Maxwell's equations derived from minimum assumptions

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Maxwell's equations, under disguise of electromagnetic fields occurred in empty space, describe dynamics of an elastic-plastic medium with point sources of the medium stress. The condition of incompressibility of the medium corresponds to the Coulomb gauge. Microscopically, the stress sources in the mechanical model of electromagnetism may appear to be the radially symmetrical point discontinuities. But this feature does not figure explicitly in the phenomenology. Because in the case of incompressible medium the dilatation term in the motion equation is replaced by the pressure. The stationary pressure field is concerned with the term of the external force entering the motion equation. The key point is how to express the external force via the density of stress sources. To this end we postulate that the increment of the external force density arising due to motion of the stress sources is proportional to the flux of the stress sources. Supplementing this postulate by the conservation of the total strength of the stress sources leads to the Coulomb pressure field.

In this report I describe in a concise style minimal requirements that an elastic medium should be subordinated in order that its dynamics be isomorphic to dynamics of the electromagnetic field. The assumptions needed and sufficient are: 1) the simplest form of the increment of the external force term in the Lame equation arising due to motion of the point source of this force in the medium, 2) the conservation of the total strength of the stress sources.

1. JELLY-LIKE MEDIUM WITH AN EXTERNAL FORCE

Dynamic equation for small displacements $\mathbf{s}(\mathbf{x},t)$ of the linear elastic medium is known to read

$$\partial_t^2\mathbf{s} = -c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s}) + c_g^2\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{s}) \tag{1}$$

where c is the speed of transverse and c_g of longitudinal waves. The terms in the right-hand side of (1) describe the force exerted on the volume element of the linear-elastic medium by the environment. In case of an incompressible still liable to shear deformations medium, i.e. the jelly, the dilatation term can be replaced by the pressure p:

$$\varsigma\partial_t^2\mathbf{s} = -\varsigma c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s}) - \boldsymbol{\nabla}p \tag{2}$$

where ς is the density of the medium, with the incompressibility condition

$$\boldsymbol{\nabla}\cdot\mathbf{s} = 0. \tag{3}$$

Taking the curl of (2):

$$\varsigma\boldsymbol{\nabla}\times\left[\partial_t^2\mathbf{s} + c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s})\right] = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}p) = 0. \tag{4}$$

Taking the divergence of $\partial_t^2\mathbf{s} + c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s})$ gives with the account of (3)

$$\boldsymbol{\nabla}\cdot\left[\partial_t^2\mathbf{s} + c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s})\right] = 0. \tag{5}$$

From what both the curl and divergence of a vector being nullified follows that the very vector equals to zero:

$$\partial_t^2\mathbf{s} + c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s}) = 0. \tag{6}$$

Thus in the elastic continuum we have only waves, and in the incompressible medium these waves are transverse. In this event, by (6) with (2), the background pressure remains unperturbed: $p = \mathsf{const}$.

We may formally introduce into motion equation (1) or (2) the volume density $\mathbf{f}(\mathbf{x},t)$ of an external force:

$$\varsigma\partial_t^2\mathbf{s} = -\varsigma c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{s}) - \boldsymbol{\nabla}p + \mathbf{f}. \tag{7}$$

An external force is regarded as the cause of the elastic deformation of the medium. It plays an important role in the elastic model of electromagnetism. Differentiating the motion equation (7) by the time we obtain

$$\varsigma\partial_t^2\mathbf{u} = -\varsigma c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u}) - \boldsymbol{\nabla}\partial_t p + \partial_t\mathbf{f} \tag{8}$$

where

$$\mathbf{u} = \partial_t\mathbf{s}. \tag{9}$$

Then from the divergence of (8) and curl of $\boldsymbol{\nabla}\partial_t p$ follows that $\partial_t\mathbf{f} = 0$ entails $\partial_t p = \mathsf{const}$ $(= 0)$. That is the stationary external force generates in the medium a stationary pressure field, irrespective of elastic waves occurred. Next, we will specify the unknown external force so that the set of equations obtained [1] will be isomorphic to Maxwell's equations.

2. THE GAS, OR PLASM, OF THE POINT STRESS SOURCES

The term of the external force in the dynamic equation (7) is connected with sources of stress of the medium [2]. The collection of the point stress sources can be treated as a gas featured by the volume density $\varrho(\mathbf{x}', t)$ and the velocity field $\mathbf{v}(\mathbf{x}', t)$. The evolution of the gas is described by the usual set of kinematic and dynamic equations appropriate for a mechanical continuum:

$$\partial_t \varrho = -\boldsymbol{\nabla}' \cdot (\varrho \mathbf{v}), \tag{10}$$

$$\varrho \frac{d\mathbf{v}}{dt} = \mathbf{F}. \tag{11}$$

Now we have mathematical frames for two interconnected mediums: the elastic substratum, governed by (3) and (7), and the gas of point sources of stress, the secondary medium, governed by (10) and (11). Both sets of equations include unknown force fields $\mathbf{f}(\mathbf{x}, t)$ and $\mathbf{F}(\mathbf{x}', t)$ which still should be connected with the parameters of the mediums.

Obviously, in statics the density $\mathbf{f}(\mathbf{x}, t)$ of the external force is concerned with the volume density $\varrho(\mathbf{x}', t)$ of the gas of stress sources. The dynamic equation for $\mathbf{f}(\mathbf{x}, t)$ must include also the velocity field $\mathbf{v}(\mathbf{x}', t)$.

In case if a stress source is capable to be split and distributed over the medium, the set of equations (10), (11) may describe the Madelung fluid: after specifying the force term $\mathbf{F}(\mathbf{x}', t)$, with some math manipulation it can be reduced to the form isomorphic to Shroedinger equation [3]. Macroscopically, the dynamic equation (11) can be approximated by the Newton mechanics with the force term depending on the velocity field $\mathbf{u}$ and pressure p of the primary medium and velocity $\mathbf{v}$ of the stress source: $\mathbf{F}(\varrho, \mathbf{u}, p, \mathbf{v})$.

Probably, the stress sources are related with the discontinuities of the medium, e.g. the singularity of the force field $\mathbf{f}$ can be underlain by the singularity of the dilatation $\boldsymbol{\nabla} \cdot \mathbf{s}$. However, the latter does not appear explicitly in the motion equation (7). Since in the incompressible medium the dilatation term is replaced by the pressure. (The singularity of $\boldsymbol{\nabla} \cdot \mathbf{s}$ is possibly concerned with the electric interaction. But this question is beyond the scope of this article.) So, we will further speak about sources of stress, singularities of the force field or may be defects but not discontinuities of the medium. Nevertheless, the motion and possibly splitting of the point stress sources in the medium evidence to that we deal with an elastic-plastic medium.

3. SUBSTRATUM FOR ELECTRODYNAMICS

The microscopic structure of the defect is currently not known to us. So, we will construct the phenomenology of its dynamics from minimum assumptions, and then compare it with electrodynamics.

Firstly, consider the point defect at $\mathbf{x}'$. Insofar as the substratum is isotropic we may expect that in statics the force field $\mathbf{f}(\mathbf{x} - \mathbf{x}')$ generated by the point defect will be radially symmetrical. The radial field is potential, so that its curl is zero. At divergence the solenoidal terms will be vanishing, and thus we have from (7) in statics

$$\boldsymbol{\nabla} p = \mathbf{f} \tag{12}$$

i.e. the pressure adjusts itself to spatial field $\mathbf{f}$ of the external force, while the solenoidal (transverse wave) part of (7) is independent on (12).

Let the source of stress move in aether, as it is appropriate for defects of an elastic-plastic medium. If for a small interval of time δt the stress source has passed a distance $\delta \mathbf{x}'$ when we may expect that the increment $\delta \mathbf{f}$ of $\mathbf{f}$ produced due to this movement will be directed along the vector $\delta \mathbf{x}' = \mathbf{v} \delta t$, where $\mathbf{v}(\mathbf{x}', t)$ is the instantaneous speed of motion of the stress source. The fundamental feature of the system is that the increment $\delta \mathbf{f}$ of the force mentioned possesses the properties of the point, or lumped force, we postulate

$$\delta \mathbf{f} = 4\pi a \mathbf{v} \delta t \delta(\mathbf{x} - \mathbf{x}') \tag{13}$$

where a is the strength of the force, and 4π is introduced for convenience in presenting the final result in the solution. Dividing (13) by δt gives the dynamic equation for the external force

$$\partial_t \mathbf{f} = 4\pi a \delta(\mathbf{x} - \mathbf{x}') \mathbf{v}. \tag{14}$$

Suppose the point defects are distributed in the medium with the normalized volume density $\varrho(\mathbf{x}', t)$, and the total strength a of the defects being conserved:

$$\int \varrho(\mathbf{x}', t) d\mathbf{x}' = 1. \tag{15}$$

The continuity equation (10) is actually the consequence of (15). Then the instantaneous force field generated by this distribution can be found multiplying equation (14) by $\varrho(\mathbf{x}', t)$ and integrating it over $\mathbf{x}'$ with the account of (15):

$$\partial_t \mathbf{f} = 4\pi a \varrho \mathbf{v} \tag{16}$$

where now all fields in the right-hand part of (16) depend on $\mathbf{x}$. The motion of defects represents a microscopic mechanism of plasticity of a solid elastic medium. Equation (16) corresponds to a convolution of the Prandtl-Reuss model of an elastic-ideal-plastic medium.

Continuity equation (10) can be rewritten from $\mathbf{x}', t$ to variables $\mathbf{x}, t$:

$$\partial_t \varrho = -\boldsymbol{\nabla} \cdot (\varrho \mathbf{v}). \tag{17}$$

In case if we know the velocity field $\mathbf{v}(\mathbf{x}', t)$ for the flow of the point defects, relations (16) and (17) close the set of equations (3), (7) describing the evolution of the incompressible elastic-plastic medium with point defects.

From the pair of equations, (16) and (17), we may deduce another equation, which enables us to obtain directly the force field generated by the static point defect. Using (16) in (17) gives

$$4\pi a \partial_t \varrho = -\partial_t \boldsymbol{\nabla} \cdot \mathbf{f}. \tag{18}$$

From (18) we may conclude that

$$\boldsymbol{\nabla} \cdot \mathbf{f} = -4\pi a \varrho. \tag{19}$$

Equations (3), (7), (19), (16) and (11) with yet unknown dependence $\mathbf{F}(\varrho, \mathbf{u}, p, \mathbf{v})$ gives the full dynamics of the incompressible elastic-plastic medium with point defects.

Taking in (19) $\varrho = \delta(\mathbf{x} - \mathbf{x}')$ we see that when the point source is at rest the field $\mathbf{f}(\mathbf{x})$ is indeed radially symmetrical. If the radial external force $\mathbf{f}$ generates in the medium a displacement field $\mathbf{s}$ (that does not appear in Maxwell's equations), the latter also must be a radial function. In statics the first term of (7) equals to zero, the second one vanishes as well since a radial field is potential. In the result (7) degenerates to (12). Hence, in the absence of elastic waves, the solenoidal displacement field arises only when the source of the external field moves in the medium. This may be a reason for transferring the deformation term from (7) to the equation (16) describing the evolution of the force field. To this end we redefine the term of the external force as

$$\boldsymbol{\epsilon} = \mathbf{f} - \varsigma c^2 \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{s}). \tag{20}$$

Using (20) in (7), (16) and (19), and taking into account (9), we may obtain the set of equations in terms of the velocity field $\mathbf{u}(\mathbf{x}, t)$:

$$\begin{aligned} \varsigma \partial_t \mathbf{u} &= \boldsymbol{\epsilon} - \boldsymbol{\nabla} p, & (21)\\ \partial_t \boldsymbol{\epsilon} &= 4\pi a \varrho \mathbf{v} - \varsigma c^2 \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}), & (22)\\ \boldsymbol{\nabla} \cdot \boldsymbol{\epsilon} &= -4\pi a \varrho. & (23) \end{aligned}$$

4. MAXWELL'S EQUATIONS

Vector and scalar fields $\mathbf{A}$, φ and $\mathbf{E}$ can be defined:

$$\begin{aligned} \mathbf{A} &= \kappa c \mathbf{u}, & (24)\\ \varsigma \varphi &= \kappa p, & (25)\\ \varsigma \mathbf{E} &= -\kappa \boldsymbol{\epsilon}. & (26) \end{aligned}$$

Submitting definitions (24), (25) and (26) equations (21), (22) and (23) acquire the electromagnetic form

$$\begin{aligned} \partial_t \mathbf{A}/c + \mathbf{E} + \boldsymbol{\nabla} \varphi &= 0, & (27)\\ \partial_t \mathbf{E} - c \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) + 4\pi \rho \mathbf{v} &= 0, & (28)\\ \boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi \rho & (29) \end{aligned}$$

where $\rho = \kappa a \varrho / \varsigma$ and the electric charge is defined via the strength a of the external force source as

$$q = \kappa a / \varsigma. \tag{30}$$

Thus, the whole set of Maxwell's equations is reproduced, with the Coulomb gauge

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{31}$$

obtained from the incompressibility condition (3) using in it the definitions of the vector potential (24) and velocity field (9) of the linear elastic medium.

5. INTERNALIZATION OF THE EXTERNAL FORCE

In order that the above described elastic-plastic model of electrodynamics be plausible, we must find the internal mechanism for supporting the "external force" field in the medium. Consider the ideal fluid. The motion of the ideal fluid is governed by the Euler equation

$$\varsigma\partial_t u_i + \varsigma u_k \partial_k u_i = -\partial_i p \tag{32}$$

where summation over recurrent index is implied throughout. Rendering (32) in the Gromeka-Lamb form

$$\varsigma\partial_t \mathbf{u} + \varsigma\boldsymbol{\nabla} u^2/2 - \varsigma\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}) + \boldsymbol{\nabla} p = 0 \tag{33}$$

we see that at $\partial_t\mathbf{u} = 0$ there holds along the flow line the Bernoulli equation

$$\varsigma u^2/2 + p = \mathsf{const.} \tag{34}$$

Relation (34) secures stationary $\mathbf{u}$ and p fields.

In the turbulent fluid $\mathbf{u}$ and p can be treated as random quantities. Averaging (32):

$$\varsigma\partial_t \langle u_i\rangle = -\varsigma\langle u_k\partial_k u_i\rangle - \partial_i\langle p\rangle\,. \tag{35}$$

We see that even if $\langle\mathbf{u}\rangle = 0$ the convection term in (35) can be nonzero. This means the occurring in the turbulent fluid of the volume force

$$\epsilon_i = -\varsigma\langle u_k\partial_k u_i\rangle = -\varsigma\partial_k\langle u_i u_k\rangle \tag{36}$$

which does not vanish in the stationarity. The force (36) is concerned with a non-uniform distribution of the turbulent energy. Equation (23) suggests that the turbulent force $\boldsymbol{\epsilon}$ is supported if only there are sources of stress in the medium. Microscopically this may be inclusions of voids, with the boundary condition $\langle p\rangle = 0$ on the wall of the cavity. Because of the cross-correlations of turbulent fluctuations the turbulent fluid exhibits the instantaneous shear elasticity.

6. DISCUSSION

Summarizing, the following main features of the mechanical medium underlying the classical electrodynamics should be accentuated.

In statics there are no shear stresses in the incompressible elastic-plastic medium modeling the electromagnetic substratum. The shear stresses appear only when the point defect moves in the medium. The instantaneous increment $\delta\mathbf{s}$ of the displacement due to the motion obeys the Lame equation (7) with the external force (13):

$$\varsigma\partial_t^2\delta\mathbf{s} + \varsigma c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\delta\mathbf{s}) + \boldsymbol{\nabla}\delta p = a\mathbf{v}\delta t\delta(\mathbf{x}-\mathbf{x}') \tag{37}$$

where δp is the increment of the pressure due to (13). This is the reason why the stationary magnetic field is generated by the (uniformly) moving electrical charges only. So, the form (21)-(22) is indeed more adequate than (7). The singular term in the right-hand side of the equation (37) does not mean for this derivation. We may state instead that the increment of the "external" force is proportional to the flux $a\rho\mathbf{v}$ of the point stress sources.

The pressure center modeling the electrical charge differs drastically from the dilatation center occurring in the simple elastic medium. Nearby the dilatation center of the simple elastic medium the pressure is known to change with the distance as $\sim 1/r^3$, which is due to shear strain, while near the pressure center in question as $\sim 1/r$. This becomes possible because the electromagnetic substratum combines in itself the features of the liquid and solid mediums, i.e. it is an elastic medium with relaxation of shear stresses.

Concluding, we have the following structure of the problem. The kinematic equation for the primary medium:

$$\partial_t\varsigma + \boldsymbol{\nabla}\cdot(\varsigma\mathbf{u}) = 0 \tag{38}$$

which for the incompressible medium

$$\varsigma = \mathsf{const} \tag{39}$$

reduces to

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{40}$$

The dynamic equation for the primary medium:

$$\varsigma\partial_t\mathbf{u} + \boldsymbol{\nabla}p = \boldsymbol{\epsilon} \tag{41}$$

which is really the equation for the ideal fluid with the turbulence force $\boldsymbol{\epsilon}$. The equation for evolution of the "external" force $\boldsymbol{\epsilon}$:

$$\partial_t\boldsymbol{\epsilon} + \varsigma c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u}) = 4\pi a\varrho\mathbf{v} \tag{42}$$

which includes the flux $\varrho\mathbf{v}$ of the secondary medium and the term of the instantaneous shear elasticity. The kinematic equation for the secondary medium:

$$\partial_t\varrho + \boldsymbol{\nabla}\cdot(\varrho\mathbf{u}) = 0 \tag{43}$$

and the dynamic equation for the secondary medium:

$$\varrho\frac{d\mathbf{v}}{dt} = \mathbf{F} \tag{44}$$

with yet an unknown force $\mathbf{F}(\varrho, \mathbf{u}, p, \mathbf{v})$ which is concerned with the interaction of the stress sources.

We do not specify here the origin of the electromagnetic interaction since it probably involves the term of the medium discontinuity which does not explicitly enter Maxwell's equations. The Reynolds turbulence in the ideal fluid underlying the plasticity of the electromagnetic substratum [4] is also not described in details here. The only point that has been exposed is that the elasticity and the term of the external force $\boldsymbol{\epsilon}$ are concerned with the (correlated) turbulent fluctuations, i.e. with a purely kinetic factor.

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