Analytical Derivation of the Drag Curve $C_D = C_D\left(\mathcal{R}\right)$

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Through a convenient mathematical approach for the Navier-Stokes equation, we obtain the quadratic dependence v^2 of the drag force F_D on a falling sphere, and the drag coefficient, C_D, as a function of the Reynolds number. Viscosity effects related to the turbulent boundary layer under transition, from laminar to turbulent, lead to the tensorial integration related to the flux of linear momentum through a conveniently choosen control surface in the falling reference frame. This approach turns out to provide an efficient route for the drag force calculation, since the drag force turns out to be a field of a non-inertial reference frame, allowing an arbitrary and convenient control surface, finally leading to the quadratic term for the drag force.

DEFINING THE MATHEMATICAL PROBLEM

Regarding the application of the Newton second law to a small closed subsystem σ with boundary $\partial(\delta V)$ and volume δV of a continuum fluid in an inertial reference frame, one obtains at an instant t:

$$\int_{\sigma} d\vec{F}_{ext} = \int_{\delta V} \rho(\vec{r},t)\vec{f}(\vec{r},t)dV + \oint_{\partial(\delta V)} \mathbf{T}\cdot\hat{n}dS, \tag{1}$$

where $\vec{f}(\vec{r},t)$ is a locally external acceleration field, $\rho(\vec{r},t)$ the scalar density field, and $\mathbf{T}$ is the most general tensor due to the effects of the surrounding fluid on σ, being given by:

$$T_{ik} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda}\right) + \zeta\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda}, \tag{2}$$

where p is the local thermodynamic pressure field, being $\mathbf{T}$ written in terms of its components under the summation convention on repeated indices and where $\mathbf{T}$ was obtained from the combination of effects due to strain and shear:

$$\mathbf{\Gamma} = \alpha(\nabla\vec{v})_{ts} + \beta(\nabla\vec{v})_c = \alpha\left[(\nabla\vec{v})_s - \frac{1}{3}\vec{\nabla}\cdot\vec{v}\mathbf{1}\right] + \zeta\,\vec{\nabla}\cdot\vec{v}\mathbf{1}, \tag{3}$$

from which one defines the viscosity coefficients, $\alpha = 2\eta$ (this latter relation following from the coupling to the planar flow case, in which one defines the dynamical viscosity η) and ζ, under an isotropic assumption. Back to the Eq. (1), one obtains the Navier-Stokes equation:

$$\rho(\vec{r},t)\dot{\vec{v}}(\vec{r},t) - \rho(\vec{r},t)\vec{f}(\vec{r},t) + \vec{\nabla}p(\vec{r},t) - \eta\vec{\nabla}^2\vec{v}(\vec{r},t) + \\ - \left(\frac{1}{3}\eta + \zeta\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}(\vec{r},t)\right) = \vec{0}. \tag{4}$$

One may expand the total derivative in relation to time:

$$\dot{\vec{v}} = \frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \dot{x}\frac{\partial\vec{v}}{\partial x} + \dot{y}\frac{\partial\vec{v}}{\partial y} + \dot{z}\frac{\partial\vec{v}}{\partial z} = \frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v},$$

where $\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}$ is the convective term, with the advection operator $\left(\vec{v}\cdot\vec{\nabla}\right)$, this latter related to the acceleration of a fluid element from one point to another in an adopted inertial reference frame, from which the Eq. (4) may be written, the Navier-Stokes equation, under the form:

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}\vec{v} - \rho\vec{f} + \vec{\nabla}p - \eta\vec{\nabla}^2\vec{v} = \vec{0},$$

also under a divergence-free hypothesis for the velocity field (constant density turns out to be a sufficient condition). Hence, one has got, just using the more compact notation as in the Eq. (4), in the ground reference frame, the following mathematical problem:

$$\begin{cases} \rho\dot{\vec{v}} - \rho\vec{g} + \vec{\nabla}p - \eta\vec{\nabla}^2\vec{v} = \vec{0}, \quad \vec{\nabla}\cdot\vec{v} = 0; \\ \\ \lim\limits_{|\vec{r}|\to\infty} \vec{v} = \vec{0}, \quad \vec{v}\left(\partial\, sphere\right) = \dot{h}(t)\hat{e}_z\ nonslip, \end{cases} \tag{5}$$

where $\vec{g}$ is the local gravitational field and $\dot{h}(t)$ is the scalar velocity of the center of a falling sphere within the fluid.

GEDANKENEXPERIMENT

One measures the local gravitational field in the non-inertial frame attached to the falling sphere from the following gedankenexperiment: hollow sphere having got mass m, with an internal weighing apparatus (with negligible mass) to measure the normal force $\vec{N}$ that the ground of the hollow sphere exerts on a proof mass m_0. By isolating the system $m + m_0$, and, subsequently, by isolating the system m_0, one obtains:

$$\frac{\vec{N}}{m_0} = \frac{\vec{F}'_{\text{drag}}}{(m+m_0)} = \frac{\vec{F}_{\text{drag}} + \delta\vec{F}_{\text{drag}}(m_0)}{(m+m_0)}, \tag{6}$$

where $\vec{F}_{\text{drag}}$ is the force the fluid exerts on the hollow sphere, without the proof mass m_0, and $\delta\vec{F}_{\text{drag}}(m_0)$ is the increment to this force - due to the consideration of the internal proof mass m_0. Hence, the gravitational field $\vec{g}_0$ within the hollow sphere is given by:

$$\vec{g}_0 = \lim_{m_0 \to 0} -\frac{\vec{N}}{m_0} = \lim_{m_0 \to 0} -\frac{\vec{F}_{\text{drag}} + \delta\vec{F}_{\text{drag}}(m_0)}{(m+m_0)} = -\frac{\vec{F}_{\text{drag}}}{m}, \tag{7}$$

from which the force we want to calculate turns out to be a property of the non-inertial reference frame attached to the sphere. Adopting this falling reference frame, we have got the mathematical problem:

$$\begin{cases} \rho\dot{\vec{v}} + \rho\dfrac{\vec{F}_{\text{drag}}}{m} + \vec{\nabla}p - \eta\vec{\nabla}^2\vec{v} = \vec{0}, \quad \vec{\nabla}\cdot\vec{v} = \vec{0}; \\ \lim\limits_{|\vec{r}|\to\infty} \vec{v} = -\dot{h}(t)\hat{e}_z, \quad \vec{v}\,(\partial\, sphere) = \vec{0}\ nonslip. \end{cases} \tag{8}$$

Comparing the Eqs. (5) and (8), one infers the force we want to calculate (divided by the sphere mass m) turns out to be an acceleration field in the adopted reference frame, given by the Eq. (7). This turns out to be a field at each point of the fluid in the falling reference frame attached to the sphere, from which one may choose a convenient control surface of integration surrounding the sphere, not only the surface of the sphere. This provides an identity, derived in the next section, from which one may extract a convenient information from the adopted control surface.

CALCULATING $\vec{g}_0$

Applying the continuity equation in its most general form, calculating the instantaneous time rate of linear momentum variation within an arbitrary control volume, fixed and undeformable, one reaches the expression for the calculation of $\vec{g}_0$:

$$-\vec{g}_0 = \frac{\vec{F}_{\text{drag}}}{m} = \frac{1}{\int \rho dV}\left(\oint \Pi\cdot\hat{n}dS - \frac{\partial}{\partial t}\int \rho\vec{v}dV\right), \tag{9}$$

$$\Pi = \left[-\mathbf{1}p + \Gamma - \rho\left(\vec{v}\otimes\vec{v}\right)\right], \tag{10}$$

since $\vec{g}_0$ does not depend on the spatial coordinates within the fluid, once this field equally permeates each point of the fluid in the falling reference frame at any given instant t.

OBTAINING THE DRAG FORCE F_D AND THE DRAG COEFFICIENT C_D

Applying the Eqs. (9) and (10) to the control region $FGBAF$ depicted in the Fig. 1, at the stationary flow regime $t \to \infty$, one obtains:

$$\vec{F}^{\infty} = \frac{m}{m+m_{BL}}\left\{-\left[\int_{FG} + \int_{GB} + \int_{BA} + \int_{AF}\right]\mathbf{1}p_\infty\cdot\hat{n}dS - \int_{FG}\left[\rho\left(\vec{v}_\infty\otimes\vec{v}_\infty\right)\right]\cdot\hat{n}dS\right\}, \tag{11}$$

where m_{BL} is the mass of the boundary layer attached to the sphere. The pressure field on FG can be obtained, since this choosen surface (FG) does not violate the laminarity condition for a sufficiently thin boundary layer $\partial p/\partial r \approx 0$, in relation to DC, in virtue of the internal confinement of turbulence within the region $GBAFG$ (AB touching the boundary layer).

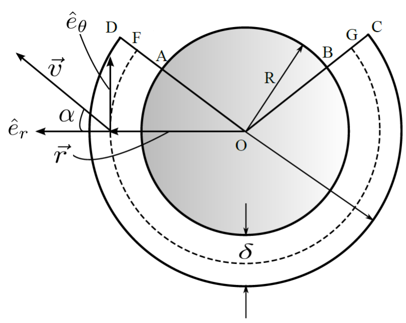

Fig. 1: Figure for the integration.

Hence:

$$p_\infty^{FG} = -\frac{\rho}{m}\varphi_{FG}^{\infty} + p_\infty^0 - \frac{9}{8}\rho\left(\dot{h}^\infty(t)\right)^2\sin^2\theta, \tag{12}$$

where φ^∞ is a scalar field due to the vanishing rotational of the force the fluid exerts on the sphere. The pressure field on AB (AB touching the *wake*, the rear region of the flow) is obtained from the condition of broken equilibrium at the separation point $S \approx B$. The obtention of the velocity field profile internal to the boundary layer is accomplished by the time average on the ensemble of turbulence within the boundary layer with the Fourier representation of the velocity field on FG by the step function, since we are firstly interested in the contribution term at high Reynolds number provided a full turbulent flow within the boundary layer at the brink of the drag crisis, where the drag force will suddenly decrease.

Hence:

$$p_{\infty}^{S}=-\frac{\rho}{m}\varphi_{GBAF}^{\infty}+p_{\infty}^{0}-\frac{9}{16}\rho\left(\dot{h}^{\infty}(t)\right)^{2}\sin^{2}\theta_{S}, \tag{13}$$

$$\langle(\vec{v}\otimes\vec{v})_{FG}\cdot\hat{n}\rangle_{t}=(\vec{v}_{\infty}\otimes\vec{v}_{\infty})\cdot\hat{n}=$$

$$\left\langle v_{FG}^{2}\left(R+\delta',\theta,t\right)\left(\cos^{2}\alpha(t)\,\hat{e}_{r}+\cos\alpha(t)\sin\alpha(t)\,\hat{e}_{\theta}\right)\right\rangle_{t}=$$

$$\frac{9}{16}\left(\dot{h}^{\infty}(t)\right)^{2}\sin^{2}\theta\,\hat{e}_{r}, \tag{14}$$

where θ_S is the separation angle, depicted in the Fig. 2.

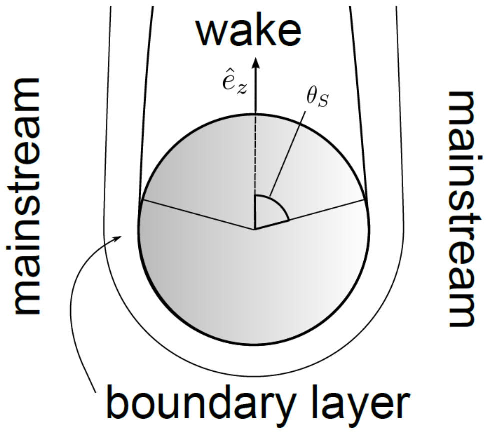

Fig. 2: Depicted elements.

Using these results within the Eq. (11), one obtains the quadratic contribution for the drag force via straightforward integration:

$$\left(1+\frac{m_{BL}}{m}\right)\vec{F}^{\infty}=\text{Buoyancy}+\frac{m_{BL}}{m}\vec{F}^{\infty}+$$

$$+\frac{9\pi}{32}\rho\left(\dot{h}^{\infty}(t)\right)^{2}R^{2}\sin^{4}\theta_{S}\,\hat{e}_{z}\Rightarrow$$

$$\vec{F}_{D}=\frac{9\pi}{32}\rho\left(\dot{h}^{\infty}(t)\right)^{2}R^{2}\sin^{4}\theta_{S}\,\hat{e}_{z}. \tag{15}$$

Renaming $\dot{h}^{\infty}(t)\equiv v$, knowing that the drag force points along the $\hat{e}_z$ direction, we simply write for the quadratic drag force contribution, the quadratic scalar component:

$$F_{D}=\frac{9\pi}{32}\rho v^{2}R^{2}\sin^{4}\theta_{S}. \tag{16}$$

One should notice this contribution arises from our consideration regarding the turbulent profile within the boundary layer, from which we see there is not any linear contribution arising at this flow regime. Writing the drag force as a series on v:

$$F_{D}\left(v\right)=\sum_{k=0}^{\infty}a_{k}v^{k}, \tag{17}$$

we know from the low Reynolds number regime that the linear contribution is given by the Stokes force [4]:

$$a_{0}=0,\quad a_{1}v=6\pi\eta Rv. \tag{18}$$

Hence, up to the drag crisis, the drag force reads:

$$F_{D}=6\pi\eta Rv+\frac{9\pi}{32}\rho\left(\sin^{4}\theta_{S}\right)R^{2}v^{2}. \tag{19}$$

The drag coefficient, C_D, and the Reynolds number, $\mathcal{R}$, are defined by:

$$C_{D}=\frac{2F_{D}}{\pi\rho R^{2}v^{2}},\quad\mathcal{R}=\frac{2\rho Rv}{\eta}. \tag{20}$$

Hence, from the Eqs. (19) and (20), one obtains the drag coefficient as a function of the Reynolds number, up to the drag crisis:

$$C_{D}\left(\mathcal{R}\right)=\frac{24}{\mathcal{R}}+\frac{9}{16}\sin^{4}\theta_{S}. \tag{21}$$

Fig. 3 shows the graph for the Eq. (21), for $\theta_S = 70.4°$. This is the separation angle obtained from the Froessling method [1]. One sees this dependence on the Reynolds number agrees with the experimental one over the entire range of Reynolds numbers up to the drag crisis, as one verifies, e.g., in [2] and [3].

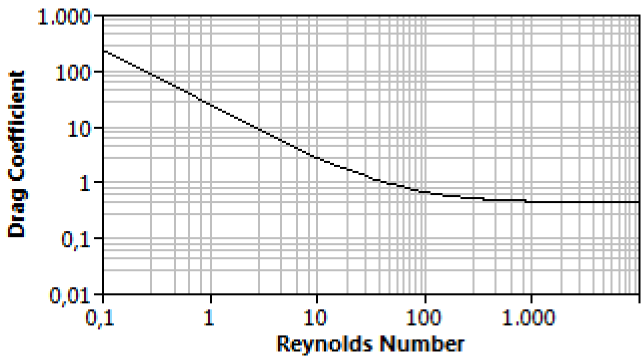

Fig. 3: Drag coefficient vs. Reynolds number, Eq. (21).

COVER LETTER

Dear Editor,

I would like to keep the same reasons I had expressed in my previous cover letter to justify my purpose of submission as being the one of a letter, but explaining, better, my reasons.

The number of pages of the paper is secondary, but I would like to point out this turns out to fit the requisite for a letter. Actually, except for eventual detailed calculation to be required under a refereeing process, the paper is short.

I had previously mentioned two main reasons: the result and the method, sustaining the importance of the paper for the field.

The result given by the Eq. (21) is new, original, fundamental, subtle, and its has got a final simple form. Expressions that turn out to arise in other works to fit the drag curve are frequently much more cumbersome, the expressions per se, these arising from fitting techniques, mostly not primarily concerned with the physics behind the phenomenon, just methematical. That ones that arise from first principles, being the Oseen method the canonical quadratic correction in the literature, turn out to be restricted to the region of low Reynolds numbers also avoiding the complications within the Navier-Stokes equation non linear characteristic, and there is not any simple expression arising from first physical principles, up to my knowledge, that fits the entire extent of Reynolds numbers up to the drag crisis. My derivation very seems to fill this void, giving ansatz to explore the phenomenon in other situations, as raised below.

In fact, the second term at the right-hand side of Eq. (21), the quadratic correction, also seems to be important to justify the letter, once this second order correction is different from the one obtained, e.g., by the Oseen method, and turns out to fit the entire range of Reynolds numbers up to the drag crisis. This term has got, consequently, a parameter, the separation angle, and it is not constant through this range. Its asymptotic behaviour is well known, and one knows that, experimentally, it turns out to exhibit fluctuations around the asymptotic value. This latter is taken as the rapidly reached laminar one one obtains via the Froessling method, and this is sufficient to give the entire behaviour over the mentioned range of Reynolds numbers. On this, one also has got an opportunity of research on the separation angle to plot the experimental curve via my Eq. (21) to obtain an exact fitting. This latter being an experimental scope, with which my paper is not primarily concerned, albeit the phenomenon raising the investigation is, of course, intrinsically under scope.

In relation to the method, it has got a very fundamental difference in relation to the route one canonically takes to calculate the drag force, which has got a profound motivation: the principle of equivalence, generally limited to the scope of general relativity. The method is related to the equivalence between an acceleration (the force we want to calculate divided by the sphere mass) and a field. The important point is the field at each point of a non inertial reference frame. This field may be factored out from the linear momenta transference integral, once it turns out to permeate the whole fluid. Even at terminal velocity, one has got the very constant gravitational field at the surface of the Earth in the adopted falling frame, but the equivalence principle states this frame turns out to be accelerated. One may think the sphere is under constant terminal velocity, but the gravitational field turns out to be its equivalent acceleration (with the minus sign, of course) which contains the force we want to calculate (divided by the sphere mass). Although this result may appear as being quite trivial, it is profound, since it provides the information, the quadratic correction we are interested. Such "triviality", up to my knowledge, was never published before. Hence the equivalent interpretation via field gives the opportunity to explore convenient control surfaces, instead of the very surface of the sphere (or a cilinder, or other shapes) one canonically adopts in an inertial frame to calculate the drag force. Again, this turns out to be subtle, but this property due to the equivalence principle emerges as a new method to explore convenient regions, which actually extracts information from an identity in which, asseverating again, the force we want to calculate may be factored from the linear momentum transference integral since this field turns out to equally permeate any point within the fluid, as explained in my paper. The gedankenexperiment may give a new technique to measure the drag force from internal weighing apparatus, instead of the use of an external dynamometer, keeping the exact shape of the immersed body under investigation as well as the exact field surrounding it. This result turns out to be a demonstration of the equivalence principle in action via a fluid dynamics phenomenon.

Hence, I respectfully insist through my previous cover letter, rewritten below:

Dear Editor,

For an eventual case of refereeing process:

This paper contains detailed derivations that are not included, since they follow from the principles discussed through the text. For me, some of them were not trivial ones, specially related to my previous investigations regarding the thermodynamics of stability and other instances of investigation related to the domain of validity of the superposition principle (low

Reynolds number), as well as the specific derivations related to the paper per se, viz., the high Reynolds number regime and the profile of turbulence within the boundary layer. Hence, under a request, I will be glad to send and explain them. Also, I was wondering a short paper, since I think the originally (up to my knowledge) obtained result and method seem to be the relevant results, instead of the derivation details, in a case of eventual acceptance and publication.

With my best respects,

Armando V.D.B. Assis.

ACKNOWLEDGMENTS

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[1] H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill Books, 1979), 2nd ed., english Edition.

[2] L. Landau and E. Lifshitz, *Course of Theoretical Physics, Vol.6.* (Pergamon Press, London-Paris-Frankfurt, 1959), 2nd ed., english Edition. 536 pp.

[3] K. S. George Constantinescu, Physics of Fluids **16**, 1449 (2004).

[4] The Oseen force will turn out to be within the quadratic contribution we obtained, viz., included within the Eq. (16). One may infer the Oseen force is a quadratic contribution.