Non-Standard Numbers and TGD

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Abstract

The chapter represents a comparison of ultrapower fields (loosely surreals, hyper-reals, long line) and number fields generated by infinite primes having a physical interpretation in Topological Geometrodynamics. Ultrapower fields are discussed in very physicist friendly manner in the articles of Elemer Rosinger and these articles are taken as a convenient starting point. The physical interpretations and principles proposed by Rosinger are considered against the background provided by TGD. The construction of ultrapower fields is associated with physics using the close analogies with gauge theories, gauge invariance, and with the singularities of classical fields. Non-standard numbers are compared with the numbers generated by infinite primes and it is found that the construction of infinite primes, integers, and rationals has a close similarity with construction of the generalized scalars. The construction replaces at the lowest level the index set $\Lambda = \mathbb{N}$ of natural numbers with algebraic numbers $\mathbb{A}$, Frechet filter of $\mathbb{N}$ with that of $\mathbb{A}$, and $\mathbb{R}$ with unit circle S^1 represented as complex numbers of unit magnitude. At higher levels of the hierarchy generalized -possibly infinite and infinitesimal- algebraic numbers emerge. This correspondence maps a given set in the dual of Frechet filter of $\mathbb{A}$ to a phase factor characterizing infinite rational algebraically so that correspondence is like representation of algebra. The basic difference between two approaches to infinite numbers is that the counterpart of infinitesimals is infinitude of real units with complex number theoretic anatomy: one might loosely say that these real units are exponentials of infinitesimals.

1 Introduction

This chapter represents some comments on articles of Elemer E. Rosinger as a physicist from the point of view of Topological Geometrodynamics. To a large extent a comparison of two possible generalizations of reals is in question: the surreal numbers introduced originally by Robinson [2] and infinite primes and corresponding generalization of reals inspired by TGD approach [8] , [3] . The articles which have inspired the comments below are following:

- How Far Should the Principle of Relativity Go?
- Quantum Foundations: Is Probability Ontological?
- Group Invariant Entanglements in Generalized Tensor Products
- Heisenberg Uncertainty in Reduced Power Algebras
- Surprising Properties of Non-Archimedean Field Extensions of the Real Numbers
- No-Cloning in Reduced Power Algebras

I have a rather rudimentary knowledge about non-standard numbers and my comments are very subjective and TGD centered. I however hope that they might tell also something about Rosinger's work [3, 4, 5, 6, 7] . My interpretation of the message of articles relies on associations with my own physics inspired ideas related to the notion of number. I divide the articles to physics related and purely mathematical ones. About the latter aspects I am not able to say much.

The construction of ultrapower fields (generalized scalars) is explained using concepts familar to physicist using the close analogies with gauge theories, gauge invariance, and with the singularities of classical fields. Some questions related to the physical applications of non-standard numbers are discussed including interpretational problems and the problems related to the notion of definite integral. The non-Archimedean character of generalized scalars is discussed and compared with that of p-adic numbers. Rosinger considers several physical ideas inspired by ultrapower fields including the generalization of general covariance to include the independence of the formulation of physics on the choice of generalized scalars, the question whether generalized scalars might allow to understand the infinities of quantum field theories, and the question whether the notion of measurement precision could realized in terms of scale hierarchy with levels related by infinite scalings. These ideas are commented in the article by comparison to p-adic variants of these ideas.

Non-standard numbers are compared with the numbers generated by infinite primes. It is found that the construction of infinite primes, integers, and rationals has a close similarity with construction of the generalized scalars. The construction replaces at the lowest level the index set $\Lambda = \mathbb{N}$ of natural numbers with algebraic numbers $\mathbb{A}$, Frechet filter of $\mathbb{N}$ with that of $\mathbb{A}$, and $\mathbb{R}$ with unit circle S^1

represented as complex numbers of unit magnitude. At higher levels of the hierarchy generalized -possibly infinite and infinitesimal- algebraic numbers emerge. This correspondence maps a given set in the dual of Frechet filter of $\mathbb{A}$ to a phase factor characterizing infinite rational algebraically so that correspondence is like representation of algebra. The basic difference between two approaches to infinite numbers is that the counterpart of infinitesimals is infinitude of real units with complex number theoretic anatomy: one might loosely say that these real units are exponentials of infinitesimals.

With motivations coming from quantum computation, Rosinger discusses also a possible generalization of the notion of entanglement [5] allowing to define it also for what could be regarded as classical systems. Entanglement is also number theoretically very interesting notion. For instance, for infinite primes and integers the notion of number theoretical entanglement emerges and relates to the physical interpretation of infinite primes as many particles states of second quantized super-symmetry arithmetic QFT. What is intriguing that the algebraic extension of rationals induces de-entanglement. The de-entanglement corresponds directly to the replacement of a polynomial with rational coefficients with a product of the monomials with algebraic roots in general.

2 Brief summary of basic concepts from the points of view of physics

Many of Rosinger's ideas relate to generalized scalars as he calls the number fields and division algebras obtained as reduced power algebras. Generalized scalars include as a special case non-standard numbers. The definition is comprehensible also for a physicist since heavy technicalities are avoided. The conceptual problems are mentioned in passing. For instance, the question whether the transfer principle stating that all that can be expressed using first order logics for reals should have similar expression for nonstandard numbers is central question.

Non-standard numbers (at least generalized scalars, hyperreals, surreals, and long line are alternative nicknames for them) probably induce feelings of awe and fear in physicist. The construction used is however structurally very familiar to a physicist who has understood the notion of gauge invariance. The correspondences are following.

- Gauge transformations and gauge potentials defined in space-time $\leftrightarrow$ real valued functions defined in a discrete set such as natural numbers.
- Gauge potentials which differ by a mere gauge transformation are physically equivalent $\leftrightarrow$ functions which are same in the set of subsets of Λ called filter are identified in quotient construction.
- Fields vanishing in a complement of lower-dimensional manifolds is physically equivalent with everywhere vanishing field $\leftrightarrow$ function vanishing in the complement of finite set is equivalent to vanishing everywhere. Filter itself can correspond to complements of lower-dimensional manifolds in the physical situation.
- Functions vanishing for set the filter and equivalent with zero element of the resulting algebra $\leftrightarrow$ gauge potentials, which are pure gauge correspond to vanishing gauge fields except in some lower-dimensional sub-manifold. Vacuum extremals would be TGD counterpart for these regions.

A more precise construction recipe [6] should be easy to understand on basis of these correspondence rules.

1. One considers real valued functions in a discrete set Λ - typically natural numbers N. For everywhere non-vanishing functions f the local algebraic inverse $1/f$ is well-defined but if the function has zeros $1/f$ is infinite at zeros. This need not be regarded as a problem if the set of zeros is finite. This motivates the construction of fields or division algebras by mapping to zero those functions which are non-vanishing at finite number of points only. Field or division algebra would be obtained as the quotient space of the function algebra with respect to ideal defined by functions which are non-vanishing in a set whose complement is finite.
2. The notion of filter defined as a set of subsets of Λ, which are equivalent with Λ itself "for practical purposes" is essential for the construction (see the appendix of [6]).

- The sets of filter $\mathcal{F}$ are ordered by inclusion; if set belongs to $\mathcal{F}$ also sets containing it belong to $\mathcal{F}$; the intersections and unions of subsets of $\mathcal{F}$ belong to $\mathcal{F}$; empty set does not belong to $\mathcal{F}$.
- Ultrafilter $\mathcal{U}$ has the additional property that for any subset of Λ either the subset or its complement belongs to $\mathcal{U}$. Frechet filter $\mathcal{U}_F$ consisting of sets whose complements are finite sets defines ultrafilter and any ultrafilter by definition contains $\mathcal{U}_F$. The existence of the ultrafilters is guaranteed by the axiom of choice (which could be challenged in physics: for instance, one could argue that only rational points can be pinpointed by a physical procedure).

3. One assigns to the filter $\mathcal{F}$ an ideal $\mathcal{I}_{\mathcal{F}}$ of function algebra $\mathbb{F}$ as the set of functions $f : \Lambda \rightarrow \mathbb{R}$ which vanish for some set of filter and thus "almost everywhere". Reduced power algebras are defined as quotients $\mathbb{F}/\mathcal{I}_{\mathcal{F}}$ of the function algebra $\mathbb{F}$ with respect to $\mathcal{I}_{\mathcal{F}}$ which means that two functions are equivalent if they coincide in some set of the filter. Functions vanishing in some set of $\mathcal{F}$ correspond to zero.

   Ultrafilters give ultrapower fields and Frechet filter consisting of sets with finite complement defines one particular ultrafilter. Functions equivalent with zero vanish in some set with a finite complement which indeed is rather natural. The algebraic inverses of functions vanishing in a finite subset of Λ have a finite number of infinite values and define infinitely large generalized scalars. Reals can be imbedded to these algebras as constant functions and one can order the elements of the resulting number field and define the analog of real line by using natural definition.

   One can order the elements of ultrapower: $f \leq g$ iff $g(\lambda) \leq g(\lambda)$ in some set of $\mathcal{U}$ and thus "almost everywhere". This allows to classify the elements of field infinitesimals, finite numbers, and infinite numbers. One has infinite number of infinitesimals identified as functions whose values are in the range $(-r, r)$ for any $r > 0$ in some set of $\mathcal{U}$. These functions do not vanish in any set of $\mathcal{U}$ as physicist might first think: only the infimum of $|f|$ over sets of $\mathcal{U}$ vanishes Infinite numbers correspond to algebraic inverses of functions having finite number of zeros.

4. The resulting generalized scalars are much more structured than reals and have complex self-similar structure. The notion of walkable world illustrates these properties. Non-Archimedean number fields can be defined as fields for which the numbers $x + nv$ for given x and v and arbitrary n have element y such $x + nv < y$ for all values of n. v defines the step of the walk and n the number of steps. The shifts of x generate "walkable worlds" reached by making arbitrary number of unit steps and they do not span the entire number field in non-Archimedean case. One can say that y is infinite relative to x.

   Already in the case of p-adic numbers [9] , [2] walkable worlds define only subsets of p-adic numbers: the reason is that the p-adic norm of $x + nv$, n p-adic integer cannot be larger than the norm of x is larger than one or one. Hence one cannot walk out from the ball defined by the numbers x with norm smaller than p^k. Now y has finite p-adic norm whereas for generalized scalars y would have infinite real norm.

   One interesting implication is that p-adic variants of translations as continuous transformations are well-defined inside p-adically finite ball so that plane waves representing eigenstates of translations can be restricted to a finite p-adic volume. Already in p-adic case the walkable worlds define a fractal structure with many basic properties possessed also by surreal walkable worlds. It is however clear that infinitesimals and infinite numbers are not realized in the p-adic context.

One can turn around the analogy with gauge theories and ask whether the notion of filter defined as the set of complements for lower-dimensional manifolds of space-time could be useful. In this case fields vanishing in open sets of space-time would be equivalent with vanishing fields and fields singular in lower-dimensional sub-manifold would be analogous to infinite numbers. If the infimum of field in the set of filters vanishes it would be analogous to infinitesimal. The singularities could be associated with Higgs fields and gauge fields. Interestingly, in quantum physics inspired theories for knots, knot-cobordisms and 2-knots essential role is played by 2-dimensional singularities of gauge fields in 4-D space-time [4] and having physical interpretation as analogs of string world sheets.

3 Could the generalized scalars be useful in physics?

The basic question is whether the generalized scalars could replace reals in theoretical physics. It is best to proceed by making questions.

3.1 Are reals somehow special and where to stop?

The following questions relate to the interpretation of generalized scalars.

1. Why reals should be so special? The possible answer is that reals, complex numbers and quaternions form associative continua. Classical number fields are indeed in central role in TGD [10] , [1] . Already p-adic number fields consist of disconnected pieces in the sense that one cannot connect two arbitrary points by a continuous curve (p-adic norm of point must change discontinuously at some point of curve is the norms of end points are different).

2. What -if anything physical- it means to replace temperature at space-time point with a function of a natural number? Doesn't this mean the replacement of real numbers with $R \times N$ and replacement of Minkowski space with $M^4 \times N^4$?

3. What is the physical meaning of generalized scalar understood as an equivalence class of real functions of natural number modulo functions vanishing in some set belonging to a filter (possibly ultrafilter)? What could be the physical meaning of filter? Could the quotient construction be interpreted as some sort of gauge invariance or could it just realize the idea "almost-everywhere is everywhere physically"?

4. Can one stop if the step replacing reals with generalized scalars is taken? Recall that quantization means replacement of the configuration space with the function space associated with it. Second quantization brings in function space associated with this space and so on. This hierarchy of quantizations is involved with the construction of infinite primes (and rationals) in TGD framework [8] , [3] and in this case one has a concrete physical interpretation in terms of many-sheeted space-time.

   Should one replace natural numbers with the power set of natural numbers consisting of finite subsets of natural numbers (dual of the Frechet filter for $\mathbb{N}$) at the next step and perform similar construction. This could be continued ad infinitum. Does one obtain an infinite hierarchy of increasingly surreal numbers in this manner? One can imagine also other kinds of constructions but it is this construction with would be analogous to that for the hierarchy of infinite primes.

3.2 Can one generalize calculus?

The obvious question of physicist is whether one can generalize differential and integral calculus - necessary for physics as we know it. Surreals were actually introduced to justify the notion of infinitesimal so that differential calculus should not be a problem. The notion of integral function is neither a problem but definite integral might be due to the loss of Archimedean property. One could try to define the notion of integral in terms of the imbedding of real numbers as constant functions and define definite integral algebraically as a substitution of the integral function between real limits. For arbitrarily limits one cannot order the limits and it seems that one should restrict the considerations to real limits.

What might also pose a problem is the definition of numerical integration - in terms of Riemann sum in its simplest form. One should divide the integration range to short ordered pieces and approximate the integral with sum. But there exists infinite number of paths connecting two functions to each other and one cannot order the pieces in general. Should one generalize complex analyticity so that functions of surreals would be expressible as power series of function and the integrals would not depend on integration path unless the surreal analytic function has singularities such as poles? Does this mean that one can choose one particular path which corresponds a path restricted to real axis so that the integral would reduce to the ordinary real integral.

In p-adic context non-Archimedean property implies that the notion of definite integral is indeed problematic [7] . The basic problem is that one cannot in general tell which one of the two p-adic numbers with the same norm is the larger one and therefore one cannot define the notion boundary essential in variational calculus. One could use algebraic definition of definite integral as a substitution of integral function and in complex case residue calculus could help. One could use the ordering of rational numbers imbedded to p-adic numbers fields to induce the ordering of p-adic rationals. The p-adic existence of the integral function poses additional conditions encountered already for the integrals of rational functions which can give logarithms of rationals leading out from the realm of rationals. These difficulties have served as a key guiding principle in the attempts to fuse real and p-adic physics to a larger structure.

3.3 Generalizing general covariance

What happens to the notion general covariance (or Principle of Relativity in the terminology used by Rosinger, see the article How Far Should the Principle of Relativity Go? [3])? Here I would like to do some nitpicking by distinguishing between Principle of Relativity which refers to the isometries of Minkowski space and General Coordinate Invariance analogous to gauge symmetry. Various symmetry groups make sense also in the surreal context since they are defined algebraically. A generalization of General Coordinate Invariance meaning that the formulation of physics becomes independent of the choice of generalized scalars is proposed by Rosinger. This notion could be interpreted as a form invariance or as the condition that the physics is indeed the same irrespective of what number field is used in which case the introduction of generalize scalars would not bring in anything new.

Rosinger chooses the non-trivial option which means that the formulation of the laws of physics should make sense irrespective of the number field chosen and considers various examples as applications of the generalized view. He shows that no-cloning theorem of quantum computation holds true also for generalized scalars because the theorem depends on the linearity of quantum theory alone (cloning would map state to two of its copies, something essentially nonlinear).

In TGD framework the notion Number Theoretical Universality interpreted as number field independent formulation of physics seems to relate closely to this principle.

1. All constructions making sense in real context should makes sense also in the p-adic context [9] , [2] . Real and p-adic physics meet in the intersection of real and p-adic worlds and result from each other by a kind of algebraic continuation. Simplifying somewhat, at the level of space-time surfaces the intersection would correspond to rational points in some preferred coordinates shared by real and p-adic surfaces and at the level of "world of classical worlds" (WCW) to surfaces expressible in terms of rational functions expressible using polynomials with rational coefficients so that real and p-adic variants of this kind of surfaces are can be identified.

2. Number Theoretic Universality leads to extremely powerful conditions on the geometry of WCW since both its real and p-adic sectors should exist and integrate to a larger structure [3] . Rationals defining the intersection of reals and various p-adics play a key role and one ends up with a generalization of number concept obtained by gluing reals and p-adics as well as their algebraic extensions to single book like structure [9] , [2] .

3. One is also forced to adopt a more refined view about General Coordinate Invariance since the coordinate transformations must respect the algebraic extensions of p-adic numbers used. This brings also non-uniqueness: there are several choices of coordinate frames not transformable to each other. The interpretation would be that that they serve as correlates of cognition. Mathematician is not an outsider and the choice of coordinate system affects the reality albeit in very delicate manner.

This allows to see a relationship between TGD inspired fusion of real and p-adic physics and Rosingers's proposal as roughly following correspondence.

Reals and p-adic number fields resp. rationals defining the intersection of reals and p-adic worlds $\leftrightarrow$ various generalized scalars resp. reals defining the intersection of various surreals worlds.

The independence on the choice of generalized scalars might give powerful constraints on the formulation of the theory.

If surreal number fields are important for theoretical physics, physical systems must be characterized by the generalized scalars. What determines this number field or algebra? Can one speak about some kind of quantal evolution in which physical systems evolve more and more complex number theoretically. Could the field of generalized scalars be replaced with a new one in quantum jump taking place via reals common to different generalized scalars?

The attempt to fuse real physics as physics of matter and p-adic physics as physics of cognition one ends up with this kind of picture and one can say that the prime characterizing p-adic number field and the algebraic numbers defining its extension (say roots of unity) characterize its evolutionary level. During evolution the algebraic complexity of the systems steadily increases.

3.4 The notion of precision and generalized scalars

Rosinger proposes [6] that the notion of precision of experiment could be assigned to the self-similar structure of the generalized scalars meaning a hierarchy of scales which differ from each other by infinite scale factors if real norm is used as a measure for the scale. There would be infinite hierarchy of precisions and what looks infinitesimal, finite, or infinite would depend on the precision used and characterized by what generalized scalars are used. Thus one can speak about relative precision.

That one could have units of (say length) differing by infinite scaling in real sense looks rather weird idea. In TGD framework one interpretation for the hierarchy of infinite primes would be that there is infinite hierarchy of variants of Minkowski space such that at the given level of the hierarchy lower levels represent infinitesimals. This would mean fractal cosmology in which the conscious entities above us in the hierarchy would be literally God like as compared to us. No hopes about testing this at LHC!

In p-adic context similar notion emerges but the infinities at different levels are not related by infinite scalings with respect to the p-adic measure for size. Given walkable world correspond in p-adic context to p-adic numbers with fixed norm and in this operational sense p-adic primes with larger norm are infinite. p-Adic prime p indeed characterizes length scale resolution and the roots of unitary used in algebraic extension of p-adics characterize the angle resolution.

Even more, if one accepts that p-adic space-time surfaces serve as correlates for cognition one is forced to conclude that cognition cannot be localized in a finite space-time volume and that "thought bubbles" have actually the size of the entire Universe. Only cognitive representations defined by rational intersections of real and p-adic space-time surfaces would be localized to a finite real volume. Maybe the infinite hierarchy of Rosinger could be assigned to the levels of existence that we are used to assign with cognition and matter corresponds to the lowest level.

3.5 Further questions about physical interpretation

Rosinger raises further interesting questions about physical interpretation.

1. In the article *Does Heisenberg Uncertainty Principle make sense in reduced power algebras?* [6] Rosenberg shows that the answer to the question of the title is affirmative. Rosinger asks in the same article whether the values of fundamental constants like c and $\hbar$ depend on the choice of generalized scalars. For instance, could $\hbar$ be infinitesimal for some generalized scalars? Could c have a well-defined infinite value for some generalized scalars.

   In the case of c one could argue that it is just a conversion factor so that one can put $c = 1$ always by a suitable choice of units. Most physicists would argue that the same is true for $\hbar$. I have however proposed a different vision explaining some strange findings in both astrophysics and biology.

2. Could the fact that infinitesimal and infinite numbers have precise meaning for generalized scalars allow to resolve the problems caused by the infinities of local quantum field theories? Rosinger argues that this might be the case [6] . The notion of infinity is relative one for generalized scalars and one could replace reals with some other generalized scalars and this could make infinite finite. As a matter fact, in p-adic context for a given p-adic number all p-adic numbers with larger norm represent an operational infinity in the sense that they cannot be reached by walks consisting of integer valued steps. As p-adic numbers they are however finite. It seems that one must be very careful how one defines the infinite: does one use norm or does on use reachability by integer valued steps as the criterion.

   One can counter argue that reals can be distinguished uniquely by their topological properties just like rationals can be distinguished by their number theoretic properties uniquely. Skeptic might say that the situation would become even worse since one would had infinite number of different kind of infinities. The infinities would be completely well-defined functions with finite number of poles but what it means to replaces temperature at space-time point with a function of natural number? Doesn't this mean that space-time point is replaced with natural numbers.

I have myself considered the possibility that p-adic mathematics for which integers infinite in real sense can make sense p-adically and have norm not larger than unity could allow to resolve the problem of infinities. In particular ultrametric topology implies that the sum of n numbers is never larger than the maximum of the largest number involved -this is just what walkable universe expresses- raises optimism. It turned however that these ideas did not work in my hands.

4 How generalized scalars and infinite primes relate?

The comparison of Rosinger's ideas with the number theoretic ideas of TGD inspires further questions.

1. Classical number fields play a key role in the formulation of quantum TGD. Do the notions of sur-complex, sur-quaternion and and sur-octonion make sense as one might expect?
2. What happens if one replaces real functions define in Λ (say natural numbers) with p-adic valued functions. One obtains algebra also now and one can define ideals and use quotient construction using ultrafilter. Does the notion of sur-p-adic make sense?
3. In TGD framework one ends up with the notion of infinite prime having direct connection with repeated second quantization of super-symmetric arithmetic quantum field theory with fermions and bosons labelled by primes- finite primes at the lowest level of hierarchy. This notion of infinity is essentially number theoretical and implies that the number theoretic anatomy of numbers and space-time points becomes an essential aspect of physics. Can one assign number theoretic anatomy also to non-standard numbers or does the real topology wipe it out?
4. How does the hierarchy of infinite primes relate to the possibly existing hierarchy of reals, surreals, sursurreals,... obtained by replacing real number valued function with surreal number valued functions replaced in turn with ....?

The last question deserves a more detailed consideration since it could provide an improved understanding of infinite primes. Consider first the construction of infinite primes [8] , [3] .

1. Infinite primes at the lowest level of hierarchy can be generated from two fermionic vacuum states $P_{\pm} = X \pm 1$, where X is defined as a product of all finite primes having p-adic norm less than one for all finite primes p. X is analogous to Dirac sea with all negative energy states filled. Simple infinite primes are of form $mX/n + rn$, where m and n have no common divisors and r consists of same primes as n. $m = \prod p_i^{k_i}$ corresponds to many boson state with k_i bosons with "momentum" p_i. In fermionic sector the square free integer n has interpretation as many-fermion state with single fermion in the modes involved. r corresponds to many-boson states in these modes. Simple infinite primes are clearly analogous to many particle states obtained by kicking fermions from sea to get positive energy holes and adding bosons whose number is arbitrary in a given mode labelled by finite prime. Simple infinite primes have unit p-adic norm so that "infinite" is a relative notion.
2. More complex infinite primes are infinite integers obtained as sums of products of infinite primes. The interpretation is in terms of bound many-particle states.
3. In zero energy ontology (ZEO) an attractive interpretation for infinite rationals is as zero energy states with numerator and denominator representing positive and negative energy parts of the state.
4. One can continue the construction indefinitely. At the next level X is replaced with the product of all infinite primes at the first level of the hierarchy and the process is repeated. The physical interpretation would be that at the next level many particle states of previous level take the role of single particle states and one constructs free and bound many particle states of these.

The many-sheeted space-time of TGD suggests a concrete realization of this process and I have indeed proposed a concrete physical interpretation of standard model quantum numbers in terms of what I call (hyper-)octonionic primes, which would generate a structure analogous to infinite primes.

Generalized scalars define a function algebra and this inspires the question is whether one could somehow assign a function algebra also to infinite primes and in this manner to see what is common features these very different looking notions might have. Infinite primes can be indeed mapped to polynomial primes as the following argument shows.

1. Simple infinite primes are characterized by two integers which have no common divisors and can be thus mapped in a natural manner to rationals $q = rn^2/m$. They can can be also mapped to monomials $x - q$, $q = rn^2/m$, where X could be seen as a particular value of x. Complex infinite primes constructed as products of simple infinite primes can be mapped to products of these monomials and sums of their products to sums of these so that on obtains a mapping to polynomial primes at the lowest level of the hierarchy. Vacua are mapped to rationals 1 and -1. One can decompose the polynomials to products of monomials $x - r$, where r is a finite algebraic number, and the interpretation would be that one considers primes in an algebraic extension of rationals and this representation applies to infinite prime when x is substituted with X.

2. This mapping makes sense also at the next level of hierarchy at least formally. Call the product of finite and infinite primes at the first level X_1 and corresponding formal variable x_1. Infinite rationals correspond now to rational functions of x_1 and x defined as ratios of polynomials $P_k(x_1, x)$ for which the highest power of x_1 is by definition x_1^k. The roots in the product representation of polynomials are obtained by the substitution $x \to X$ in the expressions of the roots as functions of x. The roots are generalized algebraic numbers which can be infinite or vanish as real numbers. This kind of mapping makes also sense at the higher levels of hierarchy. The roots of polynomial at the n:th level of the hierarchy are obtained by substituting to their expressions as algebraic functions $x_m = X_m$, $m < n$.

3. What one obtains is a map to polynomials so that one can indeed map infinite primes and also integers and rationals to a function algebra consisting of polynomials. Ideals correspond now to polynomial ideals consisting of polynomials proportional to some polynomial prime. There are no divisors of zero so that quotient construction is not needed now.

This construction leads to intriguing observations relating the construction of infinite primes to the construction of generalized scalars and suggesting that infinite primes represent a generalization of the concept of sur-complex numbers by identifying ultrafilter in terms of complements of finite subsets of algebraic numbers (Frechet filter actually). The heuristic argument goes as follows.

1. The hierarchy of subsets of algebraic numbers defined by the infinite primes at the lowest level of hierarchy defines *complement of Frechet filter* $\mathcal{CF}$ with the following defining properties. $\mathcal{CF}$ contains empty set and all finite subsets of Λ, unions of sets of $\mathcal{CF}$ belong to $\mathcal{CF}$, and subsets of a set belonging to $\mathcal{CF}$ belong to $\mathcal{CF}$.

   Note that powers of infinite primes define the same set in $\mathcal{CF}$ as infinite prime itself so that the correspondence does not seem to be many-to-one. It is not clear whether fermionic statistics could be used as a physical excuse to exclude these powers and more generally products of infinite primes for which same finite prime appears in more than one different infinite primes. Also subsets of genuinely algebraic numbers could correspond to several infinite integers and rationals.

   If one restricts the consideration to square free integers defined by the fermionic parts of infinite primes then the sets of natural numbers assignable to infinite primes correspond to finite subsets of square free natural numbers defining a Frechet filter for them.

2. $\Lambda = \mathbb{N}$ is replaced with algebraic numbers $\mathbb{A}$ so that the function space defining generalized scalars would consist of functions $f : \mathbb{A} \to \mathbb{C}$. It is not however clear what kind of functions one should consider.

(a) The first guess is that the quantum states of supersymmetric arithmetic QFT (SAQFT) correspond to functions non-vanishing only in some finite set belonging to $\mathcal{CF}$. They would map to zero in the quotient construction of ultrapower field. The functions which do not map to zero would correspond to non-vanishing elements of the ultrapower field and would have no physical interpretation. This does not sound sensible physically.

(b) The many-particle states of arithmetic QFT could more naturally correspond to functions having values on circle S^1 -rather than $\mathbb{C}$- identified as complex numbers with unit magnitude. The value of this kind of functions would be constant - most naturally 1 - for given infinite set of $\mathcal{U}$ and root of unity in the complement of $\mathcal{U}$ defined by infinite integer or rational.

These functions would be analogous to plane waves having modulus equal to 1 and if they correspond to roots of unity they would make sense also for algebraic extensions of p-adic numbers. This conforms with the fact that p-adic norms of infinite primes and rationals are equal to unity. This would lead to a rather astonishing conclusion: there are no infinite numbers nor infinitesimals in the field generated by infinite primes in the sense of generalized scalars!

Note that functions which reduce to phases in the set of algebraic numbers are also natural in the sense that there are hopes of defining for them inner product as sum over algebraic numbers. The inner product should be consistent with the inner product induced by that for Fock states and it might be better to start directly from this inner product.

(c) It is important to realize that the complements of infinite rationals do not define support for functions but the functions themselves so that the analogy with the ultrapower construction fails.

3. The higher levels in the hierarchy of infinite primes are also present and require a further generalization of the construction. At the second level of the hierarchy algebraic numbers are replaced with the power set consisting of all finite subsets of algebraic numbers and dual of Frechet filter with that consisting of all finite subsets of this power set. Higher levels of the hierarchy would correspond a repeated replacement of the set with its power set.

4. Mathematical skeptic reader might wonder why this infinite hierarchy of constructions? Does it even lead outside the realm of algebraic numbers? What is however remarkable is that it generalizes the physics by replacing the first two quantizations with an infinite hierarchy of quantizations.

4.1 Explicit realization for the function algebra associated with infinite rationals

Consider now an explicit realizations of this algebra as a function algebra. The idea is to assigns to a given infinite rational a unique phase representing and that the algebraic structure defined by multiplication is preserved. This is like mapping rationals $q = m/n$ to phases $exp(i2\pi q)$ so that products are mapped to products. One can start from the observation that simple infinite primes can be mapped to rationals. More complex infinite primes, integers, and rationals can be mapped to collections of algebraic numbers representing the roots of corresponding polynomial primes.

1. The simplest option is that the value of the complex valued function of algebraic numbers assigned to simple infinite prime characterized by rational q is equal to $exp(i2\pi q)$ for rational q and to 1 for other algebraic numbers. The product of simple infinite integers os mapped to the product of these functions assigned to the factors. The ratio of two simple infinite integers is mapped to the ratio of corresponding functions.

2. By utilizing the decomposition the map to polynomial or rational function and its decomposition into monomials with possibly algebraic roots one could map the polynomials of rational function to factors $\prod_i exp(2\pi r_i)$ for a given infinite rational in its polynomial representation decompose to a product of monomials. This representation would map products (ratios) of infinite integers to products (ratios) but sums would not be mapped to sums but products in algebraic extension of rationals. That the images would be always non-vanishing functions would conform with the

basic properties of infinite primes and with non-existence of infinitesimals and infinite numbers in the sense of the usual ultrapower construction.

3. One would have functions in the set of algebraic numbers at the first level of hierarchy. At the next level of hierarchy one would have complex complex defined in the set of generalized rationals constructed from infinite integers. These phases are actually well defined since the infinite rational appearing in the exponent can be decomposed to a sum of terms. Only those terms which are finite contribute to the phase so that one obtains a well-defined outcome. This hierarchy would continue ad infinitum. Similar hierarchy can be associated with generalized scalars.

4. Primes are replaced with prime ideals in a more abstract approach to number theory. One could also assign to the rationals assigned to simple infinite primes the prime ideal of real or complex valued functions with value equal to one for all rationals except the selected rational. The product of simple infinite primes would correspond to the ideal consisting of functions which differ from unity for the rationals appearing in the product. The sum of simple infinite primes would in turn correspond to similar functions but differing from unity also for algebraic numbers. This would give a hierarchy of ideals with particular ideal defined in terms of functions whose value is larger than integer n for most rationals and algebraic numbers.

4.2 Generalization of the notion of real by bringing in infinite number of real units

Infinite rationals lead also to a generalization of the real numbers in the sense that given real number is replaced with infinitude of numbers having the same magnitude by multiplying it by real units which differ number theoretically [8] , [3] . There exists infinite number of rationals constructed as ratios of infinite integers at various levels of the hierarchy which as real numbers are equal to real unit but have arbitrarily complex number theoretical anatomy. Single point of real line is replaced with infinitely complex infinite-dimensional structure defined by the space of real units. This generalization applies also to other classical number fields. The role of infinitesimals would be taken by the infinitude of real units and this would extend real numbers.

This has inspired the ontological proposal that the quantum states of Universe (and even the world of classical worlds (or its sub-world defined associated with 4-surfaces inside $CD \times CP_2$) could be imbedded to this space. A less wild statement is that at least the quantum states and sub-WCW assignable to the so called causal diamond identified as the intersection of future and past directed light-cones and defining the basic structural unit in zero energy ontology can be realized in terms of the number theoretic anatomy of single space-time point.

Real units (and their generalizations to octonionic context) are analogous to quantum states. Their sum is analogous to a quantum superposition and gives a real unit by using a simple normalization. Real units are also analogous to zero energy states. By writing each infinite prime P_i at a given level of hierarchy in the form $P_i = Q_i(X_n - 1)$ (note that P_i is infinitesimal as compared to X_n), one finds that real unit condition implies that the total numbers of X_n:s in the numerator and denominator of a real unit must be same. One can apply the same procedure for the factor

$$\frac{\prod_{num} Q_i}{\prod_{den} Q_i} \quad (\text{"}num\text{" and "}den\text{" denote numerator and denominator of infinite prime})$$

to conclude that it must contain same number of X_{n-1}:s in its numerator and denominator. At the lowest level one finds that one obtains ratio of integers expressed as products of powers of finite primes p_i which must be equal to unity. The interpretation in positive energy ontology is that the total number theoretic momentum coming as integer multiple of $log(p_i)$ is same for the positive and negative energy parts of the state and therefore conserved for each finite prime p_i separately (the numbers $log(p_i)$ are algebraically independent). Conservation is indeed what one expects in arithmetic QFT.

$M^4 \times CP_2$ with structured space-time points could be able to represent all the structures of quantum theory having otherwise somewhat questionable ontological status. A given mathematical structure would "really" exist if it allows imbedding to generalized $M^4 \times CP_2$, which itself has interpretation in terms of classical number fields. Accordingly, one could talk about number theoretic Brahman=Atman identity or algebraic holography.

The above considerations suggest that the hierarchy of infinite primes and hierarchy of generalized scalars cannot be identified. It is not clear clear whether could consider the fusion of these notions. Also the fusion of real and p-adic number fields to a book like structure and of generalized scalars could be considered.

4.3 Finding the roots of polynomials defined by infinite primes

Infinite primes identifiable as analogs of bound states correspond at n:th level of the hierarchy to irreducible polynomials in the variable X_n which corresponds to the product of all primes at the previous level of hierarchy. At the first level of hierarchy the roots of this polynomial are ordinary algebraic numbers but at higher levels they correspond to infinite algebraic numbers which are somewhat weird looking creatures. These numbers however exist p-adically for all primes at the previous levels because one one can develop the roots of the polynomial in question as powers series in X_{n-1} and this series converges p-adically. This of course requires that infinite-p p-adicity makes sense. Note that all higher terms in series are p-adically infinitesimal at higher levels of the hierarchy. Roots are also infinitesimal in the scale defined X_n. Power series expansion allows to construct the roots explicitly at given level of the hierarchy as the following induction argument demonstrates.

1. At the first level of the hierarchy the roots of the polynomial of X_1 are ordinary algebraic numbers and irreducible polynomials correspond to infinite primes. Induction hypothesis states that the roots can be solved at n:th level of the hierarchy.

2. At $n+1$:th level of the hierarchy infinite primes correspond to irreducible polynomials

$$P_m(X_{n+1}) = \sum_{s=0,...,m} p_s X_{n+1}^s \quad .$$

   The roots R are given by the condition

$$P_m(R) = 0 \quad .$$

   The ansatz for a given root R of the polynomial is as a Taylor series in X_n:

$$R = \sum r_k X_n^k \quad ,$$

   which indeed converges p-adically for all primes of the previous level. Note that R is infinitesimal at $n+1$:th level. This gives

$$P_m(R) = \sum_{s=0,...,m} p_s (\sum r_k X_n^k)^s = 0 \quad .$$

   (a) The polynomial contains constant term (zeroth power of X_{n+1} given by

$$P_m(r_0) = \sum_{s=0,...,m} p_r r_0^s \quad .$$

   The vanishing of this term determines the value of r_0. Although r_0 is infinite number the condition makes sense by induction hypothesis.
   One can indeed interpret the vanishing condition

$$P_{m\times m_1}(r_0) = 0$$

   as a vanishing of a polynomial at the n:th level of hierarchy having coefficients at $n-1$:th level. Here m_1 is determined by the dependence on infinite primes of lower level expressible in terms of rational functions. One can continue the process down to the lowest level of hierarchy obtaining $m \times m_1 ... \times m_k$:th order polynomial at k:th step. At the lowest level of the hierarchy one obtains just ordinary polynomial equation having ordinary algebraic numbers as roots.

One can expand the infinite primes as a Taylor expansion in variables X_i and the resulting number differs from an ordinary algebraic number by an infinitesimal in the multi-P infinite-P p-adic topology defined by any choice of n-plet of infinite-P p-adic primes $(P_1, ..., P_n)$ from subsequent levels of the hierarchy appearing in the expansion. In this sense the resulting number is infinitely near to an ordinary algebraic number and the structure is analogous to a completion of algebraic numbers to reals. Could one regard this structure as a possible alternative view about reals remains an open question. If so, then also reals could be said to have number theoretic anatomy.

(b) If one has found the values of r_0 one can solve the coefficients r_s, $s > 0$ as linear expressions of the coefficients r_t, $t < s$ and thus in terms of r_0.

(c) The naive expectation is that the fundamental theorem of algebra generalizes so that that the number of different roots r_0 would be equal to m in the irreducible case. This seems to be the case. Suppose that one has constructed a root R of P_m. One can write $P_m(X_{n+1})$ in the form

$$P_m(X_{n+1}) = (X_{n+1} - R) \times P_{m-1}(X_{n+1}) \ ,$$

and solve P_{m-1} by expanding P_m as Taylor polynomial with respect to $X_{n+1} - R$. This is achieved by calculating the derivatives of both sides with respect to X_{m+1}. The derivatives are completely well-defined since purely algebraic operations are in question. For instance, at the first step one obtains $P_{m-1}(R) = (dP_m/dX_{n+1})(R)$. The process stops at m:th step so that m roots are obtained.

What is remarkable that the construction of the roots at the first level of the hierarchy forces the introduction of p-adic number fields and that at higher levels also infinite-p p-adic number fields must be introduced. Therefore infinite primes provide a higher level concept implying real and p-adic number fields. If one allows all levels of the hierarchy, a new number X_n must be introduced at each level of the hierarchy. About this number one knows all of its lower level p-adic norms and infinite real norm but cannot say anything more about them. The conjectured correspondence of real units built as ratios of infinite integers and zero energy states however means that these infinite primes would be represented as building blocks of quantum states and that the points of imbedding space would have infinitely complex number theoretical anatomy able to represent zero energy states and perhaps even the world of classical worlds associated with a given causal diamond.

5 Further comments about physics related articles

In the following I represent comments on the physics related articles of Rosinger not directly related to generalized scalars. I have not commented the purely mathematics related more technical articles since I do not have the competence to say anything interesting about them.

5.1 Quantum Foundations: Is Probability Ontological

In this highly interesting article [4] Rosinger poses the question whether the notion of probability is ontological or only epistemic. Are probabilities basic aspect of existence or are they are "a useful construct of mind only". My own very first reaction is a counter question. Can one speak about "mere construct of mind"? "Mind" is a part of existence and the future physics must include it to its world order. If mind is able to construct a notion like probability this notion could have some quantal correlate.

Rosinger introduces the notions of deterministic (classical typically) and non-deterministic systems and distinguishes probabilistic, fuzzy and chaotic systems as special cases of non-deterministic systems. For fuzzy and chaotic systems probability is clearly a fictive but useful notion. For probabilistic systems, in particular quantum systems the situation is not clear at all.

As a mathematician Rosinger raises purely mathematical objections against the ontological status of probability. Rosinger mentions the technical difficulties with the description of stochastic processes with continuous time and objections against axiomatizations -say in terms of Kolmogorov axioms. Rosinger mentions also frequency interpretation and somewhat fuzzy propensity interpretation of probabilities and that the notion of infinity is unavoidable also now. I cannot say much about these technical aspects and can only represent the comments based on my own physics inspired belief system.

To my very subjective view the situation is far from settled from the point of view of theoretical physics and one can consider several deformations of the notion of probability.

1. Khrennikov [1] has formulated the notion of p-adic valued probability and also I have considered p-adic thermodynamics based model for particle masses (see the first part of [6]) whose predictions, which are basically due to number theoretic existence constraints- are mapped to real numbers by a canonical correspondence between reals and p-adics.

2. Also the notion of quantum spinors related in TGD framework to the description of finite measurement resolution [11] raises the possibility that the probability itself becomes observable instead of spin (by the finite precision associated with the determination of quantization axes) and has a universal spectrum.

3. The findings of Russian biologist Shnoll [1] , [1] , [1] suggesting that the expected single peaked distributions for fluctuations of various process described by probability distributions for integer valued observable are replaced by many-peaked distributions encourage to think that the time scale of experiment is essential and the usual idea about smooth approach to probabilities as the duration of experiment increases is not correct. I have proposed an explanation of these findings in terms of the deformations of probability distributions depending on rational valued parameters so that they make sense also p-adically. This predicts precise and universal deviations which can be tested.

Rosinger relates [4] the famous Bohr-Einstein debate to the ontological status of probability concept. The divisor line between Bohr and Einstein was the attitude towards non-determinism. Neither of them could accept the idea that the determinism of Schrödinger equation could fail temporarily. Bohr was ready to give up the notion of objective reality altogether whereas Einstein refused to accept state function reduction since it would have meant giving up also the deterministic dynamics of the space-time geometry. According to Rosinger, Copenhagenist would regard probability and probability amplitudes as a fundamental aspect of existence whereas Einstein would have given for probability only episthemic role.

To my opinion both Einstein and Bohr were both right and wrong. If one accepts the view that quantum states actually correspond to superpositions of deterministic histories (generalized Bohr orbits) -as suggested also by holography principle- the problem disappears. Quantum jump recreates the quantum state as quantum superposition of entire deterministic time evolution rather than tinkering with a particular time evolution. There is no contradiction between the determinism of field equation and non-determinism of quantum jump and genuine evolution emerges as a by-product.

In this framework one also ends up with the identification of theory as a mathematical objects with the reality itself. There is no need to assume reality behind the quantum states as mathematical objects. Reality is its mathematical description as quantum state and therefore nothing but this "construct of mind". Probability amplitudes receive a firm ontological status and in TGD framework correspond to what I call spinors fields of WCE having purely geometric interpretation. Whether probabilities defined in terms of density matrix have independent ontological status is not quite clear. In quantum theory continuous stochastic process would not really occur and could be seen as a mere idealization of a process which takes as discrete quantum jumps. The technical difficulties in their description would not represent argument against the ontological status of probability amplitudes.

Thermodynamical probability is usually regarded as having only episthemic status but in zero energy ontology - one characteristic aspect of TGD quantum - positive energy quantum states are replaced with zero energy states which can be regarded mathematically as complex square roots of density matrices -which I call M-matrices- decomposable to diagonal matrix representing square roots of probabilities and unitary S-matrix. M-matrices can be organized to orthogonal rows of unitary U-matrix defining the theory. Does this mean thermodynamical holography in the sense that single particle states are able to represent the mathematics of thermodynamical ensembles in terms of their quantum states?

5.2 Group Invariant Entanglements in Generalized Tensor Products

Rosinger proposes [5] a generalization of the notion of entanglement from Hilbert space context to much more general context. The motivation is that it might allow quantum computation like operations even in classical physics context so that the problems caused by the fragility of quantum entanglement could be circumvented.

Recall that ordinary quantization leads from Cartesian product to tensor product as one replaces the points of Cartesian factors with quantum states localized at these points and forms all possible tensor products and also their superpositions. In quantum theory entanglement would emerge at the level of the function space associated with Cartesian space. Already ordinary functions of several variables allow entanglement in this sense. Un-entangled functions of several variables correspond to products of functions of single variable and the sums of these products are in general entangled. Quite generally the special functions of mathematical physics emerges as separable/un-entangled solutions of linear partial differential equations and non-linearity typically implies entanglement in this sense.

The goal of Rosinger is to generalize this framework that is to find spaces - which he calls non-Cartesian spaces- containing Cartesian product as a sub-space with the points in the complement of Cartesian product identified entangled states. Rosinger defines what he calls group invariant entanglement for a Cartesian product and shows that group operations respect the property of being entangled. As an example sequences of point pairs of Cartesian product with algebraic operation analogous to tensor product defined by convolution are considered.

The notion of entanglement has turned out to be highly interesting and non-trivial also in TGD framework.

1. A rather abstract view about entanglement is in terms of correlations. In TGD framework quantum classical correspondence realized as holography defines a very abstract form of entanglement. In this case, the quantum states assignable to the partonic 2-surfaces plus 4-D tangent space-data correspond to classical physics in the interior of space-time surface so that one obtains entanglement through this correlation. This kind of entanglement would give rise to quantum classical correspondence.

2. For infinite primes [8] , [3] the notion of entanglement emerges naturally from number theory. This is not so surprising because they can be interpreted in terms of Fock state basis for second quantized arithmetic quantum field theory. The point is that the sum of infinite integers cannot be done by using fingers since we do not possess infinite number of fingers. Therefore the sum of infinite integers is just as it is written: one cannot in general eliminate the plus from the expression unless one leaves the realm of rationals in which case one can decompose the infinite integer to a product of infinite primes. The sums of infinite integers are like superpositions of quantum states and one cannot indeed use reals as field multiplying the infinite primes. Since the products of infinite primes at the lowest level of hierarchy involve parts which can be organized to a polynomial in powers of the variable X defined by the product of finite primes identifiable formally as a variable of polynomial , one can find the expansion of infinite integer as sums over products of infinite primes and this representation is very much like the representation of entangled state.

   What is interesting is that a decomposition into unentangled state product state is obtained if one allows algebraic extension of rationals and the question is whether something like this could be achieved also for quantum states quite generally by some extension of state space concept.

Entanglement has also other number theoretic aspects.

1. One could speak about irreducible entanglement in a given extension of rationals or p-adic numbers in the sense that entanglement is reducible only if the diagonalization of the density matrix is possible in the number field considered.

2. Shannon entropy has also infinite number of number theoretic variants of entanglement probabilities are rational and even algebraic numbers [5] . The number theoretic Shannon entropy is obtained by replacing the probabilities p_i in the argument of $log(p_i)$ with their p-adic norms and changing the overall sign in the definition of Shannon entropy. The resulting entanglement negentropy can be negative and achieves negative minimum for a unique prime. This means

a possibility of information carrying entanglement conjecture to characterize the difference between living and inanimate matter identified as something residing in the intersection of real and p-adic worlds. Negentropy Maximization Principle [5] stating that state function reduction reduces entanglement entropy would indeed make this kind of entanglement stable under state function reduction.

3. The stability of entanglement could also follow from the hypothesis that physical systems are ordered with respect to the hierarchy of algebraic extensions of rationals assigned with them if one believes on number theoretically irreducib le entanglement. The hierarchy of Planck constants with arbitrarily large values of Planck constants [2] would provide a further stabilization mechanism since quantum time scales typically scale like $\hbar$. The implications for quantum computation for which the fragility of entanglement is the basic obstacle are obvious.

4. A further aspect is related to finite measurement resolution which I have suggested to be realized in terms of inclusions of hyper-finite factors [11] . The basic idea is that complex rays of state space are replaced with the orbits of included algebra characterizing measurement resolution. This leads to the replacement of complex numbers with non-commutative algebra as generalized scalars and generalizes the proposal of Rosinger in another direction. In this framework quantum spinors appear as finite-dimensional non-commutative spinors characterized by fractal dimension and probability becomes the observable instead of spin. One can speak also about quantum entanglement in given measurement resolution defined by the included algebra.

Books related to TGD

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