THE DECREASING TUNNEL BY PROFESSOR FLORENTIN SMARANDACHE

A geometrical attempt is developed for approaching the Open question 1 [The Orthic triangle] concerning the the locus of centerpoints $H_{1 \ldots n}$ *and the point of convergency . A first Algebraic analysis follows .*

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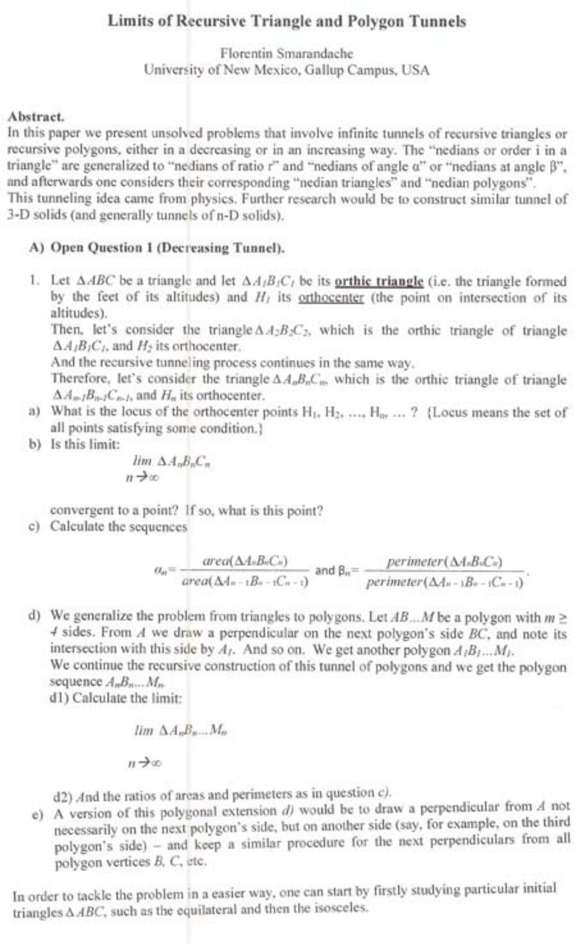

Limits of Recursive Triangle and Polygon Tunnels

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Abstract.
In this paper we present unsolved problems that involve infinite tunnels of recursive triangles or recursive polygons, either in a decreasing or in an increasing way. The "nedians or order i in a triangle" are generalized to "nedians of ratio r" and "nedians of angle α" or "nedians at angle β", and afterwards one considers their corresponding "nedian triangles" and "nedian polygons".
This tunneling idea came from physics. Further research would be to construct similar tunnel of 3-D solids (and generally tunnels of n-D solids).

A) Open Question 1 (Decreasing Tunnel).

1. Let ΔABC be a triangle and let $\Delta A_1B_1C_1$ be its **orthic triangle** (i.e. the triangle formed by the feet of its altitudes) and H_1 its orthocenter (the point on intersection of its altitudes).
Then, let's consider the triangle $\Delta A_2B_2C_2$, which is the orthic triangle of triangle $\Delta A_1B_1C_1$, and H_2 its orthocenter.
And the recursive tunneling process continues in the same way.
Therefore, let's consider the triangle $\Delta A_nB_nC_n$, which is the orthic triangle of triangle $\Delta A_{n-1}B_{n-1}C_{n-1}$, and H_n its orthocenter.

a) What is the locus of the orthocenter points $H_1, H_2, \ldots, H_n, \ldots$? {Locus means the set of all points satisfying some condition.}

b) Is this limit:

$$\lim_{n \to \infty} \Delta A_nB_nC_n$$

convergent to a point? If so, what is this point?

c) Calculate the sequences

$$\alpha_n = \frac{area(\Delta A_nB_nC_n)}{area(\Delta A_{n-1}B_{n-1}C_{n-1})} \text{ and } \beta_n = \frac{perimeter(\Delta A_nB_nC_n)}{perimeter(\Delta A_{n-1}B_{n-1}C_{n-1})}.$$

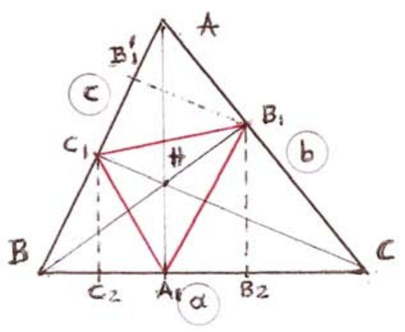
d) We generalize the problem from triangles to polygons. Let $AB\ldots M$ be a polygon with $m \geq 4$ sides. From A we draw a perpendicular on the next polygon's side BC, and note its intersection with this side by A_1. And so on. We get another polygon $A_1B_1\ldots M_1$.
We continue the recursive construction of this tunnel of polygons and we get the polygon sequence $A_nB_n\ldots M_n$.
d1) Calculate the limit:

$$\lim_{n \to \infty} \Delta A_nB_n\ldots M_n$$

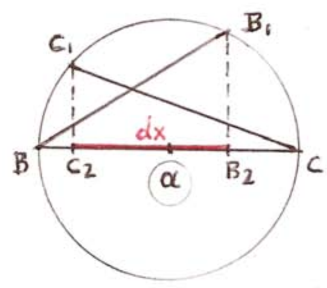
d2) And the ratios of areas and perimeters as in question *c)*.

e) A version of this polygonal extension *d)* would be to draw a perpendicular from A not necessarily on the next polygon's side, but on another side (say, for example, on the third polygon's side) – and keep a similar procedure for the next perpendiculars from all polygon vertices B, C, etc.

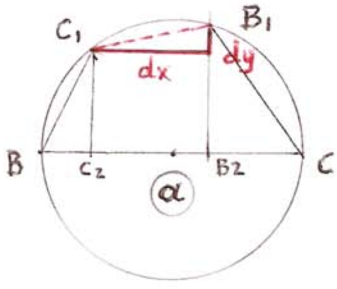
In order to tackle the problem in a easier way, one can start by firstly studying particular initial triangles ΔABC, such as the equilateral and then the isosceles.



F.1(1)



F.1(2)



F.1(3)

THE ORTHOCENTER (H) OF A TRIANGLE (ABC)

Let Δ ABC be any triangle with vertices A , B , C and let **Δ $A_1B_1C_1$** be its orthic triangle .

AA_1 , BB_1 , CC_1 are the altitudes of the vertices and points A_1 , B_1 , C_1 the feet of them .
Altitudes $AA_1 \perp BC$, $BB_1 \perp AC$, $CC_1 \perp AB$ and triangles AA_1B , AA_1C , BB_1A , BB_1C CC_1A , CC_1B are all right angled . It is also holding $B_1B'_1 \perp AB$

E_{ABC} = E = Area of triangle ABC and let be
$E_A = E_{AB_1C_1}$ = Area of triangle AB_1C_1 ,
$E_B = E_{BC_1A_1}$ = Area of triangle BC_1A_1 ,
$E_C = E_{CA_1B_1}$ = Area of triangle AC_1B_1 .

For shortness , we use the known ratios on any right angle triangle ABC at C :
The ratio of opposite side BC = a to the hypotenuse AB = c is called , sine of angle A , (sinA = a / c) and the ratio of the adjacent side AC = b to the hypotenuse AB is called cosine of angle A (cosA = a / c) . It is known ,

$E_A = E_{AB_1C_1} = ½ . AC_1 . B_1B'_1 = ½ . AC_1 .[AB_1 . \sin A] = ½ . AC_1 . AB_1 .\sin A$ .... (1)
$E_{ABC} = ½ . AB . CC_1 = ½ . AB.AC. \sin A$ …………………….. (2)

By division : $E_A / E_{ABC} = [AC_1 . AB_1 .\sin A] / [AB .AC. \sin A] = [AC_1 . AB_1.] / [AB .AC]$

and for $AC_1 = AC. \cos A = [$ **b** $.\cos A]$, $AB_1 = AB. \cos A =$ **[c** $.\cos A]$ and also

Using rotational similarity then :

E_A / E_{ABC} = [b .cosA . c .cosA] / [b . c] = $\cos^2 A$ ……….. (3a)

E_B / E_{ABC} = [c .cosA . a .cosA] / [c . a] = $\cos^2 B$ ……….. (3b)

E_C / E_{ABC} = [a .cosA . b .cosA] / [a . b] = $\cos^2 C$ ……….. (3c)

The area of the three triangles is ,

$E_A = E_{ABC} . \cos^2 A = E . \cos^2 A$
$E_B = E_{ABC} . \cos^2 B = E . \cos^2 B$
$E_C = E_{ABC} . \cos^2 C = E . \cos^2 C$ by summation :

$E_A+E_B+E_C = E.\cos^2A + E.\cos^2B + E.\cos^2C = E . [\cos^2A + \cos^2B + \cos^2C]$, so

Area of triangle **$A_1B_1C_1$** is **E_1** $= E – [E_A+E_B+E_C] =$ **$E . [1 - \cos^2A - \cos^2B - \cos^2C]$** or

$E_1 = E . [1 - \cos^2A - \cos^2B - \cos^2C] = E. [1 – (AB_1/c)^2 - (BC_1/a)^2 - (CA_1/b)^2]$ …. (4)

Replacing the results of (6.abc) in (4) , then using ,

$$\left|\frac{AB_1}{c}\right|^2 = \frac{|b^2+c^2- a^2|^2}{4. b^2.c^2} = \frac{a^4 + b^4 +c^4 -2. a^2.b^2 -2.a^2.c^2 + 2.b^2.c^2}{4. b^2.c^2}$$

$$\left|\frac{BC_1}{a}\right|^2 = \frac{|a^2+c^2- b^2|^2}{4. a^2.c^2} = \frac{a^4 + b^4 +c^4 -2. a^2.b^2 +2.a^2.c^2 - 2.b^2.c^2}{4. a^2.c^2}$$

$$\left|\frac{CA_1}{b}\right|^2 = \frac{|a^2+b^2-c^2|^2}{4.a^2.b^2} = \frac{a^4+b^4+c^4+2.a^2.b^2-2.a^2.c^2-2.b^2.c^2}{4.a^2.b^2}$$

and rewrite equation (4) as :

$4.a^2b^2c^2.[1-\cos^2A-\cos^2B-\cos^2C] = 4.a^2b^2c^2-[a6+a^2b^4+a^2c^4-2.a^4b^2-2.a^4c^2+2.a^2.b^2.c^2]$

$-[b6-2.a^2b^4+b^2c^4-2.b^4c^2+a^4b^2+2.a^2.b^2.c^2]-[c6-2.a^2c^4+b^4c^2-2.b^2c^4+a^4c^2+2.a^2.b^2.c^2]$

$= a^4.b^2+a^4.c^2+a^2.b^4+a^2.c^4+b^4.c^2+b^2.c^4-2.a^2.b^2.c^2-a6-b6-c6.$

then the ratio of area **E1** of the Orthic triangle to the area **E** of the triangle ABC is :

$$\frac{E_1}{E} = \frac{a^4.b^2+a^4.c^2+a^2.b^4+a^2.c^4+b^4.c^2+b^2.c^4-2.a^2.b^2.c^2-a6-b6-c6}{4.a^2.b^2.c^2} \quad (5)$$

$$\frac{E_1}{E} = \frac{4.a^2.b^2.c^2-a^2[b^2+c^2-a^2]^2-b^2[a^2+c^2-b^2]^2-c^2[a^2+b^2-c^2]^2}{4.a^2.b^2.c^2} \quad \ldots\ldots\ldots\ldots(5)$$

Verification :

For equilateral triangle $a = b = c \rightarrow$ **E1 / E** = (6.a6 -2.a6-3.a6)/4.a6 = a6/ 4.a6 = ¼

For isosceles triangle a , $b = c \rightarrow$ **E1 / E** = $[2.a^4.b^2+2.a^2.b^4-2.a^2.b^4-a6]/4.a^2.b^4$

= $[2.a^4.b^2-a6]/4.a^2.b^4 = [2.a^2.b^2-a^4]/4.b^4 = a^2.(2.b^2-a^2)/4.b^4$

Extensions to Pythagoras`s theorem :

Using the general conversions of Pythagoras`s theorem is easy to measure altitude
$la = AA_1$, $lb = BB_1$, $lc = CC_1$

In the right angled triangle **ABA1** $\rightarrow$ $AA_1^2 = AB^2-BA_1^2 = AB^2-(BC-A_1C)^2$
$AA_1^2 = AB^2-(BC^2+A_1C^2-2.BC.A_1C) = AB^2-BC^2-A_1C^2+2.BC.A_1C$
In the right angled triangle **AA1C** $\rightarrow$ $AA_1^2 = AC^2-A_1C^2$ and by subtraction ,

$0 = AB^2-BC^2-A_1C^2+2.BC.A_1C-AC^2+A_1C^2 = AB^2-BC^2-AC^2+2.BC.A_1C$ or

$2.BC.A_1C = AC^2+BC^2-AB^2$, so$\rightarrow$ **A1C** $= [AC^2+BC^2-AB^2]/2.BC =$ $\mathbf{[a^2+b^2-c^2]/2a}$

$\mathbf{A_1C = [a^2+b^2-c^2]/2.a}$ …… **(6ac)**

also

$A_1B = a - A_1C = a-[a^2+b^2+c^2]/2.a = [2a^2-b^2-a^2+c^2]/2.a =$ $\mathbf{[a^2+c^2-b^2]/2a}$

$\mathbf{A_1B = [a^2+c^2-b^2]/2.a}$ …… **(6ab)**

exists also

$AA_1^2 = AC^2-CA_1^2 = (AC+CA_1).(AC-CA_1)$ and by substitution (6ac)

$4.a^2(AA_1)^2 = [a^2+b^2+2.ab-c^2].[c^2-b^2-a^2+2.ab] = 2.a^2b^2+2.a^2c^2+2.b^2c^2$
$-[a^4+b^4+c^4]$ or

$$4.a^2(AA_1)^2 = 4.a^2(la)^2 = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4] \quad .... (7a)$$

In right angled triangle $BB_1C \rightarrow BB_1^2 = BC^2 - B_1C^2$.
In right angled triangle $BB_1A \rightarrow BB_1^2 = BA^2-(AC- B_1C)^2 = BA^2- AC^2- B_1C^2 + 2.AC.B_1C$
and by subtraction , and
$0 = BC^2 - B_1C^2 - BA^2 + AC^2 + B_1C^2 - 2.AC.B_1C$ then calculating B_1C, B_1A

$$B_1C = [-AB^2+ AC^2 + BC^2] / 2.AC = [a^2+b^2-c^2] / 2.b \quad ...... (6bc)$$

$$B_1A = b - [a^2+b^2-c^2] / 2.b = [2.b^2-b^2-a^2+ c^2] / 2.b = [c^2+b^2-a^2] / 2.b \quad .... (6ba)$$

also is

$BB_1^2 = BC^2 - B_1C^2 = (BC+ B_1C) . (BC - B_1C) = (a+ B_1C).(a - B_1C)$ by substitution (6bc)

$$4.b^2(BB_1)^2 = [a^2+b^2+2.ab - c^2].[c^2-b^2- a^2+ 2.ab] = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4]$$ or

$$4.b^2(BB_1)^2 = 4.a^2(lb)^2 = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4] \quad ... (7b)$$

in the same way exists ,

$$C_1B = [CB^2+ AB^2- AC^2] / 2.AC = [a^2+c^2-b^2] / 2.c \quad ...... (6cb)$$

$$C_1A = [AC^2+ AB^2-BC^2] / 2.c = [b^2+c^2-a^2] / 2.c \quad .... (6ca)$$

also

$$4.c^2(CC_1)^2 = 4.c^2(lc)^2 = 2.a^2b^2+2.a^2c^2+2.b^2c^2 - [a^4+b^4+c^4] \quad ... (7c)$$

and altitudes la , lb , lc are

$$la = AA_1 = \frac{\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}}{2.a}$$

$$lb = BB_1 = \frac{\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}}{2.b} \qquad lc = CC_1 = \frac{\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}}{2.c}$$

Verification :

For $a = b$ then $C_1A = C_1B \rightarrow C_1A = [c^2 / 2c] = c / 2 \rightarrow C_1B = [c^2 / 2c] = c / 2$
For $a = c$ then $B_1A = B_1C \rightarrow B_1A = [b^2 / 2b] = b / 2 \rightarrow B_1C = [b^2 / 2b] = b / 2$
For $b = c$ then $A_1B = A_1C \rightarrow A_1B = [a^2 / 2a] = a / 2 \rightarrow A_1C = [a^2 / 2a] = a / 2$

Since angle $< BC_1C = BB_1C = 90°$ then :

Using Pythagoras`s theorem in F.1(2) $\rightarrow BB_1^2 = B B_2 . BC \rightarrow BB_2 = BB_1^2/BC$

$$BB_2 = [BB_1^2] / BC = \frac{2.a^2.b^2 + 2.a^2.c^2 + 2.b^2.c^2 - (a^4+ b^4+ c^4)}{4.b^2.a}$$

$$\mathbf{BC_2} = [\, BC_1^2 \,] \,/\, BC = \frac{2.a^2.c^2 - 2.a^2.b^2 - 2.b^2.c^2 + (a^4+.b^4+c^4)}{4.c^2.a}$$

For $dx = [\, BB_2 - BC_2 \,]$ then :

$4.\ ab^2c^2.\ dx = 2.a^2b^2c^2 + 2.a^2c^4 + 2.b^2c^4 - c^2a^4 - c^2b^4 - c6 - b6 - a^4b^2 - b^2c^4 + 2.a^2b^4 + 2.\ b^4c^2 - 2.a^2b^2c^2$ and

$4.\ a.b^2.c^2\ .\ [\ dx = (\ BB_2 - BC_2\ )\ ] = 2.a^2b^4 + 2.a^2c^4 + b^2c^4 + c^2b^4 - a^4b^2 - a^4c^2 - b6 - c6$

In the right angle triangle BB_1B_2 exists also ,

$(BB_2)^2 = BB_1^2 - B_1B_2^2 = [2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)]\ /\ 4.b^2$

$- [a^2+b^2-c^2\ ].\ [2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)]\ /16a^2b^4 =$

$[2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)][4.a^2b^2 - a^2 - b^2 + c^2]\ /16a^2b^4 =$

$[2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)][\ c^2 + 2.a^2b^2 - (a+b)^2]/\ 16a^2b^4$

and the previous equation is :

$4.\ a.b^2.c^2.\ dx = 2.a^2b^4 + 2.a^2c^4 - a^4b^2 - a^4c^2 + b^2c^4 + c^2b^4 - b6 - c6$ or

$$\mathbf{dx = [\ BB_2 - BC_2\ ] = [\ 2.a^2b^4 + 2.a^2c^4 - a^4b^2 - a^4c^2 + b^2c^4 + c^2b^4 - b6 - c6\ ]\ /\ 4.ab^2c^2} \quad ..(a)$$

Verification :

For $b = c \quad \rightarrow \quad BA_1 = BC_2 + [\ BB_2. - BC_2\ ]\ /\ 2 = a\ /\ 2$

$\mathbf{dx} = BB_2 - BC_2 = (\ 4.a^2b^4 - 2.a^4b^2)\ /\ 4.ab^4 = (4.ab^2 - 2.a^3)\ /4.b^2 = (\ 2.ab^2 - a^3)\ /\ 2.b^2$

$BC_2 = (a^4+b^4+b^4+2.a^2b^2-2.a^2b^2-2.b^2b^2)\ /4.ab^2 = (a^4+2.b^4-2b^4)/\ 4.ab^2 = a^3\ /\ 4.b^2$

$BA_1 = (\ a^3)\ /\ 4.b^2 + (\ 2.ab^2 - a^3)\ /\ 4.b^2 = 2.ab^2\ /\ 4.b^2 = a\ /2$

$\mathbf{dx} = BB_2 - BC_2 = (\ 4.a^2b^4 - 2.a^4b^2)\ /\ 4.ab^4 = (4.ab^2 - 2.a^3)\ /4.b^2 = (\ 2.ab^2 - a^3)\ /\ 2.b^2$

From similar triangles CAA_1, $C\ B_1B_2 \rightarrow [\ B_1B_2/\ AA_1\ ] = CB_1/b$ and $(B_1B_2)^2 = [\ CB_1^2.AA_1^2]\ /b^2$ or

$B_1B_2^2 = (a^2+b^2-c^2\ )^2\ .\ [2.a^2b^2 + 2.a^2c^2 + 2.b^2c^2 - (a^4+b^4+c^4)]\ \ /\ 4.b^4.4a^2$ and

$\mathbf{B_1B_2^2 = (a^2+b^2-c^2\ )^2.\ [\ (a+b)^2 - c^2]\ .\ [\ c^2 - (a-b)^2]\ /\ 16.\ a^2b^4}$ …………….

From similar triangles BAA_1, $BC_1C_2 \rightarrow [\ C_1C_2/\ AA_1\ ] = BC_1/c$ and $(C_1C_2)^2 = [\ BC_1^2.AA_1^2]\ /c^2$

$\mathbf{C_1C_2^2 = (a^2+c^2-b^2\ )^2\ .\ [\ (a+b)^2 - c^2]\ .\ [\ c^2 - (a-b)^2]\ \ /\ \ 16.\ b^2c^4}$ ………….

and so

$$\mathbf{B_1B_2 = (a^2+b^2-c^2\ )\ .\ \sqrt{[\ c^2 - (a+b)^2\ ]\ .\ [\ (a-b)^2 - c^2]}\ \ /\ 4.\ ab^2} \quad …………….\ (8)$$

$$\mathbf{C_1C_2 = (\ a^2+c^2-b^2\ )\ .\ \sqrt{[\ c^2 - (a+b)^2\ ]\ .\ [\ (a-b)^2 - c^2]}\ \ /\ 4.\ bc^2} \quad …………..\ (8.a)$$

Verification :

For $b = c \ \rightarrow\ B_1B_2 = C_1C_2$

$$B_1B_2 = \frac{a^2}{4.ab^2}\ \sqrt{(b^2-a^2-b^2\ -2.ab\ ).(a^2+b^2\ -2.ab\ -b^2)} = \frac{a}{4.b^2}\ \sqrt{-(\ a^2+2.ab\ ).(\ a^2-2.ab\ )}$$

$$C_1C_2 = \frac{a^2}{4.ab^2}\sqrt{(b^2-a^2-b^2\ -2.ab\ ).(a^2+b^2\ -2.ab\ -b^2)} = \frac{a}{4.b^2}\sqrt{-(\ a^2+2.ab\ ).(\ a^2-2.ab\ )}$$

.

$[AA_1]^2 = b^2 - a^2\ /4 = (4.b^2-a^2)\ /4 \quad \rightarrow \quad$ **$AA_1 = |\sqrt{4.b^2-a^2}\ |\ /\ 4$**

Let $[B_1B_2\text{-}C_1C_2] = dy$, then **dy** $= [CB_1.\ AA_1]/b - [BC_1.\ AA_1]/c = [\ AA_1].[\ c.CB_1\text{-}\ b.BC_1]\ /\ bc$

$c.CB_1\text{-}\ b.BC_1 = c.(a^2+b^2-c^2)/2.b - b.(a^2+c^2\text{-}b^2/2c = [\ a^2c^2\text{-}\ a^2b^2+b^4+\ c^4]\ /\ 2.bc$

and

$$\mathbf{dy = [\ B_1B_2\ -C_1C_2\ ] = \frac{\sqrt{[c^2-(a+b)^2]\ .\ [(a\text{-}b)^2\ -\ c^2]}}{4.a\ b^2c^2}\ [\ a^2c^2\ -\ a^2b^2\ +\ b^4+\ c^4\ ]} \qquad \textbf{or}$$

$$\mathbf{dy = [\ B_1B_2\ -C_1C_2\ ] = \frac{\sqrt{[\ (a+b)^2\ -\ c^2]\ .\ [c^2-(a\text{-}b)^2]}}{4.a\ b^2c^2}\ [\ a^2c^2\ -\ a^2b^2\ +\ b^4\ -\ c^4\ ]\ ...} \qquad \textbf{(b)}$$

B_1C_1 is measured by using Pythagoras`s theorem in triangle (B_1B_1, dy , dx) and then by squaring equation (a) and (b) ,

$dx^2\ .\ [4.a\ b^2c^2]^2 = [4.a^4b^8+ 4.a^4c^8 +a^8b^4+ a^8c^4+ b^4c^8+ b^8c^4 + b12 +c12 -4.a6.b6 -4a6b^4c^2$
$+ 4.a^2b6.c^4 + 4.a^2b^8c^2\text{-}\ 4.a^2b10 - 4.a^2b^4c6 + 8.a^4b^4c^4 - 4.a6b^2c^4 - 4.a6.c6 +4.a^2b^2c^8 +4.a^2b^4c6$
$- 4.a^2b6.c^4 -4.a^2c10 + 2a^8b^2c^2\text{-}2.a^4b^4c^4 -2.a^4b6c^2+2.a^4b^8+2.a^4b^2c6 - 2.a^4b^2c6 - 2.a^4b^4c^4 +$
$2.a^4b6c^2 + 2.a^4c^8 + 2.b6.c6 -2.b^8c^4\text{-}2.b^2c10 - 2.b10c^2\text{-}\ 2.b^4c^8 + 2.b6.c6\ ] =$

$[\ 4.a^2b^2c^8 - 4.a^2b10 - 4.a^2c10 + 4.a^2b^8c^2+ 6.\ a^4b^8+6.a^4c^8 + 4.a^4b^4c^4 - 4.a6.b^2c^4 -4.a6.b^4c^2$
$- 4.a6.b6 - 4.a6.c6 +2a^8b^2c^2 + a^8b^4+ a^8c^4-2.b^2c10 + 4.b6.c6 - b^8c^4\text{-}2.b10c^2 + b12 +c12\ ]$

$dy^2\ .\ [4.a\ b^2c^2]^2 = [\ (a+b)^2\ -\ c^2]\ .\ [c^2\text{-}\ (a\text{-}b)^2]\ .\ [\ a^2c^2\ -\ a^2b^2\ +\ b^4\text{-}c^4\ ]^2 =$
$[a^2+b^2+2.ab\ \text{-}c^2].[\ c^2\text{-}a^2\ \text{-}b^2+\ 2.ab].[\ a^2c^2\ -\ a^2b^2\ +\ b^4\text{-}c^4\ ]^2 =$
$[a^2c^2\ \text{-}a^2b^2+2.a^3b\ +b^2c^2 -b^4+ 2.ab^3 + 4.abc^2\ \text{-}c^4+ a^2c^2\ \text{-}a^2b^2\text{-}2.a^3b\ +b^2c^2 - 2.ab^3$
$\text{-}2.abc^2+\ 4.a^2b^2].[\ a^2c^2\text{-}\ a^2b^2+b^4\text{-}c^4\ ]^2 = [2.a^2c^2+2.a^2b^2 + 2.b^2c^2 - a^4 - b^4\text{-}\ c^4]\ .$
$[a^4c^4+ a^4b^4+ b^8+c^8 - 2.a^4b^2c^2 +2.a^2b^4c^2\text{-}\ 2.a^2c6 - 2.a^2b6 +2.a^2b^2c^4 - 2.b^4c^4\ ] =$

$2.a6.c6+2.a6.b^4c^2+2.a^2b^8c^2+2.a^2c10\ -4.a6b^2c^4+4.a^4\ b^4c^4\text{-}4.a^4c^8\ -4.a^4b6c^2+4.a^4b^2c6$
$\text{-}4.a^2b^4c6 +2.a6.c6 -4.a6.b^4c^2+2.a^2b^2c^8 +4.a^2b10+2.a6.b^2c^4+4.a^4\ b^4c^4\text{-}4.a^4b^8+4.a^4b6c^2$
$- 4.a^4b^2c6 - 4.a^2b6.c^4\text{-}4.a6.c6 -4.a^2b^2c^8 +2.b10c^2 + c^4+4.a^2b^4c6 - 4.a^4\ b^4c^4+2.a^4b6c^2$
$+2.a^4b^2c6\ c6 + 2.a^2b6.c^4\text{-}\ 4.a^2b^8c^2 +2.b^2c10\ - a^8c^4\ - a^8b^4\ - a^4b^8\ - a^4c^8+ 2.a^8b^2c^2$
$\text{-}2.a6.b^4c^2+ 4.a6.c6 + 2.a6.b6 - 2.a6.b^2c^4+2.a^4\ b^4c^4+ 2.a^8c^4\ \text{-}b^4c^8 - a^4b^8 - 2.a^2b6.c^4$
$2.a^4b6c^2\ \text{-}2.a^2b^8c^2+ 2.a^2b^4c6 +2.b^2c10 - a^4\ b^4c^4\text{-}\ b12\ \text{-}b^8c^4+2.b^4c^8\ \text{-}a^4c^8+2.a^4b^2c6$
$\text{-}2.a^2b^4c6 +2.a^2c10 +2.a^2b6.c^4\ \text{-}2.a^2b^2c^8 - a^4\ b^4c^4\text{-}\ c12\ .$

and segment B_1C_1 is :

$(dx^2+dy^2)\ .[\ 4.ab^2c^2]^2 = [\ 4.a\ ^4b^2c^2]\ .\ [\ a^4+ b^4+c^4\ \text{-}2.a^2c^2\text{-}2.a^2b^2\ +2.b^2c^2]$ and

$$(dx^2+dy^2) = [B_1C_1]^2 = \frac{[\ 4.a\ ^4b^2c^2]\ .\ [\ b^2+c^2\ -\ a^2]^2}{[\ 4.ab^2c^2]^2}$$

$$[B_1C_1] = \frac{2.a^2bc.[b^2+c^2-a^2]}{4.ab^2c^2} = \frac{a}{2.bc}[b^2+c^2-a^2]$$

Similarly for the other sides is holding :

$$\mathbf{a_1 = B_1C_1 = \frac{a}{2.bc}[b^2+c^2-a^2] = \frac{a}{2.bc}[b^2+c^2-a^2]} \quad \dots (9.a)$$

$$\mathbf{b_1 = A_1C_1 = \frac{b}{2.ca}[c^2+a^2-b^2] = \frac{b}{2.ac}[a^2+c^2-b^2]} \quad \dots (9.b)$$

$$\mathbf{c_1 = A_1B_1 = \frac{c}{2.ab}[a^2+b^2-c^2] = \frac{c}{2.ab}[a^2+b^2-c^2]} \quad \dots (9.c)$$

Verification :

For **a = b** then :

$$A_1C_1 = \frac{a}{2.ac}[c^2] = c/2 \quad --- \quad C_1B_1 = \frac{a}{2.ac}[c^2] = c/2 \quad \text{therefore } C_1A_1 = C_1B_1$$

For **a = c** then :

$$A_1B_1 = \frac{a}{2.ab}[b^2] = b/2 \quad --- \quad B_1C_1 = \frac{a}{2.ab}[b^2] = b/2 \quad \text{therefore } B_1A_1 = B_1C_1$$

For **b = c** then :

$$A_1B_1 = \frac{b}{2.ab}[a^2] = a/2 \quad --- \quad A_1C_1 = \frac{a}{2.ab}[b^2] = b/2 \quad \text{therefore } A_1B_1 = A_1C_1$$

For **a = b = c** then :

$$A_1B_1 = \frac{a}{2aa}a^2 = a/2 \quad --- \quad A_1C_1 = \frac{a}{2aa}a^2 = a/2 \quad ---- \quad B_1C_1 = \frac{a}{2aa}a^2 = a/2$$

The Perimeter P_o of the orthic triangle $A_1B_1C_1$ is :

$$P_o = B_1C_1 + A_1C_1 + A_1B_1 = \frac{[a^2b^2+a^2c^2-a^4+a^2b^2+b^2c^2-b^4+a^2c^2+b^2c^2-c^4]}{2.abc} =$$

$$P_o = \frac{[2.a^2b^2+2.a^2c^2+2.b^2c^2-a^4-b^4-c^4]}{2.abc} = \frac{4.b^2c^2-[a^2-b^2-c^2]^2}{2.abc} =$$

$$P_o = \frac{[2.bc+a^2-b^2-c^2].[2.bc-a^2+b+c^2]}{2.abc} = \frac{[a^2-(b-c)^2].[(b+c)^2-a^2]}{2.abc} \quad \text{or}$$

$$\mathbf{P_o} = \frac{[a+b-c].[a-b+c].[a+b+c].[b+c-a]}{2.abc} = \mathbf{\frac{[a+b+c].[-a+b+c].[a-b+c].[a+b-c]}{2.abc}} \quad \dots (10)$$

Remarks :

a) Perimeter P_o becomes zero when . $[-a+b+c] = 0$, $[a-b+c] = 0$, $[a+b-c] = 0$ or when $a = b+c$, $b = a+c$, $c = a+b$ and

This is the property of any point A on line BC *where then* Segment BC = a is equal to the parts AB = c and AC = b . The same for points B and C respectively . [5]
Since perimeter is minimized by the orthic triangle then this triangle is the only one among all inscribed triangles in the triangle ABC . This property is very useful later on .

b) Since Perimeter Po of the orthic triangle is minimized at the three lengths a , b , c then is also at the three vertices A , B , C of the original triangle ABC .

c) The ratio (R_p) of the perimeter of the orthic triangle $A_1B_1C_1$ to the triangle ABC is :

$$R_p = \frac{\mathbf{P_o}}{\mathbf{P}} = \frac{[a+b-c].[a-b+c].[b+c-a].[a+b+c]}{2.abc \; . \; [a+b+c]} = \frac{\mathbf{[-a+b+c]\,.\,[a-b+c]\,.\,[a+b-c]}}{\mathbf{2.abc}} \quad \dots (11)$$

For a = b = c then → (P_o / P) = (a.a.a) / 2.a.a.a **= 1 / 2** which is holding . Let

P_o = a+b+c is the perimeter of triangle Δ ABC
P_1 = $a_1 + b_1 + c_1$ is the perimeter of orthic triangle Δ $A_1 B_1 C_1$
P_n = $a_n + b_n + c_n$ is the perimeter of **n-**th orthic triangle

The ratio R_p of the perimeters is depended only on the **n-1** sides of orthic triangles :

$$R_p = \frac{P_n}{P_{n-1}} = \frac{a_n + b_n + c_n}{a_{n-1} + b_{n-1} + c_{n-1}} = \frac{[-a_{n-1} + b_{n-1} + c_{n-1}].[a_{n-1} - b_{n-1} + c_{n-1}].[a_{n-1} + b_{n-1} - c_{n-1}]}{2.\,a_{n-1}\, b_{n-1}\, c_{n-1}}$$

d) The ratio R_a of the area of triangle Δ $A_n B_n C_n$ to the Δ $A_{n-1} B_{n-1} C_{n-1}$ is :

$$R_a = \frac{E_{n-1}}{E_n} = \frac{4.a_{n-1}^2.b_{n-1}^2.c_{n-1}^2 - a_{n-1}^2[b_{n-1}^2+c_{n-1}^2-a_{n-1}^2]^2 - b_{n-1}^2[a_{n-1}^2+c_{n-1}^2-b_{n-1}^2]^2 - c_{n-1}^2[a_{n-1}^2+b_{n-1}^2-c_{n-1}^2]^2}{4.a_{n-1}^2.\, b_{n-1}^2.\, c_{n-1}^2} \quad \dots (11a)$$

The ratio R_a of the areas is also depended only on the **n-1** sides of orthic triangles :

e.a . It is well known that the nine point circle is the circumcircle of the orthic triangle and has circumradius R_{A1B1C1} which is one half of the radius R = R_{ABC} of the circumcircle of triangle Δ ABC .

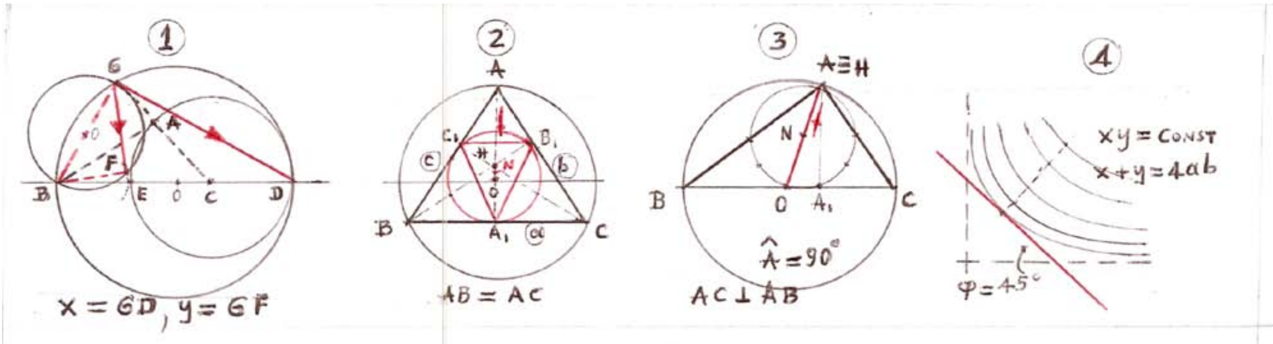
From F.4 , using Pythagoras theorem on triangle with hypotenuse equal to the diameter of the circle , then we have the law of sines as :

$$\frac{AB}{\sin C} = \frac{AC}{\sin B} = \frac{BC}{\sin A} = 2.R_{ABC} \quad \text{and} \rightarrow \quad R = \frac{AB}{2.\sin C} = \frac{AC}{2.\sin B} = \frac{BC}{2.\sin A} \quad \text{and R is}$$

$$R = \frac{BC}{2.\sin A} = \frac{a.c}{2.BB_1} = \frac{a.c\,[2.b]}{2.\sqrt{[(a+b)^2-c^2].[c^2-(a-b)^2]}} = \frac{\mathbf{abc}}{\mathbf{\sqrt{[(a+b)^2-c^2].[c^2-(a-b)^2]}}} \quad \text{and}$$

$$R_{A1B1C1} = \frac{\mathbf{abc}}{2.\,\mathbf{\sqrt{[(a+b)^2-c^2].[c^2-(a-b)^2]}}} \quad \text{…. the circumradius of orthic triangle …. (12)}$$

For $(a+b)^2 - c^2 = 0$ or $c^2 - (a-b)^2 = 0$ then $R_{A1B1C1} = \infty$, and it is $a + b = c$ and $a - b = c$ $a = b + c$ i.e. triangle ABC has the three vertices (the points) A , B , C on lines **c** and **a** respectively , a property of points on the two lines . [5]



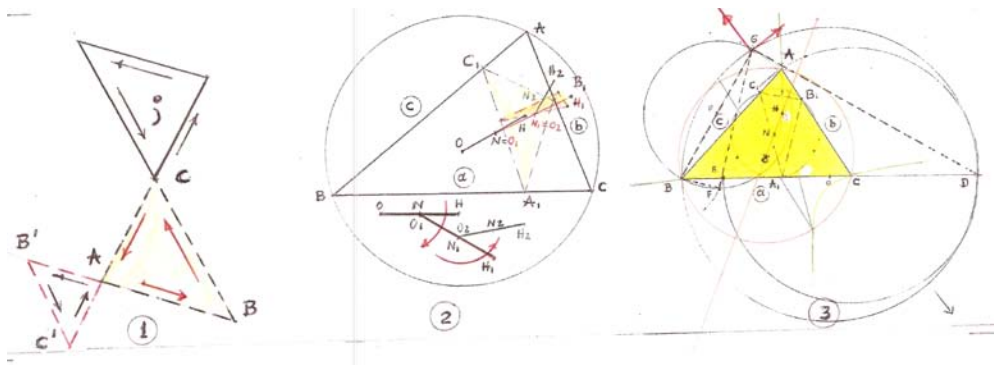
F.2(1) F.2(2) F.2(3) F.2(4)

e.b. The denominator of circum radius is of two variables $x = [(a+b)^2 - c^2]$ and $y = [c^2 - (a-b)^2]$ which have a fixed sum of lengths equal to ***4.ab*** and their product is to be made as large as possible. The product of the given variables , $x = [(a+b)^2 - c^2]$, $y = [c^2 - (a-b)^2]$ becomes maximum (this is proven) when $\boldsymbol{x = y = 2.ab}$ and exists on a family of hyperbolas curves where xy is constant for each of these curves . F.2(4)
All factors in denominator are conjugate and this is why orthogonal hyperbolas on any triangle are conjugate hyperbolas and follow Axial symmetry to their asymptotes .
The tangent on hyperbolas at point x = 2.ab formulates 45 ° angle to x , y plane system .
Since Complex numbers spring from Euclidean geometry [10] then the same result follows from complex numbers also . It is holding ,
x.y $= \rho\rho'.[\cos(\varphi+\varphi') + i.\sin(\varphi+\varphi')]$ where ρ , ρ' are the modulus and φ , φ' the angles between the positive direction of the x- axis and x , y direction .
Product **x.y** becomes maximum when the derivative of the second part is zero as ,

$$-\sin(\varphi+\varphi') + i.\cos(\varphi+\varphi') = 0 \quad \rightarrow \quad \sin(\varphi+\varphi') = i.\cos(\varphi+\varphi') \quad \text{or}$$

$$\frac{\sin(\varphi+\varphi')}{\cos(\varphi+\varphi')} = \tan(\varphi+\varphi') = +i \quad \text{or} \rightarrow \quad \tan^2(\varphi+\varphi') = -1 \quad \text{i.e.}$$

the slope of the tangent line (which is equal to the derivative) at point $\boldsymbol{x = y = 2.ab}$ is **- 1** and this happens for $(\varphi+\varphi') = 135°$ or $45°$.



F.3(1) Central Symmetry F.3(2) Vector AH F.3(3) Central and Axial Symmetry

e.c. Let`s consider the length AH and the directions rotated through point A . F.1(1)
The right angles triangles HAB_1 , BB_1C are similar because all angles are equal respectively (angle $AHB_1 = B_1BC$ because $BB_1 \perp AB_1$ and $BC \perp AH$) , so

$$\frac{AH}{BC} = \frac{AB_1}{BB_1} \quad \text{or} \quad AH = \frac{BC . AB_1}{BB_1} = \frac{a.[c^2 + b^2 - a^2]\ 2.b}{2b.\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}} = \frac{a.[c^2 + b^2 - a^2]}{\sqrt{[(a+b)^2 - c^2].[c^2 - (a-b)^2]}}$$

In any triangle ABC is holding ---- F.1(1)

$a^2 = b^2+c^2 - 2.bc.\cos A \rightarrow 2.bc.\cos A = b^2+c^2 - a^2 \rightarrow \cos A = [b^2+c^2 - a^2] / 2.bc$
$b^2 = c^2+a^2 - 2.ac.\cos B \rightarrow 2.ac.\cos B = c^2+a^2 - b^2 \rightarrow \cos B = [c^2+a^2 - b^2] / 2.ca$
$c^2 = a^2+b^2 - 2.ab.\cos C \rightarrow 2.ab.\cos C = a^2+b^2 - c^2 \rightarrow \cos C = [a^2+b^2 - c^2] / 2.ab$ so

$$\mathbf{AH} = \frac{a.[c^2+b^2- a^2].[\sqrt{[(a+b)^2-c^2].[c^2-(a-b)^2]}}{[(a+b)^2-c^2].[c^2-(a-b)^2]} = \frac{a.[c^2+b^2-a^2].\sqrt{[x.y]}}{x.y} = \frac{2.abc.\sqrt{x.y}}{x.y} (\cos A)$$

where $x= (a+b)^2-c^2$, $y= c^2-(a-b)^2$

$$\mathbf{AH} = \left|\frac{2.abc.\sqrt{x.y}}{x.y}\right| . \cos A = \left|\frac{2.abc}{\sqrt{x.y}}\right| . \cos A = \textbf{Vector AH} \quad \ldots\ldots\ldots\ldots(13)$$

From F2(1) BC = a , AC = b , AB = c , CD = CE = CA = b , BD = BC+CD = a + b ,
BA = BG = c , BE = BF = BC - EC = a – b .

With point O as center draw circle with BD = a+b as diameter and with point B as center draw circle (B , BA = BG) intersecting circle (O , OB) at point G .
Using Pythagoras theorem in triangles BDG , BFG then $GD^2 = BD^2-BG^2 = [(a+b)^2-c^2] = \mathbf{x}$
and $GF^2 = GB^2 - BF^2 = c^2 - BE^2 = c^2 - (a-b)^2 = \mathbf{y}$ and it is holding
$x+y = a^2+b^2+2.a b -c^2 + c^2-a^2-b^2 + 2.a b = 4.a.b = GD + GF$ = constant
$x . y = [(a+b)^2 - c^2].[c^2 - (a-b)^2] = GD . GF$ = constant

Since Ortocenter* H *changes Position , following orthic triangles* $A_1B_1C_1$ *, then Sector* AH_n *is altering magnitude and direction , so* AH_n *is a mathematical vector .

f. In F.2(1) GD ⊥ GB , GF ⊥ BF , therefore angle DGF = GBF so the sum of the pairs of two vectors [GD ,GE] , [BG , BE] is constant . Since addition of the two bounded magnitudes follows parallelogram rule , then their sum is the diagonal of the parallelogram .
In F2(2-4) the product of the pairs of the two vectors [GD ,GE] , [BG , BE] is constant in the direction perpendicular to diameter DE , so rectangular hyperbolas on the x , y system are 45°at the point x=y =2.ab .Where second derivative of cosA is -cosA=0 then AH is extreme

g. In F.3(2) orthic triangle $A_1B_1C_1$ of triangle ABC is circumscribed in Nine-point circle with center point N , *the middle of OH* , while point O is ABC`s circumcenter .
Since point N is the center of $A_1B_1C_1$ `s circumcenter , therefore Euler line OH is rotated through the middle point N of OH to all nested orthic triangles $A_nB_nC_n$ as is shown in F.3(2) , or are as

Sector (ONH) , ($N = O_1 N_1 H_1$) , ($N_1 = O_2 N_2 H_2$) → ($N_{n-1} = O_n N_n H_n$) → ($N_\infty = O_\infty N_\infty H_\infty$)

From vertices A,B,C and orthocenter H of triangle ABC passes rectangular circum-hyperbolas with point N as the center of the Nine-point circle . In Kiepert hyperbola , its center is the midpoint of Fermat points . In Jerabek rectangular hyperbola , its center is the intersection of Euler lines of the three triangles , of the four in triangle ABC . (the fourth is the orthic triangle).

h. In F.2(1) GD ⊥ GB and GF ⊥ BF therefore angle DGF = GBF or DGF + GBF = 360° [5] ie. angles DGF , GBF are conjugate . It is known that in an equilateral hyperbola , conjugate diameters make equal angles with the asymptotes , and because angle DGF = GBF then the sum of the two pairs of the two vectors (GD,GE) , (BG ,BE) is constant for all AH of orthic triangles .

To be vector **AH** a relative extreme (minimum or zero) , numerator of equation (13) must be zero in a fix direction to triangle ABC . Since a , b , c are constants , vector AH is getting extreme to the opposite direction by Central Symmetry through point A .

Verification :

C1. For a = b = c then $\cos A = [\,2a^2 - a^2\,] / 2.a^2 = a^2/2a^2 = 1/2$ therefore $\rightarrow$ A = 60°
$x = (a+b)^2 - c^2 = 4a^2 - a^2 = 3.a^2$, $y = c^2 - (a-b)^2 = a^2$
$x + y = 4.a^2$ and $x.y = 3.a^4$ __a
AH $= [\,2.a^3.\,a^2\sqrt{3}\,] / [\,3.a^4.2\,] =$ **$a.\sqrt{3}/3$** $= [a\sqrt{3}/2].(2/3) \rightarrow$ a | / = $a\sqrt{2}$ \ a = $a\sqrt{3}$ i.e.
Orthocenter Hn limits to → circumcenter **O → (2 / 3) . AA_1**

C2. For b = c then $\cos A = [\,2b^2 - a^2\,] / 2.b^2 = [\,b.\sqrt{2}+a\,].[\,b\sqrt{2} - a\,] / 2.b^2$ ……. (13.a)

1. For cos A = 0 then $b\sqrt{2} - a = 0$ and this for A = 90° and **AH = 0** i.e.
Orthocenter Hn limits to Vertice **A → AHn → A**

Since also cos A becomes zero with $b\sqrt{2} = -a$ (The Central symmetry to a) then
Orthocenter Hn limits always to → Vertice A or **AHn → A**

2. For cos A = 1 / 2 then A = 60° , $x = 2.ab + a^2$, $y = 2.ab - b^2$ and b = a
AH = $[2.ab^2.\sqrt{x.y}].(1/2) / xy = [\,a.b^2 / \sqrt{x.y}\,] = (ab^2) / a\sqrt{3} = a/\sqrt{3} =$ **$a\sqrt{3}/3$** = AO i.e.
Orthocenter Hn limits to circumcenter O of triangle ABC or AHn → O .

3. For cos A = $\sqrt{2}/2$ then A = 45° , $x = 2.ab + a^2$, $y = 2.ab - b^2$ and $4.la^2 = 4.b^2 - a^2$
AH $= [2.ab^2\sqrt{x.y}].(\sqrt{2}/2) / xy = [2.ab^2\sqrt{2}/\sqrt{x.y}] = [2.ab^2.\sqrt{2}] / \sqrt{(4.la^2)} = b^2/(la.\sqrt{2})$ i.e.

Orthocenter Hn limits to An on altitude la, or **AHn → b^2 / (la.$\sqrt{2}$) ,** and for $\cos A = la^2/b^2$
AHn = la i.e ***Orthocenter Hn limits to A1 , or AHn = la = AA_1 = Altitude AA_1 .***

4. For cos A = 1 then A = 0° , $x = a^2 + b^2 + 2.ab - b^2 = a.(\,a + 2.b\,)$
$y = b^2 - a^2 + 2.ab - b^2 = a.(\,2.b - a\,)$ and

AH $= [2.ab^2] / [a.\sqrt{4.b^2 - a^2}] = [2.b^2] / [\sqrt{4.b^2 - a^2}\,]$ and for A = 0° then
AH $= [\,2.b^2\,] / 2.b = b$ i.e. Orthocenter Hn limits to vertices B or C (the two vertices coincide) . or ***Orthocenter Hn limits to B , C or*** **AHn → B = C**

5. For cos A = - 1 then A = 180° , $x = a.(\,a+2.b\,)$ $y = a.(\,2.b - a\,)$ and a = 2.b

AH $= [2.b^2](-1) / [\sqrt{4.b^2 - (2.b)^2}] = [-\,2.b^2\,] / [\infty] = 0$ i.e point A coincides with the foot A_1 or $AA1 = 0$, ***Orthocenter Hn limits to B , C or*** **AHn → B = C**

C3. The first Numerator term of equation (13) and in F.2(3) is , $c^2 + b^2 - a^2$, and this *in order to be* on a right angle triangle ABC with angle < A = 90 ° , must be zero , or $a^2 = b^2 + c^2$.
The second term is $x = (a+b)^2 - c^2 \neq 0$, $a^2 + b^2 + 2.ab - c^2 = b^2 + c^2 + b^2 + 2.ab - c^2 = 2.[\,ab + b^2\,] \neq 0$ or $a + b \neq 0$ = constant → (b/a) < 0 i.e. H → A , and for $y = c^2 - (a-b)^2 \neq 0$, $c^2 - a^2 - b^2 + 2.ab = c^2 - b^2 - c^2 - b^2 + 2.ab = 2.[\,ab - b^2\,] \neq 0$ or $a - b \neq 0$ = constant , and (b/a) > 0 i.e. H → A . In F.2(3) also,

Geometrically , since B_1C_1 of orthic triangle $A_1B_1C_1$ coincides with point A , so the center N of the nine-point circle on ABC is always on OA . Since point N is $A_1B_1C_1$ ` s circum-center and represents the new $O_1$ then the new $N_1$ of $A_1B_1C_1$ `s is the center point on NA or $NH_1 = NA / 2$ Therefore point N is on OA direction at (AO/2) , (AO/4) , (AO/8) ,,,, ($AO/2^a$) points , i.e.

In any right-angle triangle ABC where angle < A = 90 °, the locus of the orthocenter points H_1........H_n of the orthic triangles $A_n B_n C_n$, is on line OA , and for n = ∞ then convergent to point A (vertice A) of the triangle ABC .

so , In an equilateral triangle ABC where a = b = c , Orthocenter H is fixed at Centroid K which coincides with Circum center O , and with the Nine-point center N .
In an Isosceles triangle ABC where a , b = c , Orthocenter H moves on altitude AA_1 (this is the locus of the orthocenter points H_1 .... H_n) and for angle A = 90° then convergent to the point A of the triangle .

Remark : In any triangle ABC rectangular hyperbolas follow Axial symmetry to their Asymptotes , in contradiction to orthocenter H , which follows Central Symmetry and Rotation through point A , the vertice opposite to the greatest side of the triangle .
This Springs out of the logic of Spaces , Anti-Spaces , Sub-Spaces of any first dimentional Unit **ds > 0** . [10] , therefore vector AH_n is limiting to the Orthocenters H_1 .... H_n of orthic triangles in triangle ABC .(Equation 13) . Since all other Conics through the vertices of triangle ABC are not passing through the Orthocenter and are not rectangular hyperbolas follow Central Symmetry . A further geometrical analysis follows .

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