spherical surface of radius r, divided by the increase in the volume of this sphere $4\pi r^2 hdr$. One therefore can write for (5)

$$div\mathbf{F} - \frac{1}{c^2}\mathbf{F}^2 = 0 \tag{6}$$

We compare this result with Newtonian gravity where (G is Newton's constant)

$$div\mathbf{F} = -4\pi G\rho \tag{7}$$

and conclude that the gravitational field **F** has a negative mass density

$$\rho_g = -\frac{F^2}{4\pi G c^2} \tag{8}$$

We can test this result by putting

$$F = -\frac{GM}{r^2} \tag{9}$$

the gravitational field of a spherical mass of radius R, for r > R. We find

$$\rho_g = -\frac{GM}{4\pi c^2 r^4} \tag{10}$$

To obtain the total amount of negative mass M_g outside of the mass M, we integrate

$$M_g = \int_R^\infty \rho_g 4\pi r^2 dr = -\frac{GM^2}{c^2 R} \tag{11}$$

or

$$M_g c^2 = -\frac{GM^2}{R} = E_{pot} \tag{12}$$

where E_{pot} is the negative gravitational potential energy of a spherical shell of radius R and mass M. This example shows, that to obtain the gravitational field mass M_g, one simple may have to equate the gravitational potential energy with $M_g c^2$.

Next we go to the two body problem treated by Bondi [1]. The Newtonian gravitational potential energy for two bodies of mass m_1 and m_2 and separated by the distance r is

$$E_{pot} = -\frac{Gm_1 m_2}{r} \tag{13}$$

and would become positive if one of the masses is negative.

Lengthy calculations with Einstein's nonlinear theory, including quantum field theoretical corrections, give for the potential energy of two masses [3]

$$E_{pot} = -\frac{Gm_1m_2}{r}\left(1 + 3\frac{G(m_1 + m_2)}{2rc^2} + \frac{41}{10\pi}\frac{r_p{}^2}{r^2}\right) \tag{14}$$

where $r_p \approx 10^{-33}$cm is the Planck length, making the last term in the bracket on the r.h.s. of (14) small for $r >> r_p$.

It is remarkable that for $m_2 = -m_1$ the second term vanishes, and one obtains there for the gravitational field energy ($m_1=|m_2|=m$)

$$E_{pot} = m_g c^2 = \frac{G|m|^2}{r}\left(1 + \frac{41}{10\pi}\frac{r_p{}^2}{r^2}\right)$$

$$\approx \frac{Gm^2}{r}, r >> r_p \tag{15}$$

Therefore, to complete Bondi's calculation one may simple add M_g to the positive mass of the positive-negative mass two body problem.

It was shown by Bopp the presence of negative masses can be accounted for in a Lagrange function, $L = \left(q_k, \dot{q}_k, \ddot{q}_k\right)$, which also depends on the acceleration. The equations of the motion are there derived from the variational principle:

$$\delta\int L(q_k, \dot{q}_k, \ddot{q}_k)dt = 0 \tag{16}$$

or from

$$\delta\int \Lambda(x_a, u_a, \dot{u}_a)ds = 0 \tag{17}$$

where $u_a = dx_a / ds$, $\dot{u}_a = du_a / ds$, $ds = (1-\beta^2)^{t/2}dt$, $\beta = \mathrm{v}/c$, $x_a = (x_1, x_2, x_3, ict)$, and where $L = \Lambda(1-\beta^2)^{1/2}dt$. With the subsidiary condition

$$F = u_a^2 = -c^2 \tag{18}$$

One obtains from (17)

$$\frac{d}{ds}\left(\frac{\partial(\Lambda + \lambda F)}{\partial u_a} - \frac{d}{ds}\frac{\partial\Lambda}{\partial u_a}\right) - \frac{\partial\Lambda}{\partial x_a} = 0 \tag{19}$$

where λ is a Lagrange multiplier. In the absence of external forces, Λ can only depend on $\dot{u}_a^2$. The simplest assumption is a linear dependence

$$\Lambda = -k_0 - (1/2)k_1\dot{u}_a^2 \tag{20}$$

whereby (19) becomes

$$\frac{d}{ds}\left(2\lambda u_a + k_1\ddot{u}_a\right) = 0 \tag{21}$$

or

$$2\dot{\lambda} u_a + 2\lambda\dot{u}_a + k_1\ddot{u}_a = 0 \tag{22}$$

Differentiating the subsidiary condition one has

$$u_a\dot{u}_a = 0,\ u_a\ddot{u}_a + \dot{u}_a^2 = 0,\ u_a\dddot{u}_a + 3u_a\ddot{u}_a = 0 \tag{23}$$

by which (22) becomes

$$-2\dot{\lambda} - 3k_1\dot{u}_a\ddot{u}_a = -2\dot{\lambda} - \frac{3}{2}k_1\frac{d}{ds}\dot{u}_a^2 = 0 \tag{24}$$

It has the integral (summation over ν)

$$2\lambda = k_0 - \frac{3}{2}k_1\dot{u}^2 \tag{25}$$

where k_0 appears as a constant of integration. By inserting (25) into (21) the Langrange multiplier is eliminated and one has

$$\frac{d}{ds}\left[(k_0 - \frac{3}{2}k_1\dot{u}_\nu^2)u_a + k_1\ddot{u}_a\right] = 0 \tag{26}$$

Writing (26) as follows:

$$\frac{dP_a}{ds} = 0,\quad P_a = (k_0 - \frac{3}{2}k_1\dot{u}^2)u_a + k\,\ddot{u}_a \tag{27}$$

where P_a are the components of the momentum-energy four-vector. For k_1=0 one has p_a=k_0u_a, which by putting k_0=m is the four-momentum of a spinless particle with rest mass m. The mass-dipole moment is therefore given by

$$P_a = k_1\dot{u}_a \tag{28}$$

As can be seen from the conservation of angular momentum

$$\frac{d}{ds}J_{\alpha\beta} = 0 \tag{29}$$

where

$$J_{\alpha\beta} = [\mathbf{x},\mathbf{P}]_{\alpha\beta} + [\mathbf{p},\mathbf{u}]_{\alpha\beta} \tag{30}$$

and where $[\mathbf{x},\mathbf{P}]_{\alpha\beta} = x_\alpha P_\beta - x_\beta P_\alpha$, that for a particle at rest (P_k=0, k=1, 2, 3) one has

$$J_{kl} = [\mathbf{p},\mathbf{u}]_{kl} = p_k u_l - p_l u_k, \quad k,l\text{=1, 2, 3} \tag{31}$$

which is just the spin angular momentum.

The energy of a pole-dipole particle at rest, and for which u=icγ, is determined by the fourth component

$$\mathbf{P}_4 = imc = i(k_0 - \frac{3}{2}k_1\dot{u}^2)c\gamma \tag{32}$$

For the transition to quantum mechanics one needs the equation of motion in canonical form. There we separate the space and time derivative, whereby $L = -\Lambda ds/dt = L(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}})$. Setting c=1 we have

$$\left.\begin{aligned} L &= -(k_0 + \frac{1}{2}k_1\dot{u}_a^2)(1-\mathrm{v}^2)^{1/2} \\ \dot{u}_a^2 &= \frac{1}{\left[(1-\mathrm{v}^2)^{1/2}\right]^4}\left[\dot{\mathbf{v}}^2 + \left(\frac{\mathbf{v}\cdot\dot{\mathbf{v}}}{(1-\mathrm{v}^2)^{1/2}}\right)^2\right] \end{aligned}\right\} \tag{33}$$

From

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{v}}}, \quad \mathbf{s} = \frac{\partial L}{\partial \dot{\mathbf{v}}} \tag{34}$$

one has to compute the Hamilton function

$$H = \mathbf{v}\cdot\mathbf{P} + \dot{\mathbf{v}}\cdot\mathbf{s} - L \tag{35}$$

From $\mathbf{s} = \partial L/\partial\dot{\mathbf{v}}$ one obtains

$$\begin{aligned} \mathrm{s} &= \frac{1}{\left[\sqrt{1-\mathrm{v}^2}\right]^3}\left[\dot{\mathrm{v}} + \frac{(\mathrm{v}\cdot\dot{\mathrm{v}})}{(1+\mathrm{v}^2)}\right] \\ \dot{\mathrm{v}} &= -\frac{\left[\sqrt{1-\mathrm{v}^2}\right]^3}{\mathrm{k}_1}\left[\mathrm{s} - (\mathrm{v}\cdot\mathrm{s})\mathrm{v}\right] \end{aligned} \tag{36}$$

by which together with (33) $\dot{\mathbf{v}}\cdot\mathbf{s}$ can be expressed in terms of **v** and **s**. In these variables the angular momentum conservation law (29) assumes the form

$$\mathbf{r}\times\mathbf{P} + \mathbf{v}\times\mathbf{s} = \mathbf{const} \tag{37}$$

with the vector **s** is equal the mass dipole moment. For the Hamilton function (35) one then finds

$$H = \mathbf{v} \cdot \mathbf{P} + k_0 (1 - \mathrm{v}^2)^{1/2} - (1/2k_1)(1 - \mathrm{v}^2)^{3/2} \left[\mathbf{s}^2 - (\mathbf{s} \cdot \mathbf{v})^2 \right] \tag{38}$$

Putting

$$\begin{aligned} &\mathbf{P} = \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}} \\ &\mathbf{v} = \mathbf{a} \\ &(1 - \mathrm{v}^2)^{1/2} = \alpha_4 \end{aligned} \tag{39}$$

where $\alpha = \{\mathbf{a}, \alpha_4\}$ are the Dirac matrices, one finally obtains the Dirac equation

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial t} + H\psi = 0 \tag{40}$$

where

$$\begin{aligned} &H = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 m \\ &\alpha_\beta \alpha_\nu + \alpha_\nu \alpha_\beta = 2\delta_{\beta\nu} \end{aligned} \tag{41}$$

with the mass given by

$$\begin{gathered} m = k_0 - (1/2k_1)(1 - \mathrm{v}^2)\left[\mathbf{s}^2 - (\mathbf{s} \cdot \mathbf{v})^2 \right] \\ m = k_0 \; for \; \mathrm{v} = c \end{gathered} \tag{42}$$

This result can be directly applied to the Planck mass plasma where positive and negative mass quasiparticles form gravitational bound Dirac particle fermions [4]

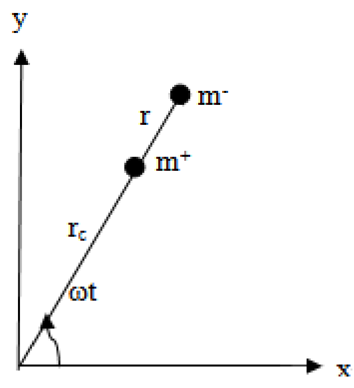

Fig. 1: Pole-dipole particle configuration.

Following Hönl and Papapetrou [5], we analyze the simple classical mechanical two body pole-dipole model shown in Figure 1. It consists of a positive mass m^+ and a negative mass m^-. In a two body problem with both masses positive and with an attractive force in between, the two bodies can execute a circular motion around their center of mass. In case one of the masses is negative, but with both together having a positive mass pole $m_0=m^+-|m^-|$, the circular motion persists, except that the center of mass is no more in between the masses, even though it is still located on the line connecting m^+ and m^-. As a consequence, the pole-dipole particle executes a rotational motion which causes the spin. This motion has the same property as the "Zitterbewegung" derived by Schrödinger from the Dirac equation [6].

If $|m^+| > |m^-|$, the distance of m^- from the center of mass is larger than for m^+, and we assume that m^+ is at a distance r_c, with m^- at a distance r_c+r. Furthermore, if $m_0 << m^+ \sim |m^-|$, one has $r << r_c$. Defining $\gamma_+ = (1 - \mathrm{v}_+^2 / c^2)^{-1/2}$, with $\mathrm{v}_+ = r_c \omega$ where ω is the angular velocity around the center of mass, and $\mathrm{v}_- = (1 - \mathrm{v}_-^2 / c^2)^{-1/2}$. With $\mathrm{v}_- = (r_c + r)\omega$, momentum conservation leads to

$$m^+ \gamma_+ r_c = | m^- | \gamma_- (r_c + r) \tag{43}$$

For $r << r_c$ and henceforth putting $\gamma_+ = \gamma$ one can expand:

$$\gamma_- = \gamma (1 + \frac{r_c r \omega^2 \gamma^2}{c^2} + ...) \tag{44}$$

For the mass dipole moment one has

$$p = m^+ r = | m^- | r = \frac{m^+ \gamma - | m^- | \gamma_-}{\gamma_-} r_c \tag{45}$$

With the help of (44) and for $\gamma >> 1$ one finds

$$r_c \approx p \gamma^2 / m_0 \tag{46}$$

and for the energy

$$E / c^2 = m = m^+ \gamma - | m^- | \gamma_- \approx p \gamma / r_c \tag{47}$$

and finally, for the angular momentum (putting $\omega r_c \sim c$):

$$J = \left[m^+ \gamma r_c^2 - | m^- | \gamma_- (r_c + r)^2 \right] \omega \approx -p \gamma c \approx -m c r_c \tag{48}$$

The correct spin angular momentum is obtained from the Dirac equation for $r_c \approx \hbar / 2mc$. From (46) and (47) one has

$$m = m_0 / \gamma \tag{49}$$

In a co-rotating reference system of the pole-dipole particle the gravitational interaction energy is positive, and for $m^+ - | m^- | << | m^\pm |$, given by

$$E = m_0 c^2 = -\frac{G m^+ m^-}{r} \approx \frac{G | m^\pm |^2}{r} \tag{50}$$

According to (34) the mass in a system at rest is

$$mc^2 = \frac{G\,|\,m^{\pm}\,|^2}{\gamma r} \tag{51}$$

With $p \sqcup |\, m^{\pm} \,|\, r$, equation (47) and $r_c = \hbar / 2mc$, one obtains

$$2\gamma \,|\, m^{\pm} \,|\, r_c = \hbar \tag{52}$$

which can be used to eliminate r from (51), with the result that

$$m = 2G \,|\, m^{\pm} \,|^3 \,/ \hbar c = 2 \,|\, m^{\pm} \,|^3 \,/ m_p^2 \tag{53}$$

where $m_p = \sqrt{\hbar c / G}$ is the Planck mass.

Equation (53) is the gravitational field mass of a positive mass interacting with a likewise negative mass, replaces the mass a zero rest mass fermion acquired in the standard model by the Higgs mechanism.

It shows how a fermion, if composed of a large positive and a large negative mass,can acquire its mass without the Higgs mechanism by the gravitational field of the large positive and negative mass.

The assumption for the hidden existence of negative masses is consistent with Schrödinger's discovery that the "Zitterbewegung" results from the interference of positive and negative energy waves [6].

Equation (53) can also be written as follows:

$$\frac{m}{m_p} = 2\left(\frac{\left|m^{\pm}\right|}{m_p}\right)^3 \tag{54}$$

But with $|\, m^{\pm} \,|\, rc \approx \hbar$ and $m_p r_p \cong \hbar$, it follows that also $m/m_p \approx (r_p/r)^3$, and according to (15), one can neglect the quantum corrections for all masses of interest.

3. Conclusion

A more detailed analysis of the positive-negative mass two body problem first carried out by Bondi, does not lead to a self acceleration. It rather leads to the finding that the Dirac spinors can be thought of as being composed of positive and negative mass particles, and rather than leading to a self-acceleration, it leads to the "Zitterbewegung," which for the Dirac particle was discovered by Schrödinger. It definitely does not lead to an unstable vacuum composed of positive and negative masses as claimed by Cavalleri and Tonni [7].

Replacing supersymmetry by the assumption that the vacuum is made up by an equal number of positive and negative masses, and replacing the Higgs field by the Einsteinian gravitational field of positive masses interacting with likewise negative masses, it can be seen as a model replacing the standard supersymmetric model of elementary particles and cosmology [8, 9].

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