Geometrodynamic Mass Decrease During Gravitational Collapse

Kenneth Sandale[*†]

Abstract The proof of Birkhoff's Theorem relies on a coordinate transformation to diagonalize the metric. This coordinate transformation is made to affect the vacuum region, but will nevertheless cause the matter in the non-vacuum region to have a different velocity in the new coordinate system than in the old coordinate system, because a coordinate transformation cannot be abruptly turned off where the two regions meet without violating the holonomy requirement. The effects of this coordinate transformation on the matter have never before been studied. In fact, the coordinate transformation turns out to cause part of a gravitationally collapsing mass distribution to at some point start to move backwards in time. This causes problems which invalidate the proof.

Furthermore, we provide an actual counterexample to Birkhoff's Theorem: In the particular circumstance of a spherically symmetric thin shell of matter collapsing it is shown that the Bianchi Identities give results contrary to Birkhoff's Theorem.

Keywords: Birkhoff's Theorem; Gravitational Collapse; Black Holes; Bianchi Identities; General Relativity

Introduction

Birkhoff's Theorem [1] implies that the metric in the exterior vacuum region of a spherically symmetric mass distribution does not vary in time, regardless of the (spherically symmetric) motion of the mass.

While interesting directly, some of its implications may be of more interest than the theorem itself:

1) One implication allows for the formation of black holes. If the metric outside the mass distribution does not vary with time, then $1-\frac{2M}{r}$ does not vary at a fixed point in the exterior region, implying that "M", the "Schwarzschild mass" does not vary with time. This time-constancy of M allows for $\frac{2M}{R}$ to become equal to unity, the condition for a black hole formation, if R, the radius of the mass

[*] sandale@alum.mit.edu
[†] Allegro Wireless Canada Inc., 2350 Matheson Blvd East, Mississauga, ON, L4W 5G9, Canada

distribution becomes small enough. If it were the case that M decreased with gravitational collapse, and the decrease was sufficiently quantitatively large, $\frac{2M}{R}$ would not be able to attain unity.

2) Another implication of Birkhoff's Theorem is that gravitational monopole radiation cannot occur.

The standard proof of Birkhoff's Theorem [2-4] begins by writing out the most general spherically symmetric line element

$$ds^2 = g_{00}(r,t)dt^2 + g_{01}(r,t)drdt + g_{11}(r,t)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$$

and then uses a coordinate transformation that mixes r and t to make g_{01} vanish, putting the line element into the form (dropping the prime from the new t) $ds^2 = g_{00}(r,t)dt^2 + g_{11}(r,t)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$. After this is done, the $R_1^0 = 0$ vacuum Einstein Equation simply yields $-g^{00}\frac{\partial \ln(g_{11})}{\partial t} = 0$, implying g_{11} has no time dependence. By subtracting the $R_1^1 - \frac{1}{2}g_1^1 R = 0$ Einstein Equation from the $R_0^0 - \frac{1}{2}g_0^0 R = 0$ Einstein Equation in the vacuum region, with an appropriate choice of time coordinate, $\frac{\partial g_{11}}{\partial t} = 0$ can be seen to imply $\frac{\partial g_{00}}{\partial t} = 0$.

The use of $g_{01} = 0$ in proofs of Birkhoff's Theorem appears to be universal, appearing in conventional proofs [2-3,5] and even in novel proofs [6]. This is not mere coincidence. The same reasoning that derived $\frac{\partial g_{11}}{\partial t} = 0$ from the assumption of a vanishing g_{01} comes very close to implying that $\frac{\partial g_{11}}{\partial t}$ *cannot* be zero if g_{01} is non-vanishing. If g_{01} does not vanish then the $R_1^0 - \frac{1}{2}g_1^0 R = 0$ vacuum Einstein Equation rather than implying $-g^{00}\frac{\partial \ln(g_{11})}{\partial t} = 0$, implies that $-g^{00}\frac{\partial \ln(g_{11})}{\partial t}$ is some quantity that does not at all appear to be zero.

Because of its essentialness to the proof of Birkhoff's Theorem, the coordinate transformation causing g_{01} to vanish deserves serious scrutiny. This however appears never to have been done. While the usefulness of the coordinate transformation in the proof pertains to the vacuum region, the coordinate transformation cannot just stop at the interface of the vacuum region and the matter region without violating the holonomy condition for coordinate transformations. So therefore the coordinate transformation will cause the matter to have a different velocity in the new coordinate system than in the old coordinate system.

The possible perniciousness of the coordinate transformation's actions on the matter can be understood from an analogous situation in electromagnetism, where we can create an obviously invalid proof that a uniformly moving charged particle does not produce a magnetic field. Consider, in one Lorentz frame, a uniformly moving charge and the magnetic field it produces. A Lorentz Transformation can be made that causes the magnetic field to vanish. So we now have a situation where in the new coordinate system the particle is not generating a magnetic field. The incorrect proof then reaches the conclusion that in some coordinate system one can have a uniformly *moving* particle not generating a magnetic field. The error of course is that the coordinate transformation that caused the magnetic field to vanish also impacted on the charged matter, causing it to no longer be moving. The vanishing magnetic field result is not a *general* result, but only really applicable to a situation where the charged matter is not moving. Likewise, because the Birkhoff coordinate transformation to make g_{01} vanish affects the motion of the matter we cannot be assured that the result of the exterior vacuum metric having a vanishing time dependence is applicable for general spherically symmetric motion of a mass distribution.

Finding ourselves in a situation where Birkhoff's Theorem was not validly proven, we are motivated to seek a possible counterexample. The Schwarzschild mass is often treated simply as a constant of integration whose value is chosen to correspond with Newtonian Theory [7-9]. But by integration of the $R_0^0 - \frac{1}{2} g_0^0 R = 0$ Einstein Equation in the non-vacuum region it can be seen to be a specific quantity $\int T_0^0 4\pi r^2 dr$ [10]. Applying the Bianchi Identities to a situation with a gravitationally collapsing shell of mass we get the result that the Schwarzschild mass is not a constant of motion, but rather is a decreasing quantity. This definitively contradicts Birkhoff's Theorem.

1. Direct Examination Of Changes In Mass Via The Bianchi Identities

1.1 Calculation Of Changes In "M" During Collapse

Let us consider the change in the stress-energy of a thin shell of pressureless matter undergoing spherically symmetric gravitational collapse, using the Schwarzschild-Birkhoff solution for the metric field.

From the Bianchi Identities

$$\frac{\partial T_0^{\mu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\sigma\mu} T_0^{\sigma} - \Gamma^{\sigma}_{0\mu} T^{\mu}_{\sigma} = 0 \tag{1}$$

$$\frac{\partial T_0^{\mu}}{\partial x^{\mu}} + \frac{\frac{\partial\sqrt{-g}}{\partial x^{\sigma}}}{\sqrt{-g}} T_0^{\sigma} - \Gamma^{\sigma}_{0\mu} T^{\mu}_{\sigma} = 0 \tag{2}$$

$$\sqrt{-g}\,\frac{\partial T_0^{\mu}}{\partial x^{\mu}} + \frac{\partial\sqrt{-g}}{\partial x^{\sigma}} T_0^{\sigma} - \sqrt{-g}\,\Gamma^{\sigma}_{0\mu} T^{\mu}_{\sigma} = 0 \tag{3}$$

$$\frac{\partial\left(\sqrt{-g}\,T_0^{\mu}\right)}{\partial x^{\mu}} - \sqrt{-g}\,\Gamma^{\sigma}_{0\mu} T^{\mu}_{\sigma} = 0 \tag{4}$$

$$\frac{\partial\left(\sqrt{-g}\,T_0^{0}\right)}{\partial x^{0}} + \sum_{i=1}^{3} \frac{\partial\left(\sqrt{-g}\,T_0^{i}\right)}{\partial x^{i}} - \sqrt{-g}\,\Gamma^{\sigma}_{0\mu} T^{\mu}_{\sigma} = 0 \tag{5}$$

$$\int \frac{\partial\left(\sqrt{-g}\,T_0^{0}\right)}{\partial x^{0}} d^3x + \int \sum_{i=1}^{3} \frac{\partial\left(\sqrt{-g}\,T_0^{i}\right)}{\partial x^{i}} d^3x - \int \sqrt{-g}\,\Gamma^{\sigma}_{0\mu} T^{\mu}_{\sigma}\, d^3x = 0 \tag{6}$$

Because $\sqrt{-g}\,T_0^i$ vanishes outside the shell, the $\int \sum_{i=1}^{3} \frac{\partial\left(\sqrt{-g}\,T_0^{i}\right)}{\partial x^{i}} d^3x$ term vanishes if the range of integration is such that all the shell of matter is within the boundaries of integration with no matter touching a boundary. Under that assumption, we get

$$\int \frac{\partial\sqrt{-g}\,T_0^{0}}{\partial x^{0}} d^3x - \int \sqrt{-g}\,\Gamma^{\sigma}_{0\mu} T^{\mu}_{\sigma}\, d^3x = 0 \tag{7}$$

Writing out the $\Gamma^{s}_{0\mu}$ in the second term explicitly in terms of the derivatives of the metric (using the notation that the index "1" refers to the radial direction) and keeping in mind that g_{10} is zero in the Schwarzschild-Birkhoff coordinates, that T_2^2 and T_3^3 are zero because the shell of matter is pressureless, and because it can see seen that the term proportional to T_0^1 turns out to exactly cancels with a term proportional to T_1^0, we get

$$\int \frac{\partial \sqrt{-g}T_0^0}{\partial x^0} d^3x - \frac{1}{2}\int \sqrt{-g}\left(g^{00}\frac{\partial g_{00}}{\partial x^0} T_0^0 + g^{11}\frac{\partial g_{11}}{\partial x^0} T_1^1 \right) d^3x = 0 \tag{8}$$

$$\int \frac{\partial \sqrt{-g}T_0^0}{\partial x^0} d^3x - \frac{1}{2}\int \sqrt{-g}\, g^{00}\frac{\partial g_{00}}{\partial x^0} T_0^0 d^3x = \frac{1}{2}\int \sqrt{-g}\, g^{11}\frac{\partial g_{11}}{\partial x^0} T_1^1 d^3x \tag{9}$$

Exploiting the spherical symmetry of our physical situation, we replace $\sqrt{-g}$ with $\sqrt{g_{00}}\sqrt{-g_{11}}\,4\pi r^2$ giving

$$\int \frac{\partial\left(\sqrt{g_{00}}\sqrt{-g_{11}}T_0^0 4\pi r^2\right)}{\partial x^0} dr - \frac{1}{2}\int \sqrt{g_{00}}\sqrt{-g_{11}}\, g^{00}\frac{\partial g_{00}}{\partial x^0} T_0^0 4\pi r^2 dr =$$

$$\frac{1}{2}\int \sqrt{g_{00}}\sqrt{-g_{11}}\, g^{11}\frac{\partial g_{11}}{\partial x^0} T_1^1 4\pi r^2 dr \tag{10}$$

The first term on the left-hand side,

$\int \frac{\partial\left(\sqrt{g_{00}}\sqrt{-g_{11}}T_0^0 4\pi r^2\right)}{\partial x^0} dr$ becomes

$$\int \frac{1}{2}\frac{\frac{\partial g_{00}}{\partial x^0}}{\sqrt{g_{00}}}\sqrt{-g_{11}}T_0^0 4\pi r^2 dr + \int \sqrt{g_{00}}\frac{\partial\left(\sqrt{-g_{11}}T_0^0 4\pi r^2\right)}{\partial x^0} dr$$

so Equation 10 becomes

$$\int \frac{1}{2}\frac{\frac{\partial g_{00}}{\partial x^0}}{\sqrt{g_{00}}}\sqrt{-g_{11}}T_0^0 4\pi r^2 dr + \int \sqrt{g_{00}}\frac{\partial\left(\sqrt{-g_{11}}T_0^0 4\pi r^2\right)}{\partial x^0} dr$$

$$-\frac{1}{2}\int \sqrt{g_{00}}\sqrt{-g_{11}}\, g^{00}\frac{\partial g_{00}}{\partial x^0} T_0^0 4\pi r^2 dr = \frac{1}{2}\int \sqrt{g_{00}}\sqrt{-g_{11}}\, g^{11}\frac{\partial g_{11}}{\partial x^0} T_1^1 4\pi r^2 dr \tag{11}$$

In the Schwarzschild-Birkhoff regime $g^{00} = g_{00}^{-1}$, therefore the first and third terms in Equation 11 cancel, giving

$$\int \sqrt{g_{00}} \frac{\partial\left(\sqrt{-g_{11}} T_0^0 4\pi r^2\right)}{\partial x^0} dr = \frac{1}{2} \int \sqrt{g_{00}} \sqrt{-g_{11}} g^{11} \frac{\partial g_{11}}{\partial x^0} T_1^1 4\pi r^2 dr \qquad (12)$$

For weak field situations, the gravitational field is approximately $\frac{\partial g_{00}}{\partial r}$, and thus the difference between the value of g_{00} on the inner surface and the value on the outer surface is of order of magnitude the strength of the gravitational field times the thickness of the shell. Thus for thin shells, to an excellent approximation, g_{00} is constant throughout the shell when the thickness is small. However, in situations where the situation approaches that of being a black hole, the gravitational field is not approximately $\frac{\partial g_{00}}{\partial r}$, and our approximation would no longer be valid.

Thus, as long as the situation is not approaching that of a black hole we can approximate g_{00} as being constant over the thin shell and pull it out from the integral to cause Equation 12 to become Equation 13. However, it is important to note that in situations approaching a black hole, this is not valid.

$$\sqrt{g_{00}} \int \frac{\partial\left(\sqrt{-g_{11}} T_0^0 4\pi r^2\right)}{\partial x^0} dr = \tfrac{1}{2} \sqrt{g_{00}} \int \sqrt{-g_{11}} g^{11} \frac{\partial g_{11}}{\partial x^0} T_1^1 4\pi r^2 dr \qquad (13)$$

$$\int \frac{\partial\left(\sqrt{-g_{11}} T_0^0 4\pi r^2\right)}{\partial x^0} dr = \tfrac{1}{2} \int \sqrt{-g_{11}} g^{11} \frac{\partial g_{11}}{\partial x^0} T_1^1 4\pi r^2 dr \qquad (14)$$

We choose our mass distribution to be such that $(v/c)^2$ is negligible, and thus the right-hand side of Equation 14 can be approximated as zero. However, for our purposes this approximation is not really even necessary. The key result of our work will be that the

Schwarzschild mass decreases during gravitational collapse. We will show in the next paragraph that the right hand side of Equation 14 is always negative for gravitational collapse in the Schwarzschild Birkhoff regime, and thus the effect of it non-vanishing is never to cancel the decrease in mass, but rather actually to increase the decrease in mass.

The right-hand side of Equation 14 can be shown to be negative by re-writing it as $\frac{1}{2}\int\sqrt{-g_{11}}\frac{\partial g_{11}}{\partial x^0}T^{11}4\pi r^2 dr$. T^{11} is positive since it is the product of the mass density (a positive number) and the square of the velocity (a positive number). $\frac{\partial g_{11}}{\partial x^0}$ in the shell can be seen to be negative as the shell collapses in the Schwarzschild Birkhoff regime, . Therefore the integral on the right-hand side of Equation 14 is always negative. Therefore, even if we do not approximate it as zero, we will still get the same result of a decreasing Schwarzschild M that we are about to see presently.

$$\frac{\partial\left(\int\sqrt{-g_{11}}T_0^0 4\pi r^2 dr\right)}{\partial x^0}=0 \tag{15}$$

$$\frac{d}{dt}\left(\int\sqrt{-g_{11}}T_0^0 4\pi r^2 dr\right)=0 \tag{16}$$

This is quite remarkable. It indicates that the quantity that is the constant of motion is $\int\sqrt{-g_{11}}T_0^0 4\pi r^2 dr$. *This is NOT the same quantity as* $\int T_0^0 4\pi r^2 dr$, *the quantity that mathematically is the Schwarzschild Mass!*

As the shell gravitationally collapses, the magnitude of $\sqrt{-g_{11}}$ *in the Schwarzschild regime increases and thus for* $\int\sqrt{-g_{11}}T_0^0 4\pi r^2 dr$ *to remain constant, the quantity* $\int T_0^0 4\pi r^2 dr$, *which is the specific quantity which must be* M *in the Schwarzschild Solution, decreases. So the Schwarzschild mass would decrease during gravitational collapse!*

So we have shown that the actual mathematical quantity for the "mass" in the Schwarzschild scenario MUST decrease during gravitational collapse, something Birkhoff's Theorem implied could never happen.[1]

[1] While the geometrodynamic change in mass density in a collapsing system has previously not been examined via the Bianchi Identities, in part because the Birkhoff's Theorem seemingly argues against the necessity of doing so, it actually has been done in another situation. In cosmological models, the mass

1.2 Extreme Gravitational Collapse and Extreme Gravitational Anti-Collapse

Seemingly if g_{11} were to have its magnitude become infinite--as what would happen if the shell were to collapse to the point of black hole formation, $\int T_0^0 4\pi r^2 dr$ would have to become zero in order for $\int \sqrt{-g_{11}} T_0^0 4\pi r^2 dr$ to remain constant. Since $\int T_0^0 4\pi r^2 dr$ is the Schwarzschild mass, what we have seemingly shown is that concomitant to a black hole forming, its mass goes to zero. Obviously a zero mass object will not generate a black hole, and thus we would see black hole formation (at least for a thin shell) in the Schwarzschild metric as being self-contradictory, and thus impossible.

The situation is not as simple though as it might seem. For the argument of the previous paragraph to work trivially, g_{11} would need to be uniform over the mass distribution, which it is not. Furthermore, as we noted earlier, our derivation of Equation 16 assumed that g_{00} was uniform over the thin shell, which it is not under conditions approaching a black hole.

It is interesting to note that Nature allows the converse process to a gravitationally collapsing body geometrodynamically losing mass. The time reversed process of mass decrease during gravitational collapse is allowed—in the Schwarzschild regime from out of virtually nothingness an expanding object of mass approaching zero can expand into having a finite mass.[2]

It might be tempting to speculate on whether such a process created the Universe, but of course such a model would not give the isotropy and homogeneity found in Friedmann-type models. (It would however obviate the question of what the Universe was like before t=0.)

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density in an expanding universe is indeed explicitly calculated via the Bianchi Identities. In such situations there is no analogue to Birkhoff's Theorem to guide (actually, to misguide) us. Perhaps more importantly in causing the need to adjust the mass density to be recognized in those situations, intuitively we expect that as the volume of the Universe expands the mass density should need to decrease in a way commensurate with the expansion in order to preserve the total mass (minus losses due to the action of pressure).

[2] We need not consider whether the initial infinitesimally small object could get enough radial momentum to expand against its own gravitational field. Relativity puts certain constraints on $T^{\mu 0}$, i.e. the value of $T^{\mu 0}$ at an infinitesimally future instant of time is completely determined by the values of $T^{\mu\nu}$ and the metric and its derivatives (via the Bianchi Identities) at an earlier instant, but the T^{ii} ($i \neq 0$) components are totally free. A sharp "unprovoked" change in T^{ii} is fully allowed in Relativity.

1.3 Monopole Radiation

We cannot assume that "information" about the decrease in the mass will be sent to distant regions instantaneously, and so we would expect monopole gravitational waves.

Indeed, one might want to consider construing the decrease of the Schwarzschild mass of a gravitationally collapsing object as being "due" to monopole radiation carrying away mass/energy, but we do not wish here to engage in heuristic characterizations.

1.4 Limitations of the Methodology

Before moving to the next section we should note that our results for gravitational collapse were done using the Schwarzschild-Birkhoff metric inserted into the Bianchi Identities. But our main conclusion was that the Schwarzschild-Birkhoff regime could not really be correct. So, while we have shown that the regime used to allow black holes is wrong, our other results are not themselves reliable, being that to get them we assumed a metric which we ended up showing was actually wrong.

Obviously we should concern ourselves with what the results would be using the correct regime.

We could try to generalize Equation 16 to what it would be without the now disproven Birkhoff's Theorem assumptions/results, to get the time evolution of $\int T_0^0 4\pi r^2 dr$. However, we no longer know that $\int T_0^0 4\pi r^2 dr$ is the "mass" quantity, anyway. It was the "mass" quantity in the Schwarzschild-Birkhoff regime because solving the equations within the Schwarzschild-Birkhoff regime yielded that it was. However those equations were stripped of terms involving time derivatives of the metric, terms we now know do not vanish, and stripped of terms involving g_{01}, terms we will soon see do not vanish. So even if we could find the time-evolution of $\int T_0^0 4\pi r^2 dr$, it would not necessarily be of use. It appears that there is no simple way to fully understand in detail what happens during gravitational collapse.

It obviously is quite plausible that a Correspondence Principle effect will occur such that in the limit of gentle collapse the results we got using the Schwarzschild-Birkhoff metric approach are correct.

What we do know is that Birkhoff's Theorem is incorrect, and that there is now no reason to believe a collapsing mass distribution will maintain a constant "mass", and thus we are no longer assured that unlimited gravitational collapse must lead to black hole formation.

2 ANALYSIS OF THE BIRKHOFF AND SCHWARZSCHILD COORDINATE TRANSFORMATIONS

If, as we calculated in the preceding section, the Schwarzschild mass decreases during gravitational collapse, then in the vacuum region exterior to the mass shell the quantities $1-\frac{2M}{r}$ and $\frac{-1}{\left(1-\frac{2M}{r}\right)}$ will change in time, in contradiction to the Birkhoff's Theorem. Logically then, either the Bianchi Identities or Birkhoff's Theorem is incorrect. There is no possibility for the Bianchi Identities to be wrong if General Relativity is correct, but it turns out that there actually is one place, and only one place, in the Birkhoff's Theorem proof that is not validly justified. The coordinate transformation used to make g_{01} vanish in the vacuum affects the motion of the mass distribution because the holonomic requirement for coordinate transformations prevents coordinate transformations from being abruptly turned off. This aspect of the proof of Birkhoff's Theorem has apparently never previously been scrutinized. What we will establish happens is that this coordinate transformation will cause the particles in the mass shell to eventually reverse their direction in time, something we will see leads to invalidation of the proof.

Reversal of time direction is an exotic phenomenon, but it turns out to be implied by a wide range of seemingly normal coordinate transformations. Consider the example in the next paragraph.

The coordinate transformation to make g_{01} vanish is of the form $dt'=\eta(Cdt-Edr)$ where C is the g_{00} in the initial coordinate system, $-E$ is the g_{01} in the initial coordinate system, and η is the integrating factor. If in the original coordinate system g_{00} was $1-\frac{2M}{r}$ and g_{01} was $-\frac{2Mv}{3r^2}$, where M is a constant, then noting that $dr=vdt$ the Birkhoff coordinate transformation would be $dt'=\eta\left[\left(1-\frac{2M}{r}\right)-\frac{2Mv^2}{3r^2}\right]dt$. For dt' to turn negative, all that is needed is for $\frac{2M}{r}+\frac{2Mv^2}{3r^2}$ to become greater than 1 and for η to remain positive at that point. Within the Schwarzschild-Birkhoff regime clearly $\frac{2M}{r}+\frac{2Mv^2}{3r^2}$ will eventually become greater than 1 during unlimited collapse. We can see that η will not switch signs when $\frac{2M}{r}+\frac{2Mv^2}{3r^2}$ becomes greater than 1 by considering the Weinberg method for constructing an integrating factor [2]. At the moment $(Cdt-Edr)$ becomes zero, assign a spatially constant non-infinitesimally positive value of η to all points on that space-like hypersurface. The holonomy condition,

$\frac{\partial}{\partial r}(\eta C) = \frac{\partial}{\partial t}(\eta E)$, since both C and E have finite derivatives on the shell (except when the shell collapses to $r = 0$, something that occurs after $\frac{2M}{r} + \frac{2Mv^2}{3r^2}$ becomes greater than 1, not while $\frac{2M}{r} + \frac{2Mv^2}{3r^2}$ is becoming greater than 1, implies that η has a finite time derivative. Since the time derivative of η is not negatively infinite it will not instantaneously go from being a finite positive number to being a negative number.

We are not arguing that the metric in the previous paragraph is the actual metric, but rather we are just trying to show that a non-exotic metric can lead to exotic time-direction reversal in the Birkhoff coordinate transformation. Our proof that the reversal occurs is not done by direct examination of the g_{00} and g_{01} in the original coordinate system—it might be very difficult to calculate the actual values of those quantities—but rather by concluding on the basis of our Bianchi Identity treatment of the collapsing mass shell in the previous section that there *must* be an error somewhere in the Birkhoff Theorem proof, and realizing that time reversal is the only place the error could be hidden. Birkhoff's Theorem is sufficiently simple that all other parts of the proof can be seen not to be flawed.

So we know that in the new coordinate system, the Birkhoff coordinate system, the particles composing the collapsing mass shell will eventually start moving backwards in time. Whether such a situation is allowable is most certainly questionable, but even if it were allowable, the situation would not be that of a shell collapsing to progressively smaller radii. It would be topologically like the standard situation in Feynman diagrams where an electron moves forward in time and then turns around and moves backwards in time. (Of course, in QED this would also involve photon creation.) So an observer observing things in the Birkhoff coordinate system would observe two mass shells heading towards each other and then vanishing. So what would be going on in the Birkhoff coordinate system would not be a single shell undergoing collapse, and thus it could not be used to evaluate what would happen in such a situation.[3]

One might attempt to salvage the Birkhoff's Theorem coordinate transformation by limiting our consideration to gravitational collapse that is just starting, and ignore the strange things that happen later on when g_{00} and g_{01} in the initial coordinate system might take on values resulting in the particles reversing their direction in time in the

[3] The need to take into account the effect of a vacuum coordinate transformation on the matter in the non-vacuum region can be underscored by the following consideration. One can fallaciously prove that a uniformly moving electric charge does not produce a magnetic field by making a the Lorentz Transformation on the $F^{\mu\nu}$ tensor that removes the B vector field. Of course the error is that this Lorentz Transformation must be carried onto the region where the charge is and causes the charge to now be at rest.

Birkhoff's coordinate system. But ignoring the event does not change it from happening. For example, if we take a quantum electrodynamics Feynman diagram where an electron goes forward in time before subsequently reversing its direction in time with the emission of a photon, and we just do not look at the time period where the pair annihilation occurred, the positron will nevertheless still be present. Personally ignoring the future time period where the mass shell reverses its direction in time will not change things—the event still occurs, and thus we will still have two mass shells with opposite velocities heading towards each other.

Another way one might be tempted to try to salvage Birkhoff's Theorem is by specifying that we will apply the Birkhoff coordinate transformation for the time period when the shell is collapsing, but then at some time before the time when the shell reverses its time-direction we no longer apply the coordinate transformation. It turns out that we cannot assume this is allowable--we have no guarantee that such a scheme can be done in a way that satisfies the holonomy[4] requirement for coordinate transformations.

CONCLUSION

The coordinate transformation used in the derivation of Birkhoff's Theorem to make g_{01} vanish is not a valid coordinate transformation.

If one follows the time-evolution of the quantity expressing the Schwarzschild "M" expressed in terms of an integral of T_0^0 one finds that it is not a constant of motion, in contradiction to what would be necessary for Birkhoff's Theorem to be true.

[1] G. D. Birkhoff: Relativity and Modern Physics. Harvard University Press, Cambridge (1923), p.244-5

[4] For a matrix $M_b^{a'}$ to truly be an acceptable coordinate transformation, $M_b^{a'}$ must satisfy the condition $\frac{\partial M_b^{a'}}{\partial x^c} = \frac{\partial M_c^{a'}}{\partial x^b}$, (because $\frac{\partial\left(\frac{\partial x^{a'}}{\partial x^b}\right)}{\partial x^c} = \frac{\partial\left(\frac{\partial x^{a'}}{\partial x^c}\right)}{\partial x^b}$). This is the "holonomy requirement". In the region where the Birkhoff coordinate transformation is being applied the holonomy requirement would be satisfied because an integrating factor has been created to make it work. In the region after where the Birkhoff's coordinate transformation has been terminated it would be satisfied because the coordinate transformation there is simply the identity coordinate transformation of $\delta_b^{a'}$. However at the boundary there is no reason to think we will have holonomy--we cannot be assured we can just abruptly "turn off" a coordinate transformation in a way that holonomy will be satisfied. Indeed we cannot even be assured that we can turn it off in a non-abrupt way with holonomy satisfied.

[2] S. Weinberg: Gravitation and Cosmology: Principles and Applications of General Relativity. John Wiley & Sons (1972), p. 337

[3] H. Stephani: General Relativity: An Introduction to the Theory of Gravitational Field Cambridge University Press; (1990), pp 102-3

[4] H. Ohanian, R. Ruffini: Gravitation and Spacetime. W. W. Norton & Company, Inc. (1994), pp397-400

[5] Ibid., p. 398

[6] S. Deser, J. Franklin, *Schwarzschild and Birkhoff a la Weyl*, Am. J. Phys.73:261-264 (2005); and arXiv:gr-qc/0408067v2

[7] Ibid. 2, p. 181

[8] C. Misner, K. Thorne, J. Wheeler: Gravitation, W.H. Freeman and Co. (1973), p603

[9] Ibid. 4, p. 395

[10] L. Landau, E. Lifshitz: The Classical Theory of Fields. Pergamon, London (1975), Equation 100.23, Page 303