What could be the generalization of Yangian symmetry of $\mathcal{N} = 4$ SUSY in TGD framework?

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January 18, 2012

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1 Introduction

Lubos [38] told for some time ago about last impressive steps in the understanding of $\mathcal{N} = 4$ maximally sypersymmetric YM theory possessing 4-D super-conformal symmetry. This theory is related by AdS/CFT duality to certain string theory in $AdS_5 \times S^5$ background. Second stringy representation was discovered by Witten and based on 6-D Calabi-Yau manifold defined by twistors. In the following I will discuss briefly the notion of Yangian symmetry and suggest its generalization in TGD framework by replacing conformal algebra with appropriate super-conformal algebras. Also a possible realization of twistor approach and the construction of scattering amplitudes in terms of Yangian invariants defined by Grassmannian integrals is considered in TGD framework and based on the idea that in zero energy ontology one can represent massive states as bound states of massless particles. There is also a proposal for a physical interpretation of the Cartan algebra of Yangian algebra allowing to understand at the fundamental level how the mass spectrum of n-particle bound states could be understood in terms of the n-local charges of the Yangian algebra.

1.1 Background

I am outsider as far as concrete calculations in $\mathcal{N} = 4$ SUSY are considered and the following discussion of the background probably makes this obvious. My hope is that the reader had patience to not care about this and try to see the big pattern.

The developments began from the observation of Parke and Taylor [25] that n-gluon tree amplitudes with less than two negative helicities vanish and those with two negative helicities have unexpectedly simple form when expressed in terms of spinor variables used to represent light-like momentum. In fact, in the formalism based on Grassmanian integrals the reduced tree amplitude for two negative helicities is just "1" and defines Yangian invariant. The article *Perturbative Gauge Theory As a String Theory In Twistor Space* [26] by Witten led to so called Britto-Cachazo-Feng-Witten (BCFW) recursion relations for tree level amplitudes [28, 29] allowing to construct tree amplitudes using the analogs of Feynman rules in which vertices correspond to maximally helicity violating tree amplitudes (2 negative helicity gluons) and propagator is massless Feynman propagator for boson. The progress inspired the idea that the theory might be completely integrable meaning the existence of infinite-dimensional un-usual symmetry. This symmetry would be so called Yangian symmetry [17] assigned to the super counterpart of the conformal group of 4-D Minkowski space.

Drumond, Henn, and Plefka represent in the article *Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory* [35] an argument suggesting that the Yangian invariance of the scattering amplitudes ins an intrinsic property of planar $\mathcal{N} = 4$ super Yang Mills at least at tree level.

The latest step in the progress was taken by Arkani-Hamed, Bourjaily, Cachazo, Carot-Huot, and Trnka and represented in the article *Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory* [34]. At the same day there was also the article of Rutger Boels entitled *On BCFW shifts of integrands and integrals* [36] in the archive. Arkani-Hamed *et al* argue that a full Yangian symmetry of the theory allows to generalize the BCFW recursion relation for tree amplitudes to all loop orders at planar limit (planar means that Feynman diagram allows imbedding to plane without intersecting lines). On mass shell scattering amplitudes are in question.

1.2 Yangian symmetry

The notion equivalent to that of Yangian was originally introduced by Faddeev and his group in the study of integrable systems. Yangians are Hopf algebras which can be assigned with Lie algebras as the deformations of their universal enveloping algebras. The elegant but rather cryptic looking definition is in terms of the modification of the relations for generating elements [17]. Besides ordinary product in the enveloping algebra there is co-product Δ which maps the elements of the enveloping algebra to its tensor product with itself. One can visualize product and co-product is in terms of particle reactions. Particle annihilation is analogous to annihilation of two particle so single one and co-product is analogous to the decay of particle to two. Δ allows to construct higher generators of the algebra.

Lie-algebra can mean here ordinary finite-dimensional simple Lie algebra, Kac-Moody algebra or Virasoro algebra. In the case of SUSY it means conformal algebra of M^4- or rather its super counterpart. Witten, Nappi and Dolan have described the notion of Yangian for super-conformal algebra in very elegant and and concrete manner in the article *Yangian Symmetry in D=4 superconformal Yang-Mills theory* [27]. Also Yangians for gauge groups are discussed.

In the general case Yangian resembles Kac-Moody algebra with discrete index n replaced with a continuous one. Discrete index poses conditions on the Lie group and its representation (adjoint representation in the case of $\mathcal{N}=4$ SUSY). One of the conditions conditions is that the tensor product $R\otimes R^*$ for representations involved contains adjoint representation only once. This condition is non-trivial. For $SU(n)$ these conditions are satisfied for any representation. In the case of $SU(2)$ the basic branching rule for the tensor product of representations implies that the condition is satisfied for the product of any representations.

Yangian algebra with a discrete basis is in many respects analogous to Kac-Moody algebra. Now however the generators are labelled by non-negative integers labeling the light-like incoming and outgoing momenta of scattering amplitude whereas in in the case of Kac-Moody algebra also negative values are allowed. Note that only the generators with non-negative conformal weight appear in the construction of states of Kac-Moody and Virasoro representations so that the extension to Yangian makes sense.

The generating elements are labelled by the generators of ordinary conformal transformations acting in M^4 and their duals acting in momentum space. These two sets of elements can be labelled by conformal weights $n=0$ and $n=1$ and and their mutual commutation relations are same as for Kac-Moody algebra. The commutators of $n=1$ generators with themselves are however something different for a non-vanishing deformation parameter h. Serre's relations characterize the difference and involve the deformation parameter h. Under repeated commutations the generating elements generate infinite-dimensional symmetric algebra, the Yangian. For $h=0$ one obtains just one half of the Virasoro algebra or Kac-Moody algebra. The generators with $n>0$ are $n+1$-local in the sense that they involve $n+1$-forms of local generators assignable to the ordered set of incoming particles of the scattering amplitude. This non-locality generalizes the notion of local symmetry and is claimed to be powerful enough to fix the scattering amplitudes completely.

2 How to generalize Yangian symmetry in TGD framework?

As far as concrete calculations are considered, I have nothing to say. I am just perplexed. It is however possible to keep discussion at general level and still say something interesting (as I hope!). The key question is whether it could be possible to generalize the proposed Yangian symmetry and geometric picture behind it to TGD framework.

1. The first thing to notice is that the Yangian symmetry of $\mathcal{N}=4$ SUSY in question is quite too limited since it allows only single representation of the gauge group and requires massless particles. One must allow all representations and massive particles so that the representation of symmetry algebra must involve states with different masses, in principle arbitrary spin and arbitrary internal quantum numbers. The candidates are obvious: Kac-Moody algebras [19] and Virasoro algebras[18] and their super counterparts. Yangians indeed exist for arbitrary super Lie algebras. In TGD framework conformal algebra of Minkowski space reduces to Poincare algebra and its extension to Kac-Moody allows to have also massive states.

2. The formal generalization looks surprisingly straightforward at the formal level. In zero energy ontology one replaces point like particles with partonic two-surfaces appearing at the ends of light-like orbits of wormhole throats located to the future and past light-like boundaries of causal diamond ($CD\times CP_2$ or briefly CD). Here CD is defined as the intersection of future and past directed light-cones. The polygon with light-like momenta is naturally replaced with a polygon with more general momenta in zero energy ontology and having partonic surfaces as its vertices. Non-point-likeness forces to replace the finite-dimensional super Lie-algebra with infinite-dimensional Kac-Moody algebras and corresponding super-Virasoro algebras assignable to partonic 2-surfaces.

3. This description replaces disjoint holomorphic surfaces in twistor space with partonic 2-surfaces at the boundaries of $CD\times CP_2$ so that there seems to be a close analogy with Cachazo-Svrcek-Witten picture. These surfaces are connected by either light-like orbits of partonic 2-surface or space-like 3-surfaces at the ends of CD so that one indeed obtains the analog of polygon.

What does this then mean concretely (if this word can be used in this kind of context;-)?

1. At least it means that ordinary Super Kac-Moody and Super Virasoro algebras associated with isometries of $M^4 \times CP_2$ annihilating the scattering amplitudes must be extended to a co-algebras with a non-trivial deformation parameter. Kac-Moody group is thus the product of Poincare and color groups. This algebra acts as deformations of the light-like 3-surfaces representing the light-like orbits of particles which are extremals of Chern-Simon action with the constraint that weak form of electric-magnetic duality holds true. I know so little about the mathematical side that I cannot tell whether the condition that the product of the representations of Super-Kac-Moody and Super-Virasoro algedbras ontains adjoint representation only once, holds true in this case. In any case, it would allow all representations of finite-dimensional Lie group in vertices whereas $\mathcal{N}=4$ SUSY would allow only the adjoint.

2. Besides this ordinary kind of Kac-Moody algebra there is the analog of Super-Kac-Moody algebra associated with the light-cone boundary which is metrically 3-dimensional. The finite-dimensional Lie group is in this case replaced with infinite-dimensional group of symplectomorphisms of $\delta M^4_{+/-}$ made local with respect to the internal coordinates of partonic 2-surface. A coset construction is applied to these two Virasoro algebras so that the differences of the corresponding Super-Virasoro generators and Kac-Moody generators annihilate physical states. This implies that the corresponding four-momenta are same: this expresses the equivalence of gravitational and inertial masses. A generalization of the Equivalence Principle is in question. This picture also justifies p-adic thermodynamics applied to either symplectic or isometry Super-Virasoro and giving thermal contribution to the vacuum conformal and thus to mass squared.

3. The construction of TGD leads also to other super-conformal algebras and the natural guess is that the Yangians of all these algebras annihilate the scattering amplitudes.

4. Obviously, already the starting point symmetries look formidable but they still act on single partonic surface only. The discrete Yangian associated with this algebra associated with the closed polygon defined by the incoming momenta and the negatives of the outgoing momenta acts in multi-local manner on scattering amplitudes. It might make sense to speak about polygons defined also by other conserved quantum numbers so that one would have generalized light-like curves in the sense that state are massless in 8-D sense.

2.1 Is there any hope about description in terms of Grassmannians?

At technical level the successes of the twistor approach rely on the observation that the amplitudes can be expressed in terms of very simple integrals over sub-manifolds of the space consisting of k-dimensional planes of n-dimensional space defined by delta function appearing in the integrand. These integrals define super-conformal Yangian invariants appearing in twistorial amplitudes and the belief is that by a proper choice of the surfaces of the twistor space one can construct all invariants. One can construct also the counterparts of loop corrections by starting from tree diagrams and annihilating pair of particles by connecting the lines and quantum entangling the states at the ends in the manner dictated by the integration over loop momentum. These operations can be defined as operations for Grassmannian integrals in general changing the values of n and k. This description looks extremely powerful and elegant and nosta importantly involves only the external momenta.

The obvious question is whether one could use similar invariants in TGD framework to construct the momentum dependence of amplitudes.

1. The first thing to notice is that the super algebras in question act on infinite-dimensional representations and basically in the world of classical worlds assigned to the partonic 2-surfaces correlated by the fact that they are associated with the same space-time surface. This does not promise anything very practical. On the other hand, one can hope that everything related to other than M^4 degrees of freedom could be treated like color degrees of freedom in $\mathcal{N}=4$ SYM and would boil down to indices labeling the quantum states. The Yangian conditions coming from isometry quantum numbers, color quantum numbers, and electroweak quantum numbers are of course expected to be highly non-trivial and could fix the coefficients of various singlets resulting in the tensor product of incoming and outgoing states.

2. The fact that incoming particles can be also massive seems to exclude the use of the twistor space. The following observation however raises hopes. The Dirac propagator for wormhole throat is massless propagator but for what I call pseudo momentum. It is still unclear how this momentum relates to the actual four-momentum. Could it be actually equal to it? The recent view about pseudo-momentum does not support this view but it is better to keep mind open. In any case this finding suggests that twistorial approach could work in in more or less standard form. What would be needed is a representation for massive incoming particles as bound states of massless partons. In particular, the massive states of super-conformal representations should allow this kind of description.

Could zero energy ontology allow to achieve this dream?

1. As far as divergence cancellation is considered, zero energy ontology suggests a totally new approach producing the basic nice aspects of QFT approach, in particular unitarity and coupling constant evolution. The big idea related to zero energy ontology is that all virtual particle particles correspond to wormhole throats, which are pairs of on mass shell particles. If their momentum directions are different, one obtains time-like continuum of virtual momenta and if the signs of energy are opposite one obtains also space-like virtual momenta. The on mass shell property for virtual partons (massive in general) implies extremely strong constraints on loops and one expect that only very few loops remain and that they are finite since loop integration reduces to integration over much lower-dimensional space than in the QFT approach. There are also excellent hopes about Cutkoski rules.

2. Could zero energy ontology make also possible to construct massive incoming particles from massless ones? Could one construct the representations of the super conformal algebras using only massless states so that at the fundamental level incoming particles would be massless and one could apply twistor formalism and build the momentum dependence of amplitudes using Grassmannian integrals.

   One could indeed construct on mass shell massive states from massless states with momenta along the same line but with three-momenta at opposite directions. Mass squared is given by $M^2 = 4E^2$ in the coordinate frame, where the momenta are opposite and of same magnitude. One could also argue that partonic 2-surfaces carrying quantum numbers of fermions and their superpartners serve as the analogs of point like massless particles and that topologically condensed fermions and gauge bosons plus their superpartners correspond to pairs of wormhole throats. Stringy objects would correspond to pairs of wormhole throats at the same space-time sheet in accordance with the fact that space-time sheet allows a slicing by string worlds sheets with ends at different wormhole throats and definining time like braiding.

The weak form of electric magnetic duality indeed supports this picture. To understand how, one must explain a little bit what the weak form of electric magnetic duality means.

1. Elementary particles correspond to light-like orbits of partonic 2-surfaces identified as 3-D surfaces at which the signature of the induced metric of space-time surface changes from Euclidian to Minkowskian and 4-D metric is therefore degenerate. The analogy with black hole horizon is obvious but only partial. Weak form of electric-magnetic duality states that the Kähler electric field at the wormhole throat and also at space-like 3-surfaces defining the ends of the space-time surface at the upper and lower light-like boundaries of the causal diamond is proportonial to Kähler magnetic field so that Kähler electric flux is proportional Kähler magnetic flux. This implies classical quantization of Kähler electric charge and fixes the value of the proportionality constant.

2. There are also much more profound implications. The vision about TGD as almost topological QFT suggests that Kähler function defining the Kähler geometry of the "world of classical worlds" (WCW) and identified as Kähler action for its preferred extremal reduces to the 3-D Chern-Simons action evaluted at wormhole throats and possible boundary components. Chern-Simons action would be subject to constraints. Wormhole throats and space-like 3-surfaces would represent extremals of Chern-Simons action restricted by the constraint force stating electric-magnetic duality (and realized in terms of Lagrange multipliers as usual).

If one assumes that Kähler current and other conserved currents are proportional to current defining Beltrami flow whose flow lines by definition define coordinate curves of a globally defined coordinate, the Coulombic term of Kähler action vanishes and it reduces to Chern-Simons action if the weak form of electric-magnetic duality holds true. One obtains almost topological QFT. The absolutely essential attribute "almost" comes from the fact that Chern-Simons action is subject to constraints. As a consequence, one obtains non-vanishing four-momenta and WCW geometry is non-trivial in M^4 degrees of freedom. Otherwise one would have only topological QFT not terribly interesting physically.

Consider now the question how one could understand stringy objects as bound states of massless particles.

1. The observed elementary particles are not Kähler monopoles and there much exist a mechanism neutralizing the monopole charge. The only possibility seems to be that there is opposite Kähler magnetic charge at second wormhole throat. The assumption is that in the case of color neutral particles this throat is at a distance of order intermediate gauge boson Compton length. This throat would carry weak isospin neutralizing that of the fermion and only electromagnetic charge would be visible at longer length scales. One could speak of electro-weak confinement. Also color confinement could be realized in analogous manner by requiring the cancellation of monopole charge for many-parton states only. What comes out are string like objects defined by Kähler magnetic fluxes and having magnetic monopoles at ends. Also more general objects with three strings branching from the vertex appear in the case of baryons. The natural guess is that the partons at the ends of strings and more general objects are massless for incoming particles but that the 3-momenta are in opposite directions so that stringy mass spectrum and representations of relevant super-conformal algebras are obtained. This description brings in mind the description of hadrons in terms of partons moving in parallel apart from transversal momentum about which only momentum squared is taken as observable.

2. Quite generally, one expects for the preferred extremals of Kähler action the slicing of space-time surface with string world sheets with stringy curves connecting wormhole throats. The ends of the stringy curves can be identified as light-like braid strands. Note that the strings themselves define a space-like braiding and the two braidings are in some sense dual. This has a concrete application in TGD inspired quantum biology, where time-like braiding defines topological quantum computer programs and the space-like braidings induced by it its storage into memory. Stringlike objects defining representations of super-conformal algebras must correspond to states involving at least two wormhole throats. Magnetic flux tubes connecting the ends of magnetically charged throats provide a particular realization of stringy on mass shell states. This would give rise to massless propagation at the parton level. The stringy quantization condition for mass squared would read as $4E^2 = n$ in suitable units for the representations of super-conformal algebra associated with the isometries. For pairs of throats of the same wormhole contact stringy spectrum does not seem plausible since the wormhole contact is in the direction of CP_2. One can however expect generation of small mass as deviation of vacuum conformal weight from half integer in the case of gauge bosons.

If this picture is correct, one might be able to determine the momentum dependence of the scattering amplitudes by replacing free fermions with pairs of monopoles at the ends of string and topologically condensed fermions gauge bosons with pairs of this kind of objects with wormhole throat replaced by a pair of wormhole throats. This would mean suitable number of doublings of the Grassmannian integrations with additional constraints on the incoming momenta posed by the mass shell conditions for massive states.

2.2 Could zero energy ontology make possible full Yangian symmetry?

The partons in the loops are on mass shell particles have a discrete mass spectrum but both signs of energy are possible for opposite wormhole throats. This implies that in the rules for constructing loop amplitudes from tree amplitudes, propagator entanglement is restricted to that corresponding to pairs of partonic on mass shell states with both signs of energy. As emphasized in [34], it is the Grassmannian integrands and leading order singularities of $\mathcal{N} = 4$ SYM, which possess the full

Yangian symmetry. The full integral over the loop momenta breaks the Yangian symmetry and brings in IR singularities. Zero energy ontologist finds it natural to ask whether QFT approach shows its inadequacy both via the UV divergences and via the loss of full Yangian symmetry. The restriction of virtual partons to discrete mass shells with positive or negative sign of energy imposes extremely powerful restrictions on loop integrals and resembles the restriction to leading order singularities. Could this restriction guarantee full Yangian symmetry and remove also IR singularities?

2.3 Could Yangian symmetry provide a new view about conserved quantum numbers?

The Yangian algebra has some properties which suggest a new kind of description for bound states. The Cartan algebra generators of $n = 0$ and $n = 1$ levels of Yangian algebra commute. Since the co-product Δ maps $n = 0$ generators to $n = 1$ generators and these in turn to generators with high value of n, it seems that they commute also with $n \geq 1$ generators. This applies to four-momentum, color isospin and color hyper charge, and also to the Virasoro generator L_0 acting on Kac-Moody algebra of isometries and defining mass squared operator.

Could one identify total four momentum and Cartan algebra quantum numbers as sum of contributions from various levels? If so, the four momentum and mass squared would involve besides the local term assignable to wormhole throats also n-local contributions. The interpretation in terms of n-parton bound states would be extremely attractive. n-local contribution would involve interaction energy. For instance, string like object would correspond to $n = 1$ level and give $n = 2$-local contribution to the momentum. For baryonic valence quarks one would have 3-local contribution corresponding to $n = 2$ level. The Yangian view about quantum numbers could give a rigorous formulation for the idea that massive particles are bound states of massless particles.

2.4 What about the selection of preferred $M^2 \subset M^4$?

The puzzling aspect of the proposed picture is the restriction of the pseudo-momenta to M^2 and quite generally the the selection of preferred plane $M^2 \subset M^4$. This selection is one the key aspects of TGD but is not too well understood. Also the closely related physical interpretation of the 2-D pseudo-momenta in M^2 is unclear.

2.4.1 The avatars of $M^2 \subset M^4$ in quantum TGD

The choice of preferred plane $M^2 \subset M^4$ pops u again and again in quantum TGD.

1. There are very strong reasons to believe that the solutions of field equations for the preferred extremals assign M^2 to each point of space-time surface and the interpretation is as the plane of non-physical polarizations. One can also consider the possibility that M^2 depends on the point of space-time surface but that the different choices integrate to 2-D surface analogous to string world sheet - very naturally projection of stringy worlds sheets defining the slicing of the space-time surface.

2. The number theoretic vision- in particular $M^8 - H$ duality ($H = M^4 \times CP_2$) providing a purely number theoretic interpretation for the choice $H = M^4 \times CP_2$ - involves also the selection of preferred M^2. The duality states that the surfaces in H can be regarded equivalently as surfaces in M^8. The induced metric and Kähler form are identical as also the value of Kähler function. The description of the duality is following.

    (a) The points of space-time surface in $M^8 = M^4 \times E^4$ in M^8 are mapped to points of space-time surface in $M^4 \times CP_2$. The M^4 part of the map is just a projection.

    (b) CP_2 part of the map is less trivial. The idea is that M^8 is identified as a subspace of complexified octonions obtained by adding commutative imaginary unit, I call this sub-space hyper-octonionic. Suppose that space-time surface is hyper-quaternionic (in appropriate sense meaning that one can attach to its each point a hyper-quaternionic plane, not necessary tangent plane). Assume that it also contains a preferred hypercomplex plane M^2 of M^8 at each point -or more generally a varying plane M^2 planes whose distribution however integrates to form 2-surface analogous to string world sheet. The interpretation is as a preferred plane of non-physical polarizations so that basic aspect of gauge symmetry would have a number theoretic interpretation. Note that one would thus have a local hierarchy of octonionic, quaternionic, and complex planes.

(c) Under these assumptions the tangent plane (if action is just the four-volume or its generalization in the case of Kähler action) is characterized by a point of $CP_2 = SU(3)/U(2)$ where $SU(3)$ is automorphism group of octonions respecting preferred plane M^2 of polarizations and $U(2)$ is automorphism group acting in the hyper-quaternionic plane. This point can be identified as a point of CP_2 so that one obtains the duality.

3. Also the definition of CDs and the proposed construction of the hierarchy of Planck constants involve a choice of preferred M^2, which corresponds to the choice of rest frame and quantization axis of angular momentum physically. Therefore the choice of quantization axis would have direct correlates both at the level of CDs and space-time surface. The vector between the tips of CD indeed defines preferred direction of time and thus rest system. Similar considerations apply in the case of CP_2.

4. Preferred M^2 -but now at this time at momentum space level - appears as the plane of pseudo-momenta associated with the generalized eigen modes of the modified Dirac equation associated with Chern-Simons action. Internally consistency requires a restriction to this plane. This looks somewhat mysterious since this would mean that all exchanged virtual momenta would be in M^2 if the choice is same for all lines of the generalized Feynman graph. This would restrict momentum exchanges in particle reactions to single dimension and does not make sense. One must however notice that in the description of hadronic reactions in QCD picture one makes a choice of longitudinal momentum direction and considers only longitudinal momenta. It would seem that the only possibility is that the planes M^2 are independent for independent exchanged momenta. For instance, in $2 \rightarrow 2$ scattering the exchange would be in plane defined by the initial and final particles of the vertex. There are also good arguments for a number theoretic quantization of the momenta in M^2.

   The natural expectation from $M^8 - H$ duality is that the selection of preferred M^2 implies a reduction of symmetries to those of $M^2 \times E^6$ and $M^2 \times E^2 \times CP_2$. Could the equivalence of M^8 and H descriptions force the reduction of M^4 momentum to M^2 momentum implied also by the generalized eigen value equation for the modified Dirac operator at wormhole throats?

2.4.2 The moduli space associated with the choice of M^2

Lorentz invariance requires that one must have moduli space of CDs with fixed tips defined as $SO(3,1)/SO(1) \times SO(2)$ characterizing different choices of M^2. Maximal Lorentz invariance requires the association of this moduli space to all lines of the generalized Feynman graph. It is easy to deduce that this space is actually the hyperboloid of 5-D Minkowski space. The moduli space is 4-dimensional and has Euclidian signature of the metric. This follows from the fact that $SO(3,1)$ has Euclidian signature as a surface in the four-fold Cartesian power $H(1,3)^4$ of Lobatchevski space with points identified as four time-like unit vectors defining rows of the matrix representing Lorentz transformation. This surface is defined by the 6 orthogonality conditions for the rows of the Lorentz transformation matrix constraints stating the orthogonality of the 4 unit vectors. The Euclidian signature fixes the identification of the moduli space as $H(1,5)$ having Euclidian signature of metric. The 10-D isometry group $SO(1,5)$ of the moduli space acts as symmetries of 5-D Minkowski space (note that the conformal group of M^4 is $SO(2,4)$. The non-compactness of this space does not favor the idea of wave function in moduli degrees of freedom.

Concerning the interpretation of pseudo-momenta it is best to be cautious and make only questions. Should one assume that M^2 for the exchanged particle is fixed by the initial and final momenta of the particle emitting it? How to fix in this kind of situation a unique coordinate frame in which the number theoretic quantization of exchanged momenta takes place? Could it be the rest frame for the initial state of the emitting particle so that one should allow also boosts of the number theoretically preferred momenta? Should one only assume the number theoretically preferred mass values for the exchanged particle but otherwise allow the hyperbolic angle characterizing the energy vary freely?

2.5 Does $M^8 - H$ duality generalize the duality between twistor and momentum twistor descriptions?

$M^8 - H$ duality is intuitively analogous to the duality of elementary wave mechanics meaning that one can use either x-space or momentum space to describe particles. M^8 is indeed the tangent space of H and one could say that $M^8 - H$ duality assigns to a 4-surface in H its "momentum" or tangent as a 4-surface in M^8. The more concrete identification of M^8 as cotangent bundle of H so that its points would correspond to 8-momenta: this very naive picture is of course not correct.

$M^8 - H$ duality suggests that the descriptions using isometry groups of $M^4 \times E^4$ and $M^4 \times CP_2$ -or as the special role of M^2 suggests - those of $M^2 \times E^6$ and $M^2 \times E^2 \times CP_2$ should be equivalent. The interpretation in hadron physics context would be that $SO(4)$ is the counterpart of color group in low energy hadron physics acting on strong isospin degrees of freedom and $SU(3)$ that of QCD description useful at high energies. $SO(4)$ is indeed used in old fashioned hadron physics when quarks and gluons had not yet been introduced. Skyrme model is one example.

The obvious question is whether the duality between descriptions based on twistors and momentum space twistors generalizes to $M^8 - H$ duality. The basic objection is that the charges and their duals should correspond to the same Lie algebra- or rather Kac-Moody algebra. This is however not the case. For the massless option one has $SO(2) \times SU(3)$ at H-side and $SO(2) \times SO(4)$ or $SO(6)$ and M^8 side.This suggests that $M^8 - H$ duality is analogous to the duality between descriptions using twistors and momentum space twistors and transforms the local currents J_0 to non-local currents J_1 and vice versa. This duality would be however be more general in the sense that would relate Yangian symmetries with different Kac-Moody groups transforming locality to non-locality and vice versa. This interpretation is consistent with the fact that the groups $SO(2) \times SO(4)$, $SO(6)$ and $SO(2) \times SU(3)$ have same rank and the stanard construction of Kac-Moody generators in terms of exponentials of the Cartan algebra involves only different weights in the exponentials.

If $M^8 - H$ duality has something to do with the duality between descriptions using twistors and momentum space twistors involved with Yangian symmetry, it should be consistent with the basic aspects of the latter duality. The following arguments provide support for this.

1. $SO(4)$ should appear as a dynamical symmetry at $M^4 \times CP_2$ side and $SU(3)$ at M^8 side (where it indeed appears as both subgroup of isometries and as tangent space group respecting the choice of M^2. One could consider the breaking of $SO(4)$ to the subgroup corresponding to vectorial transformations and interpreted in terms of electroweak vectorial $SU(2)$: this would conform with conserved vector current hypothesis and partially conserved axial current hypothesis. The $U(1)$ factor assignable to Kähler form is also present and allows Kac-Moody variant and an extension to Yangian.

2. The heuristics of twistorial approach suggests that the roles of currents J_0 and their non-local duals J_1 in Minkowski space are changed in the transition from H description to M^8 description in the sense that the non-local currents J_1 in H description become local currents in 8-momentum space (or 4-momentum +strong isospin) in M^8 description and J_0 becomes non-local one. In the case of hadron physics the non-local charges assignable to hadrons as collections of partons would become local charges meaning that one can assign them to partonic 2-surfaces at boundaries of CDs assigned to M^8: this says that hadrons are the only possible final states of particle reactions. By the locality it would be impossible decompose momentum and strong isospin to a collection of momenta and strong isospins assigned to partons.

3. In H description it would be impossible to do decompose quantum numbers to those of quarks and gluons at separate uncorrelated partonic 2-surfaces representing initial and final states of particle reaction. A possible interpretation would be in terms of monopole confinement accompanying electroweak and color confinement: single monopole is not a particle. In $M^4 \times E^4$ monopoles must be also present since induced Kähler forms are identical. The Kähler form represents magnetic monopole in E^4 and breaks its translational symmetry and also selects unique $M^4 \times E^4$ decomposition.

4. Since the physics should not depend on its description, color should be confined also now. Indeed, internal quantum numbers should be assigned in M^8 picture to a wave function in $M^2 \times E^6$ and symmetries would correspond to $SO(1,1) \times SO(6)$ or - if broken- to those of $SO(1,1) \times G$,

$G = SO(2) \times SO(4)$ or $G = SO(3) \times SO(3)$. Color would be completely absent in accordance with the idea that fundamental observable objects are color singlets. Instead of color one would have $SO(4)$ quantum numbers and $SO(4)$ confinement: note that the rank of this group crucial for Kac-Moody algebra construction is same as that of $SU(3)$.

It is not clear whether the numbers of particle states should be same for $SO(4)$ and $SU(3)$. If so, quark triplet should correspond to doublet and singlet for strong vectorial isospin in M^8 picture. Gluons would correspond to $SU(2)_V$ multiplets contained by color octet and would therefore contain also other representations than adjoint. This could make sense in composite particle interpretation.

5. For $M^2 \times E^2$ longitudinal momentum and helicity would make sense and one could speak of massless strong isospin at M^8 side and massless color at H-side: note that massless color is the only possibility. For $M^2 \times SO(6)$ option one would have 15-D adjoint representation of $SO(6)$ decomposing as $3\times 3+3\times 1+1\times 3$ under $SO(3)\times SO(3)$. This could be interpreted in terms of spin and vectorial isopin for massive particles so that the multiplets would relate to weak gauge bosons and Higgs boson singlet and triplet plus its pseudoscalar variant. For 4-D representation of $SO(6)$ one would have 2×2 decomposition having interpretation in terms of spin and vectorial isospin.

   Massive spin would be associated as a local notion with $M^2 \times E^3$ and would be essentially 5-D concept. At H side massive particle would make sense only as a non-local notion with four-momentum and mass represented as a non-local operator.

These arguments indeed encourage to think that M^8-H duality could be the analog for the duality between the descriptions in terms of twistors and momentum twistors. In this case the Kac-Moody algebras are however not identical since the isometry groups are not identical.

3 Some mathematical details about Grasmannian formalism

In the following I try to summarize my amateurish understanding about the mathematical structure behind the Grassmann integral approach. The representation summarizes my own understanding based on the articles of Arkani-Hamed and collaborators [33, 34]. These articles are rather sketchy and the article of Bullimore provides additional details [37] related to soft factors. The article of Mason and Skinner provides excellent introduction to super-twistors [35] and dual super-conformal invariance.

Before continuing a brief summary about the history leading to the articles of Arkani-Hamed and others is in order. This summary covers only those aspects which I am at least somewhat familiar with and leaves out many topics about existence which I am only half-conscious.

1. It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda_a\tilde{\lambda}_{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$\begin{aligned} \langle\lambda,\mu\rangle &= \epsilon_{ab}\lambda^a\mu^b \ , \\ \left[\tilde{\lambda},\tilde{\mu}\right] &= \epsilon_{a'b'}\lambda^{a'}\mu^{b'} \ , \\ p\cdot q &= \langle\lambda,\mu\rangle\left[\tilde{\lambda},\tilde{\mu}\right] \ , \ (q_{aa'}=\mu_a\tilde{\mu}_{a'}) \ . \end{aligned} \tag{3.1}$$

2. Tree amplitudes are considered and it is convenient to drop the group theory factor $Tr(T_1T_2...T_n)$. The starting point is the observation that tree amplitude for which more than $n-2$ gluons have the same helicity vanish. MHV amplitudes have exactly $n-2$ gluons of same helicity- taken by a convention to be negative- have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle\lambda_x,\lambda_y\rangle^4}{\prod_{i=1}^n\langle\lambda_i,\lambda_{i+1}\rangle} \tag{3.2}$$

When the sign of the helicities is changed $\langle..\rangle$ is replaced with $[..]$.

3. The article of Witten [26] proposed that twistor approach could be formulated as a twistor string theory with string world sheets "living" in 6-dimensional CP_3 possessing Calabi-Yau structure and defining twistor space. In this article Witten introduced what is known as half Fourier transform allowing to transform momentum integrals over light-cone to twistor integrals. This operation makes sense only in space-time signature $(2,2)$. Witten also demonstrate that maximal helicity violating twistor amplitudes (two gluons with negative helicity) with n particles with k negative helicities and l loops correspond in this approach to holomorphic 2-surfaces defined by polynomials defined by polynomials of degree $D = k-1+l$, where the genus of the surface satisfies $g \leq l$. AdS/CFT duality provides a second stringy approach to $\mathcal{N}=4$ theory allowing to understand the scattering amplitudes in terms of Wilson loops with light-like edges: about this I have nothing to say. In any case, the generalization of twistor string theory to TGD context is highly attractive idea and will be considered later.

4. In the article [28] Cachazo, Svrcek, and Witten propose the analog of Feynman diagrammatics in which MHV amplitudes can be used as analogs of vertices and ordinary $1/P^2$ propagator as propagator to construct tree diagrams with arbitrary number of negative helicity gluons. This approach is not symmetric with respect to the change of the sign of helicities since the amplitudes with two positive helicities are constructed as tree diagrams. The construction is non-trivial because one must analytically continue the on mass shell tree amplitudes to off mass shell momenta. The problem is how to assign a twistor to these momenta. This is achieved by introducing an arbitrary twistor $\eta^{a'}$ and defining λ_a as $\lambda_a = p_{aa'}\eta^{a'}$. This works for both massless and massive case. It however leads to a loss of the manifest Lorentz invariance. The paper however argues and the later paper [29] shows rigorously that the loss is only apparent. In this paper also BCFW recursion formula is introduced allowing to construct tree amplitudes recursively by starting from vertices with 2 negative helicity gluons. Also the notion which has become known as BCFW bridge representing the massless exchange in these diagrams is introduced. The tree amplitudes are not tree amplitudes in gauge theory sense where correspond to leading singularities for which 4 or more lines of the loop are massless and therefore collinear. What is important that the very simple MHV amplitudes become the building blocks of more complex amplitudes.

5. The nex step in the progress was the attempt to understand how the loop corrections could be taken into account in the construction BCFW formula. The calculation of loop contributions to the tree amplitudes revealed the existence of dual super-conformal symmetry which was found to be possessed also by BCFW tree amplitudes besides conformal symmetry. Together these symmetries generate infinite-dimensional Yangian symmetry [35].

6. The basic vision of Arkani-Hamed and collaborators is that the scattering amplitudes of $\mathcal{N}=4$ SYM are constructible in terms of leading order singularities of loop diagrams. These singularities are obtained by putting maximum number of momenta propagating in the lines of the loop on mass shell. The non-leading singularities would be induced by the leading singularities by putting smaller number of momenta on mass shell are dictated by these terms. A related idea serving as a starting point in [33] is that one can define loop integrals as residue integrals in momentum space. If I have understood correctly, this means that one an imagine the possibility that the loop integral reduces to a lower dimensional integral for on mass shell particles in the loops: this would resemble the approach to loop integrals based on unitarity and analyticity. In twistor approach these momentum integrals defined as residue integrals transform to residue integrals in twistor space with twistors representing massless particles. The basic discovery is that one can construct leading order singularities for n particle scattering amplitude with k negative helicities as Yangian invariants $Y_{n,k}$ and invariants constructed from them by canonical operations changing n and k.

3.1 Yangian algebra and its super counterpart

The article of Witten [27] gives a nice discussion of the Yangian algebra and its super counterpart. Here only basic formulas can be listed and the formulas relevant to the super-conformal case are given.

1. Yangian algebra

Yangian algebra $Y(G)$ is associative Hopf algebra. The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n = 0$ and $n = 1$. The first half of these relations discussed in very clear manner in [27] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f_C^{AB} J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f_C^{AB} J^{(1)C} \quad . \tag{3.3}$$

Besides this Serre relations are satisfied. These have more complex and read as

$$\begin{aligned} &\left[J^{(1)A}, \left[J^{(1)B}, J^C\right]\right] + \left[J^{(1)B}, \left[J^{(1)C}, J^A\right]\right] + \left[J^{(1)C}, \left[J^{(1)A}, J^B\right]\right] \\ &= \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM}\{J_D, J_E, J_F\} \ , \\ &\left[\left[J^{(1)A}, J^{(1)B}\right], \left[J^C, J^{(1)D}\right]\right] + \left[\left[J^{(1)C}, J^{(1)D}\right], \left[J^A, J^{(1)B}\right]\right] \\ &= \frac{1}{24} f^{AGL} f^{BEM} f_K^{CD} \\ &+ f^{CGL} f^{DEM} f_K^{AB}) f^{KFN} f_{LMN}\{J_G, J_E, J_F\} \ . \end{aligned} \tag{3.4}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A, B, C\}$ denotes the symmetrized product of three generators.

Repeated commutators allow to generate the entire algebra whose elements are labeled by non-negative integer n. The generators obtain in this manner are n-local operators arising in (n-1) commutator of $J^{(1)}$:s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purposed of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exists also for continuum one-dimensional index).

Discrete one-dimensional lattice provides under certain consistency conditions a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} are given as

$$J^{(1)A} \quad = \quad f_{BC}^A \sum_{i<j} J_i^B J_j^C \quad . \tag{3.5}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N} = 4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned} \Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \ , \\ \Delta(J^{(1)A}) &= J^{(1)A} \otimes 1 + 1 \otimes J^{(1)A} + f_{BC}^A J^B \otimes J^C \ per, \end{aligned} \tag{3.6}$$

Δ allows to imbed Lie algebra to the tensor product in non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

2. Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N}=4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [27].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anticommutator of fermionic generators can contain besides identity also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrize tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad .$$

a and d representing the bosonic part of the algebra are $n\times n$ matrices and $m\times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n\times m$ and $m\times n$ matrices, whose anticommutator is the direct sum of $n\times n$ and $n\times n$ matrices. For $n=m$ bosonic generators transform like Lie algebra generators of $SU(n)\times SU(n)$ whereas fermionic generators transform like $n\otimes\overline{n}\oplus\overline{n}\otimes n$ under $SU(n)\times SU(n)$. Supertrace is defined as $Str(x)=Tr(a)-Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n\neq m$ the super trace condition removes identity matrix and $PU(n|m)$ and $SU(n|m)$ are same. That this does not happen for $n=m$ is an important delicacy since this case corresponds to $\mathcal{N}=4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as now physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R\otimes\overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'}J^{(1)C'} = g_{CC'}f^{C'}_{AB}\sum_{i<j}J_i^AJ_j^B \\ &= g_{CC'}f^{C'}_{AB}g^{AA'}g^{BB'}\sum_{i<j}J^i_{A'}J^j_{B'} \quad . \end{aligned} \tag{3.7}$$

Here $g_{AB}=Str(J_AJ_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

3. Generators of super-conformal Yangian symmetries

The explicit formula for the generators of super-conformal Yangian symmetries in terms of ordinary twistors is given by

$$
\begin{aligned}
j_B^A &= \sum_{i=1}^{n} Z_i^A \partial_{Z_i^B} \ , \\
j_B^{(1)A} &= \sum_{i<j} (-1)^C \left[Z_i^A \partial_{Z_j^C} Z_j^C \partial_{Z_j^B} \right] \ .
\end{aligned} \tag{3.8}
$$

This formula follows from completely general formulas for the Yangian algebra discussed above and allowing to express the dual generators $j_N^{(1)}$ as quadratic expression of j_N involving structures constants. In this rather sketchy formula twistors are ordinary twistors. Note however that in the recent case the lattice is replaced with its finite cutoff corresponding to the external particles of the scattering amplitude. This probably corresponds to the assumption that for the representations considered only finite number of lattice points correspond to non-trivial quantum numbers or to cyclic symmetry of the representations.

In the expression for the amplitudes the action of transformations is on the delta functions and by partial integration one finds that a total divergence results. This is easy to see for the linear generators but not so for the quadratic generators. A similar formula but with j_B^A and $j_B^{(1)A}$ interchanged applies in the representation of the amplitudes as Grassmann integrals using ordinary twistors. The verification of the generalization of Serre formula is also straightforward.

3.2 Twistors and momentum twistors and super-symmetrization

In [35] the basics of twistor geometry are summarized. Despote this it is perhaps good to collect the basic formulas here.

1. Conformally compactified Minkowski space

Conformally compactified Minkowksi space can be described as $SO(2,4)$ invariant (Klein) quadric

$$
T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0 \ . \tag{3.9}
$$

The coordinates (T, V, W, X, Y, Z) define homogenous coordinates for the real projective space RP^5. One can introduce the projective coordinates $X_{\alpha\beta} = -X_{\beta\alpha}$ through the formulas

$$
\begin{array}{lll}
X_{01} = W - V \ , & X_{02} = Y + iX \ , & X_{03} = \frac{i}{\sqrt{2}}T - Z \ , \\
X_{12} = -\frac{i}{\sqrt{2}}(T + Z) \ , & X_{13} = Y - iX \ , & X_{23} = \frac{1}{2}(V + W) \ .
\end{array} \tag{3.10}
$$

The motivation is that the equations for the quadric defining the conformally compactified Minkowski space can be written in a form which is manifestly conformally invariant:

$$
\epsilon^{\alpha\beta\gamma\delta} X_{\alpha\beta} X_{\gamma\delta} = 0 \ . \tag{3.11}
$$

The points of the conformally compactified Minkowski space are null separated if and only if the condition

$$
\epsilon^{\alpha\beta\gamma\delta} X_{\alpha\beta} Y_{\gamma\delta} = 0 \tag{3.12}
$$

holds true.

2. Correspondence with twistors and infinity twistor

One ends up with the correspondence with twistors by noticing that the condition is equivalent with the possibility to expression $X_{\alpha\beta}$ as

$$
X_{\alpha\beta} = A_{[\alpha} B_{\beta]} \ , \tag{3.13}
$$

where brackets refer to antisymmetrization. The complex vectors A and B define a point in twistor space and are defined only modulo scaling and therefore define a point of twistor space CP_3 defining a covering of 6-D Minkowski space with metric signature $(2,4)$. This corresponds to the fact that the Lie algebras of $SO(2,4)$ and $SU(2,2)$ are identical. Therefore the points of conformally compactified Minkowski space correspond to lines of the twistor space defining spheres CP_1 in CP_3.

One can introduce a preferred scale for the projective coordinates by introducing what is called infinity twistor (actually a pair of twistors is in question) defined by

$$I_{\alpha\beta} = \begin{pmatrix} \epsilon^{A'B'} & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{3.14}$$

Infinity twistor represents the projective line for which only the coordinate X_{01} is non-vanishing and chosen to have value $X_{01} = 1$.

One can define the contravariant form of the infinite twistor as

$$I^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{AB} \end{pmatrix} \ . \tag{3.15}$$

Infinity twistor defines a representative for the conformal equivalence class of metrics at the Klein quadric and one can express Minkowski distance as

$$(x-y)^2 \quad = \quad \frac{X^{\alpha\beta} Y_{\alpha\beta}}{I_{\alpha\beta} X^{\alpha\beta} I_{\mu\nu} Y^{\mu\nu}} \ . \tag{3.16}$$

Note that the metric is necessary only in the denominator. In twistor notation the distance can be expressed as

$$(x-y)^2 \quad = \quad \frac{\epsilon(A,B,C,D)}{\langle AB\rangle\langle CD\rangle} \ . \tag{3.17}$$

Infinite twistor $I_{\alpha\beta}$ and its contravariant counterpart project the twistor to its primed and unprimed parts usually denoted by $\mu^{A'}$ and λ^{A} and defined spinors with opposite chiralities.

4. Relationship between points of M^4 and twistors

In the coordinates obtained by putting $X_{01} = 1$ the relationship between space-time coordinates $x^{AA'}$ and $X^{\alpha\beta}$ is

$$X_{\alpha\beta} = \begin{pmatrix} -\frac{1}{2}\epsilon^{A'B'}x^2 & -ix^{A'}_{\ B} \\ ix_{A}^{\ B'} & \epsilon_{A,B} \end{pmatrix} \ , \quad X^{\alpha\beta} = \begin{pmatrix} \epsilon_{A'B}\ {}' x^2 & -ix_{A'}^{\ B} \\ ix^{A}_{B'} & -\frac{1}{2}\epsilon^{AB}x^2 \end{pmatrix} \ , \tag{3.18}$$

If the point of Minkowski space represents a line defined by twistors (μ_U, λ_U) and (μ_V, λ_V), one has

$$x^{AC'} = i\frac{(\mu_V \lambda_U - \mu_U \lambda_V)^{AC'}}{\langle UV\rangle} \tag{3.19}$$

The twistor μ for a given point of Minkowski space in turn is obtained from λ by the twistor formula by

$$\mu^{A'} \quad = \quad -ix^{AA'}\lambda_A \ . \tag{3.20}$$

5. Generalization to the super-symmetric case

This formalism has a straightforward generalization to the super-symmetric case. CP_3 is replaced with $CP_{3|4}$ so that Grassmann parameters have four components. At the level of coordinates this means the replacement $[W_I] = [W_\alpha, \chi_\alpha]$. Twistor formula generalizes to

$$\mu^{A'} = -ix^{AA'}\lambda_A \quad , \quad \chi_\alpha = \theta_\alpha^A \lambda_A \quad . \tag{3.21}$$

The relationship between the coordinates of chiral super-space and super-twistors generalizes to

$$(x,\theta) \quad = \quad \left(i\frac{(\mu_V\lambda_U - \mu_U\lambda_V)}{\langle UV\rangle}, \frac{(\chi_V\lambda_U - \chi_U\lambda_V)}{\langle UV\rangle} \right) \tag{3.22}$$

The above summaried formulas can be applied to super-symmetric variants of momentum twistors to deduce the relationship between region momenta x assigned with edges of polygons and twistors assigned with the ends of the light-like edges. The explicit formulas are represented in [35]. The geometric picture is following. The twistors at the ends of the edge define the twistor pair representing the region momentum as a line in twistor space and the intersection of the twistor lines assigned with the region momenta define twistor representing the external momenta of the graph in the intersection of the edges.

6. Basic kinematics for momentum twistors

The supersymmetrization involves replacement of multiplets with super-multiplets

$$\Phi(\lambda,\tilde{\lambda},\eta) = G^+(\lambda,\tilde{\lambda}) + \eta_i\Gamma^a\lambda,\tilde{\lambda}) + \; + \epsilon_{abcd}\eta^a\eta^b\eta^c\eta^d G^-(\lambda,\tilde{\lambda}) \quad . \tag{3.23}$$

Momentum twistors are dual to ordinary twistors and were introduced by Hodges. The light-like momentum of external particle a is expressed in terms of the vertices of the closed polygon defining the twistor diagram as

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu = \lambda_i\tilde{\lambda}_i \quad , \quad \theta_i - \theta_{i+1} = \lambda_i\eta_i \quad . \tag{3.24}$$

One can say that massless momenta have a conserved super-part given by $\lambda_i\eta_i$. The dual of the super-conformal group acts on the region momenta exactly as the ordinary conformal group acts on space-time and one can construct twistor space for dua region momenta.

Super-momentum conservation gives the constraints

$$\sum p_i = 0 \quad , \quad \sum \lambda_i\eta_i = 0 \quad . \tag{3.25}$$

The twistor diagrams correspond to polygons with edges with lines carrying region momenta and external massless momenta emitted at the vertices.

This formula is invariant under overall shift of the region momenta x_a^μ. A natural interpretation for x_a^μ is as the momentum entering to the the vertex where p_a is emitted. Overall shift would have interpretation as a shift in the loop momentum. x_a^μ in the dual coordinate space is associated with the line $Z_{a-1}Z_a$ in the momentum twistor space. The lines $Z_{a-1}Z_a$ and Z_aZ_{a+1} intersect at Z_a representing a light-like momentum vector p_a^μ.

The brackets $\langle abcd\rangle \equiv \epsilon_{IJKL}Z_a^IZ_b^JZ_c^KZ_d^L$ define fundamental bosonic conformal invariants appearing in the tree amplitudes as basic building blocks. Note that Z_a define points of 4-D complex twistor space to be distinguished from the projective twistor space CP_3. Z_a define projective coordinates for CP_3 and one of the four complex components of Z_a is redundant and one can take $Z_a^0 = 1$ without a loss of generality.

3.3 The general form of Grassmannian integrals

The basic object of study are the leading singularities of color-stripped n-particle N^{k}MHV amplides, where k denotes the number of loops and MHV refers to maximal helicity violating amplitudes with 2 negative helicity gluons. The discovery is that these singularities are expressible in terms

Yangian invariants $Y_{n,k}(Z_1,...,Z_n)$, where Z_i are super-twistors, are defined by residue integrals over the compact $nk-1$-dimensional complex space $G(n,k) = U(n)/U(k)\times U(n-k)$ of k-planes of complex n-dimensional space. n is the number of external massless particles, k is the number loops, and Z_a, $i = 1,...,n$ denotes the projective 4-coordinate of the super-variant $CP^{3|4}$ of the momentum twistor space CP_3 assigned to the massless external particles is following. $Gl(n)$ acts as linear transformations in the n-fold Cartesian power of twistor space.

Yangian invariant $Y_{n,k}$ is a function of twistor variables Z^a having values in super-variant $CP_{3|3}$ of momentum twistor space CP_3 assigned to the massless external particles being simple algebraic functions of the external momenta. It is also possible to define Yangian invariants $L_{n,k+2}(W_1,...,W_n)$ by using ordinary twistors W_a and identical defining formula. The two invariants are related by the formula $L_{n,k+2}(W_1,...,W_n) = M^{tree}_{MHV}\times Y_{n,k}(Z_1,...,Z_n)$. Here M^{tree}_{MHV} is the tree contribution to the maximally helicity violating amplitude for the scattering of n-particles: recall that these amplitudes contain two negative helicity gluons whereas the amplitudes containing a smaller number of them vanish [28]. The equivalence between the descriptions based on ordinary and momentum twistors states the dual conformal invariance of the amplitudes implying Yangian symmetry. It has been conjectured that Grassmannian integrals generate all Yangian invariants.

The formulas for the Grassmann integrals for twistors and momentum twistors are given by following expressions.

1. The integrals $Y_{n,k}(W_1,...,W_n)$ in the description based on ordinary twistors correspond to $k-2$ loops and are defined as

$$\begin{aligned} Y_{n,k}(W_1,...,W_n) &= \frac{1}{Vol(GL(2))}\int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)}\times \\ &\times \prod_{\alpha=1}^{k} d^{4|4}Y_\alpha \prod_{i=1}^{n}\delta^{4|4}(W_i - C_{\alpha i}Y_\alpha) \ . \end{aligned} \tag{3.26}$$

   Here $C_{\alpha a}$ denote the $n\times k$ coordinates used to parametrize the points of $G_{k,n}$.

2. The integrals $Y_{n,k}(Z_1,...,Z_n)$ in the description based on momentum twistors correspond to k loops and are defined as

$$Y_{n,k}(Z_1,...,Z_n) = \frac{1}{Vol(GL(k))}\times\int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)}\times\prod_{\alpha=1}^{k}\delta^{4|4}(C_{\alpha a}Z_a) \ . \tag{3.27}$$

   The possibility to select $Z_a^0 = 1$ implies $\sum_k C_{\alpha k} = 0$ allowing to eliminate $C_{\alpha n}$ so that the actual number of coordinates Grassman coordinates is $nk-1$.

The $4k$ delta functions reduce the number of integration variables of contour integrals from nk to $(n-4)k$ in the bosonic sector (the definition of delta functions involves some delicacies not discussed here). The n quantities $(m,...m+k)$ are $k\times k$-determinants defined by subsequent columns from m to $m+k-1$ of the $k\times n$ matrix defined by the coordinates $C_{\alpha a}$ and correspond geometrically to the k-volumes of the k-dimensional parallel-pipeds defined by these column vectors. The fact that the scalings of twistor space coordinates Z_a can be compensated by scalings of $C_{\alpha a}$ deforming integration contour but leaving the residue integral invariant so that the integral depends on projective twistor coordinates only.

Since the integrand is a rational function, a multi-dimensional residue calculus allows to deduce the values of these integrals as residues associated with the poles of the integrand in a recursive manner. The poles correspond to the zeros of the $k\times k$ determinants appearing in the integrand or equivalently to singular lower-dimensional parallel-pipeds. It can be shown that local residues are determined by $(k-2)(n-k-2)$ conditions on the determinants in both cases. The value of the integral depends on the explicit choice of the integration contour for each variable $C_{\alpha a}$ left when delta functions are taken into account. The condition that a correct form of tree amplitudes is obtained fixes the choice of the integration contours.

For the ordinary twistors W the residues correspond to projective configurations in CP_{k-1}, or more precisely in the space $CP_{k-1}^n/Gl(k)$, which is $(k-1)n-k^2$-dimensional space defining the support for the residues integral. CP_{k-1} comes from the fact that C_{α_k} are projective coordinates: the amplitudes are indeed invariant under the scalings $W_i \rightarrow t_i W_i$, $C_{\alpha i} \rightarrow tC_{\alpha i}$. The coset space structure comes from the fact that $Gl(k)$ is a symmetry of the integrand acting as $C_{\alpha i} \rightarrow \Lambda_\alpha^{\ \beta} C_{\beta_i}$. The reduction to CP_{k-1} implies the reduction of the support of the integral to line in the case of MHV amplitudes and to plane in the case of NMHV as one sees from the expression $d\mu = \prod_\alpha d^{4|4}Y_\alpha \prod_{i=1}^n \delta^{4|4}(W_i - C_{\alpha i}Y_\alpha)$. For $(i_1, ..., i_k) = 0$ the vectors $i_1, ..i_k$ belong to $k-2$-dimensional plane of CP_{k-1}. In the case of $NMHV$ (N^2MHV) amplitudes this translates at the level of twistors to the condition that the corresponding twistors $\{i_1, i_2, i_3\}$ $(\{i_1, i_2, i_3, i_4\})$ are collinear (in the same plane) in twistor space. This can be understood from the fact that the delta functions in $d\mu$ allow to express W_i in terms of $k-1$ Y_α:s in this case.

The action of conformal transformations in twistor space reduces to the linear action of $SU(2,2)$ leaving invariant Hermitian sesquilinear form of signature $(2,2)$. Therefore the conformal invariance of the Grassmannian integral and its dual variant follows from the possibility to perform a compensating coordinate change for $C_{\alpha a}$ and from the fact that residue integral is invariant under small deformations of the integration contour. The above described relationship between representations based on twistors and momentum twistors implies the full Yangian invariance.

3.4 Canonical operations for Yangian invariants

There are several operations that one can perform for Yangian invariants $Y_{n,k}$ and all these operations appear in the recursion formula for planar all loop amplitudes. These operations are described in [34] much better than I could do it so that I will not go to any details. It is possible to add and remove particles, to fuse two Yangian invariants, to merge particles, and to construct from two Yangian invariants a higher invariant containing so called BCFW bridge representing single particle exchange using only twistorial methods.

1. Inverse soft factors add to the diagam a massless collinear particles between particles a and b and by definition one has

$$O_{n+1}(a, c, b, ...) = \frac{\langle ab \rangle}{\langle ac \rangle \langle cb \rangle} O_n(a'b') \ . \tag{3.28}$$

   At the limit when the momentum of the added particle vanishes both sides approach the original amplitude. The right-handed spinors and Grassmann parameters are shifted

$$\begin{aligned} \tilde{\lambda}'_a &= \tilde{\lambda}_a + \frac{\langle cb \rangle}{\langle ab \rangle} \tilde{\lambda}_c \ , \quad \tilde{\lambda}'_b = \tilde{\lambda}_b + \frac{\langle ca \rangle}{\langle ba \rangle} \tilde{\lambda}_c \ , \\ \eta'_a &= \eta_a + \frac{\langle cb \rangle}{\langle ab \rangle} \eta_c \ , \quad \eta'_b = \eta_b + \frac{\langle ca \rangle}{\langle ba \rangle} \eta_c \ . \end{aligned} \tag{3.29}$$

   (a) The addition of particle leaving the number k of loops unchanged means just the re-interpretation

$$Y'_{n,k}(Z_1, ..., Z_{n-1}, Z_n) = Y_{n-1,k}(Z_1, ..., Z_{n-1}) \tag{3.30}$$

   without actual dependence on Z_n. There is however a dependence on the momentum of the added particle since the relationship between momenta and momentum twistors is modified by the addition obtained by applying the basic rules relating region super momenta and momentum twistors (light-like momentum determines λ_i and twistor equations for x_i and λ_i, η_i determine (μ_i, χ_i)) is exrpessible assigned to the external particles [37]. Modifications are needed only for the new vertex and its neighbours.

(b) The addition of a particle increasing the number of loops with single unit is a more complex operation which can be understood in terms of a residue of $Y_{n,k}$ proportional to $Y_{n-1,k-1}$ and Yangian invariant $[z_1...z_5]$ with five arguments constructed from basic Yangian invariants with four arguments. The relationship between the amplitudes is now

$$Y'_{n,k}(..,Z_{n-1}Z_n,Z_1...) = [n-2\ n-1\ n\ 1\ 2]\times Y_{n-1,k-1}(...\hat{Z}_{n-1},\hat{Z}_1,...) \ . \quad (3.31)$$

Here

$$[abcde] = \frac{\delta^{0|4}(\eta_a\langle bcde\rangle + cyclic)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle} \ . \quad (3.32)$$

Here $\langle abcd\rangle$ are the fundamental conformal invariants associate with four twistors defined in terms of the permutation symbol. $\langle abcde\rangle$ is the fundamental R-invariant appearing in one loop corrections of MHV amplitudes.
$\hat{Z}_{n-1},\hat{Z}_1$ are deformed momentum twistor variables. The deformation is determined from the relationship between external momenta, region momenta and momentum twistor variables. $\hat{Z}^1$ is the intersection of the the line (12) with the plane $(n-2\ n-1\ 2)$- briefly $\hat{Z}^1 = (n-2\ n-1\ 2)\cap(12)$ $\hat{Z}^{n-1}$ is the intersection of the the line $(n-2\ n-1)$ with the plane $(12n)$, or $\hat{Z}^1 = (12n)\cap(n-2\ n-1)$. The interpretation for the intersections at the level of ordinary Feynman diagrams is in terms of the collineary of the four-momenta involved with the underlying box diagram with parallel on mass shell particles. These result from unitary conditions obtained by putting maximal number of loop momenta on mass shell to give the leading singularities. The explicit expressions for the momenta are

$$\begin{aligned} \hat{Z}^1 &= Z_1\langle 2\ n-2\ n-1\ n\rangle + Z_2\langle n-2\ n-1\ n\ 1\rangle \ , \\ \hat{Z}^{n-1} &= Z_{n-2}\langle n-2\ n-1\ n\ 2\rangle + Z_{n-1}\langle n\ 1\ 2\ n-2\rangle \ , \end{aligned} \quad (3.33)$$

2. Particles can be also removed. The first manner to remove particle is by integrating over the twistor variable characterizing the particle. This reduces also the number of loops by one unit. Merge operation preserves the number of loops but removes a particle particle by identifying the twistor variables of neighboring particles. This operation corresponds to an integral over on mass shell loop momentum at the level of tree diagrams and by Witten's half Fourier transform can be transformed to twistor integral.

3. The product

$$Y'(Z_1,...Z_n) = Y_1(Z_1,...Z_m)\times Y_2(Z_{m+1},...Z_n) \quad (3.34)$$

of two Yangian invariants is again a Yangian invariant. This is not quite trivial since the dependence of region momenta and momentum twistors on the momenta of external particles makes the operation non-trivial.

Merge operation allows to construct more interesting invariants from the products of Yangian invariants. One begins from a product of Yangian invariants (Yangian invariant trivially) represented cyclically as points of circle and identifies the last twistor argument of given invariant with the first twistor argument of the next invariant and performs integrals over the momentum twistor variables appearing twice. The soft k-increasing and preserving operations can be described also in terms of this operation for Yangian invariants such that the second invariant corresponds to 3-vertex. The cyclic merge operation applied to four MHV amplitudes gives NMHV amplitudes associated with on mass shell momenta in box diagrams. By applying similar operation to NMHV amplitudes and MHV amplitues one obtains 2-loop amplitudes. In [34] examples about these operations are described.

4. BCFW bridge allows to build general tree diagrams from MHV tree diagrams [29]. At the level of Feynman diagrams it corresponds to a box diagram containing general MHV tree diagrams labeled by L and R and MHV and $\overline{MHV}$ 3-vertices ($\overline{MHV}$ 3-vertex allows expression in terms of MHV diagrams) with the lines of the box on mass shell so that the three momenta emanating from the vertices are parallel and give rise to a one-loop leading singularity.

   At the level of Feynman diagrams BCFW bridge corresponds to so called "two-mass hard" leading singularities associated with box diagrams with light-like momenta at the four lines of the diagram [33]. The motivation for the study of these diagrams comes from the hypothesis the leading order singularities obtained by putting as many particles as possible on mass shell contain the data needed to construct scattering amplitudes of $\mathcal{N} = 4$ SYM completely. This representation of the leading singularities generalizes to arbitrary loops. The recent article is a continuation of this program to planar amplitudes.

   Also BCFW bridge allows an interpretation as a particular kind fusion for Yang invariants and involves all the basic operations. One starts from the amplitudes $Y^L_{n_L,k_L}$ and $Y^R_{n_R,k_R}$ and constructs an amplitude $Y'_{n_L+n_R,k_L+k_R+1}$ representing the amplitude which would correspond to a generalization of the MHV diagrams with the two tree diagrams connected by the MHV propagator (BFCW bridge) replaced with arbitrary loop diagrams. Particle "1" *resp.* "j+1" is added by the soft k-increasing factor to Y_{n_L+1,k_L+1} *resp.* Y_{n_R+1,k_R+1} giving amplitude with $n+2$ particles and two more loops. The subsequent operations must remove one loop. First repeated "1" and "j+1" are identified with their copies by loop conserving merge operation, and then one performs an integral over the twistor variable associated with the internal line obtained and reducing the number of loops by on unit.

5. An especially interesting operation is the one which removes two particles with opposite four-momenta and involves integration over two twistor variables Z_a and Z_b with the additional assumption that the momenta are opposite. This operation allows addition of a loop to a given amplitude and this is the king idea idea of the approach. The line Z_aZ_b represents loop momentum on one hand and the dual x-coordinate identified as momentum propagating along the line on the other hand. The integration over these variables is equivalent to an integration over loop momentum as the explicit calculation of [34] demonstrates. If the integration contours are products in the product of twistor spaces associated with a and b the and gives lower order Yangian invariant as answer. It is also possible to choose the integration contour to be entangled in the sense that it cannot be reduced to a product of integration contours in the product of twistor spaces. In the entangled case the integration gives a loop integral. In the removal operation Yangian invariance can be broken by IR singularities associated with the integration contour and the procedure does not produce genuine Yangian invariant always.

   This operation is used to increase the number of loops by one. The physical picture is that one starts from say 1-loop amplitude and cuts one line by assigning to the pieces of the line opposite light-like momenta having interpretation as incoming and outgoing particles. The resulting amplitude is called forward limit. The only reasonable interpretation seems to be that the loop integration is expressed by unitarity as forward limit meaning cutting of the line carrying the loop momentum. This operation can be expressed in a manifestly Yangian invariant way as entangled removal of two particles with the merge operation meaning the replacement $Z_n \rightarrow Z_{n-1}$. Particle $n+1$ is added adjacent to A, B as a k-increasing inverse soft factor and then A and B are removed by entangled integration, and after this merge operation identifies $n+1$ and 1.

3.5 Limitations of the approach

The following comments are an attempt to summarize my far from complete understanding about what is involved with the representation as contour integrals.

Consider first the limitations of the approach.

1. The basis idea is that the representation for tree amplitudes generalizes to loop amplitudes. On other words, the amplitude defined as a sum of Yangian invariants expressed in terms of Grassmann integrals represents the sum of loops up to some maximum loop number. The problem is here that shifts of the loop momenta are essential in the UV regularization procedure.

Fixing the coordinates $x_1, ...x_n$ having interpretation as momenta associated with lines in the dual coordinate space allows to eliminate the non-uniqueness due to the common shift of these coordinates.

2. It is not however not possible to identify loop momentum as a loop momentum common to different loop integrals unless one restricts to planar loops. Non-planar diagrams are obtained from a planar diagram by permuting the coordinates x_i but this means that the unique coordinate assignment is lost. Therefore the representation of loop integrands as Grassmann integrals makes sense only for planar diagrams. From TGD point of view one could argue that this is one good reason for restricting the loops so that they are for on mass shell particles with non-parallel on mass shell four-momenta and possibly different sign of energies for given wormhole contact representing virtual particle.

3. IR regularization is needed even in $\mathcal{N} = 4$ for SYM given by "moving out on the Coulomb branch theory" so that IR singularities remain the problem of the theory.

3.6 What is done?

The article proposes a generalization of the BCFW recursion relation for tree diagrams of $\mathcal{N} = 4$ for SYM so that it applies to planar diagrams with a summation over an arbitrary number of loops.

1. The basic goal of the article is to generalize the recursion relations of tree amplitudes so that they would apply to loop amplitudes. The key idea is following. One can formally represent loop integrand as a contour integral in complex plane whose coordinate parameterizes the deformations $Z_n \rightarrow Z_n + \epsilon Z_{n-1}$ and re-interpret the integral as a contour integral with oppositely oriented contour surrounding the rest of the complex plane which can be imagined also as being mapped to Riemann sphere. What happens only the poles which correspond to lower number of loops contribute this integral. One obtains a recursion relation with respect to loop number. This recursion seems to be the counterpart for the recursive construction of the loops corrections in terms of absorptive parts of amplitudes with smaller number of loop using unitarity and analyticity.

2. The basic challenge is to deduce the Grassmann integrands as Yangian invariants. From these one can deduce loop integrals by integration over the four momenta associated with the lines of the polygonal graph identifiable as the dual coordinate variables x_a. The integration over loop momenta can induce infrared divergences breaking Yangian symmetry. The big idea here is that the operations described above allow to construct loop amplitudes from the Yangian invariants defining tree amplitudes for a larger number of particles by removing external particles by fusing them to form propagator lines and by using the BCFW bridge to fuse lower-dimensional invariants. Hence the usual iterative procedure (bottom-up) used to construct scattering amplitudes is replaced with a recursive procedure (top-down). Of course, once lower amplitudes has been constructed they can be used to construct amplitudes with higher particle number.

3. The first guess is that the recursion formula involves the same lower order contributions as in the case of tree amplitudes. These contributions have interpretation as factorization of channels involving single particle intermediate states. This would however allow to reduce loop amplitudes to 3-particle loop amplitudes which vanish in $\mathcal{N} = 4$ SYM by the vanishing of coupling constant renormalization. The additional contribution is necessary and corresponds to a source term identifiable as a "forward limit" of lower loop integrand. These terms are obtained by taking an amplitude with two additional particles with opposite four-momenta and forming a state in which these particles are entangled with respect to momentum and other quantum numbers. Entanglement means integral over the massless momenta on one hand. The insertion brings in two momenta x_a and x_b and one can imagine that the loop is represented by a branching of propagator line. The line representing the entanglement of the massless states with massless momentum define the second branch of the loop. One can of course ask whether only massless momentum in the second branch. A possible interpretation is that this state is expressible by unitarity in terms of the integral over light-like momentum.

4. The recursion formula for the loop amplitude $M_{n,k,l}$ involves two terms when one neglects the possibility that particles can also suffer trivial scattering (cluster decomposition). This term basically corresponds to the Yangian invariance of n arguments identified as Yangian invariant of $n-1$ arguments with the same value of k.

   (a) The first term corresponds to single particle exchange between particle groups obtained by splitting the polygon at two vertices and corresponds to the so called BCFW bridge for tree diagrams. There is a summation over different splittings as well as a sum over loop numbers and dimensions k for the Grassmann planes. The helicities in the two groups are opposite.

   (b) Second term is obtained from an amplitude obtained by adding of two massless particles with opposite momenta and corresponds to $n+2, k+1, l-1$. The integration over the light-like momentum together with other operations implies the reduction $n+2 \rightarrow n$. Note that the recursion indeed converges. Certainly the allowance of added zero energy states with a finite number of particles is necessary for the convergence of the procedure.

4 Could the Grassmannian program be realized in TGD framework?

In the following the TGD based modification of the approach based on zero energy ontology is discussed in some detail. It is found that the approach leading to discretization of mass squared for virtual particles and even the discretization of momenta are consistent with Grassmannian approach in the simple case considered and allow to get rid of IR divergences. Also the possibility that the number of generalized Feynman diagrams contributing to a given scattering amplitude is finite so that the recursion formula for the scattering amplitudes would involve only a finite number of steps (maximum number of loops) is considered. One especially promising feature of the residue integral approach with discretized virtual momenta is that it makes sense also in the p-adic context in the simple special case discussed since residue integral reduces to momentum integral (summation) and lower-dimensional residue integral.

4.1 What Yangian symmetry could mean in TGD framework?

The loss of the Yangian symmetry in the integrations over the coordinates x^a ($p^a = x^{a+1} - x^a$) assigned to virtual momenta seems to be responsible for many ugly features. It is basically the source of IR divergences regulated by "moving out on the Coulomb branch theory" so that IR singularities remain the problem of the theory. This raises the question whether the loss of Yangian symmetry is the signature for the failure of QFT approach and whether the restriction of loop momentum integrations to avoid both kind of divergences might be a royal road beyond QFT. In TGD framework zero energy ontology indeed leads to to a concrete proposal based on the vision that virtual particles are something genuinely real.

The detailed picture is of course far from clear but to get an idea about what is involved one can look what kind of assumptions are needed if one wants to realize the dream that only a finite number of generalized Feynman diagrams contribute to a scattering amplitude which is Yangian invariant allowing a description using a generalization of the Grassmannian integrals.

1. Assume the bosonic emergence and its super-symmetric generalization holds true. This means that incoming and outgoing states are bound states of massless fermions but the fermions can opposite directions of three-momenta making them massive. Incoming and outgoing particles would consist of fermions associated with wormhole throats and would be characterized by a pair of twistors in the general situation and in general massive. This allows also string like mass squared spectrum for bound states having fermion and antifermion at the ends of the string as well as more general n-particle bound states. Hence one can speak also about the emergence of string like objects. For virtual particles the fermions would be massive and have discrete mass spectrum. Also super partners containing several collinear fermions and antifermions at given throat are possible. Collinearity is required by the generalization of SUSY. The construction of these states bring strongly in mind the merge procedure involving the replacement $Z^{n+1} \rightarrow Z^n$.

2. The description in terms of twistors would require a modification of the description for adding loops for both members of the pair AB.

   (a) The challenge is to understand how mass shell condition affects the description in terms of twistor pairs $Z_A Z_B$ assigned with the pair of light-like momenta. The identification of x with the momentum associated with the generalized line of Feynman diagram is very natural and the discrete mass spectrum means that x is restricted to discrete mass shells. This identification is indeed consistent with the identification $x_{a+1} - x_a = p_a$ with p_a representing light-like four-momentum.

   (b) In [34] the $M_{6,4,l=0}(1234AB)$ the integration over twistor variables z_A and z_B using "entangled" integration contour leads to 1-loop MHV amplitude N^pMHV, $p = 1$. The parametrization of the integration contour is $z_A = (\lambda_A, x\lambda_A)$, $z_B = (\lambda_B, x\lambda_B)$, where x is the M^4 coordinate representing the loop momentum. This boils down to an integral over $CP_1 \times CP_1 \times M^4$ [34]. The integrals over spheres CP_1s are contour integrals so that only an ordinary integral over M^4 remains.

   (c) The obvious implication of the restriction of x on mass shells is the absence of IR divergences and one might hope that under suitable assumptions one achieves Yangian invariance. The first question is of course whether the required restriction of x to mass shells in z_A and z_B or possibly even algebraic discretization of momenta is consistent with the Yangian invariance. This seems to be the case: the integration contour reduces to entangled integration contour in $CP_1 \times CP_1$ not affected by the discretization and the resulting loop integral differs from the standard one by the discretization of masses and possibly also momenta with massless states excluded. Whether Yangian invariance poses also conditions on mass and momentum spectrum is an interesting question.

3. One can consider also the possibility that the incoming and outgoing particles - to be distinguished from massless fermions appearing as their building blocks- have actually small masses presumably related to the IR cutoff defined by the size scale of the largest causal diamond involved. p-Adic thermodynamics could be responsible for this mass. Also the binding of the wormhole throats can give rise to a small contribution to vacuum conformal weight possibly responsible for gauge boson masses. This would imply that a given n-particle state can decay to N-particle states for which N is below some limit. The fermions inside loops would be also massive. This allows to circumvent the IR singularities due to integration over the phase space of the final states (say in Coulomb scattering).

4. The representation of the off mass shell particles as pairs of wormhole throats with non-parallel four-momenta makes sense and that the particles in question are on mass shell with mass squared being proportional to inverse of a prime number as the number theoretic vision applied to the modified Dirac equation suggests. On mass shell property poses extremely powerful constraints on loops and when the number of the incoming momenta in the loop increases, the number of constraints becomes larger than the number of components of loop momentum for the generic values of the external momenta. Therefore there are excellent hopes of getting rid of UV divergences.

   A stronger assumption encouraged by the classical space-time picture about virtual particles is that the 3-momenta associated with throats of the same wormhole contact are always in same or opposite directions. Even this allows to have virtual momentum spectrum and non-trivial mass spectrum for them assuming that the three momenta are opposite.

5. The best that one can hope is that only a finite number of generalized Feynman diagrams contributes to a given reaction. This would guarantee that amplitudes belong to a finite-dimensional algebraic extension of rational functions with rational coefficients since finite sums do not lead out from a finite algebraic extension of rationals. The first problem are self energy corrections. The assumption tht the mass non-renormalization theorems of SUSYs generalize to TGD framework would guarantee that the loops contributing to fermionic propagators (and their super-counterparts) do not affect them. Also the iteration of more complex amplitudes as analogs of ladder diagrams representing sequences of reactions $M \rightarrow M_1 \rightarrow M_2 ..... \rightarrow N$ such that at each M_n in the sequence can appear as on mass shell state could give a non-vanishing contribution

to the scattering amplitude and would mean infinite number of Feynman diagrams unless these amplitudes vanish. If N appears as a virtual state the fermions must be however massive on mass shell fermions by the assumption about on-mass shell states and one can indeed imagine a situation in which the decay $M \rightarrow N$ is possible when N consists of states made of massless fermions is possible but not when the fermions have non-vanishing masses. This situation seems to be consistent with unitarity. The implication would be that the recursion formula for the all loop amplitudes for a given reaction would give vanishing result for some critical value of loops.

Already these assumptions give good hopes about a generalization of the Grassmann approach to TGD framework. Twistors are doubled as are also the Grassmann variables and there are wave functions correlating the momenta of the the fermions associated with the opposite wormhole throats of the virtual particles as well as incoming gauge bosons which have suffered massivation. Also wave functions correlating the massless momenta at the ends of string like objects and more general many parton states are involved but do not affect the basic twistor formalism. The basic question is whether the hypothesis of unbroken Yangian symmetry could in fact imply something resembling this picture. The possibility to discretize integration contours without losing the representation as residue integral quite generally is basic prerequisite for this and should be shown to be true.

4.2 How to achieve Yangian invariance without trivial scattering amplitudes?

In $\mathcal{N} = 4$ SYM the Yangian invariance implies that the MHV amplitudes are constant as demonstrated in [34]. This would mean that the loop contributions to the scattering amplitudes are trivial. Therefore the breaking of the dual super-conformal invariance by IR singularities of the integrand is absolutely essential for the non-triviality of the theory. Could the situation be different in TGD framework? Could it be possible to have non-trivial scattering amplitudes which are Yangian invariants. Maybe! The following heuristic argument is formulated in the language of super-twistors.

1. The dual conformal super generators of the super-Lie algebra $U(2,2)$ acting as super vector fields reducing effectively to the general form $J = \eta_a^K \partial/\partial Z_a^J$ and the condition that they annihilate scattering amplitudes implies that they are constant as functions of twistor variables. When particles are replaced with pairs of wormhole throats the super generators are replaced by sums $J_1 + J_2$ of these generators for the two wormhole throats and it might be possible to achieve the condition

$$(J_1 + J_2)M = 0 \tag{4.1}$$

with a non-trivial dependence on the momenta if the super-components of the twistors associated with the wormhole throats are in a linear relationship. This should be the case for bound states.

2. This kind of condition indeed exists. The condition that the sum of the super-momenta expressed in terms of super-spinors λ reduces to the sum of real momenta alone is not usually posed but in the recent case it makes sense as an additional condition to the super-components of the the spinors λ associated with the bound state. This quadratic condition is exactly of the same general form as the one following from the requirement that the sum of all external momenta vanishes for scattering amplitude and reads as

$$X = \lambda_1\eta_1 + \lambda_2\eta_2 = 0 \ . \tag{4.2}$$

The action of the generators $\eta_1\partial_{\tilde{\lambda}_1} + \eta_2\partial_{\tilde{\lambda}_2}$ forming basic building blocks of the super generators on $p_1 + p_2 = \lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2$ appearing as argument in the scattering amplitude in the case of bound states gives just the quantity X, which vanishes so that one has super-symmetry. The generalization of this condition to n-parton bound state is obvious.

3. The argument does not apply to free fermions which have not suffered topological condensation and are therefore represented by CP_2 type vacuum extremal with single wormhole throat. If one accepts the weak form of electric-magnetic duality, one can circumvent this difficulty. The free fermions carry Kähler magnetic charge whereas physical fermions are accompanied by a bosonic wormhole throat carrying opposite Kähler magnetic charge and opposite electroweak isospin so that a ground state of string like object with size of order electroweak length scale is in question. In the case of quarks the Kähler magnetic charges need not be opposite since color confinement could involve Kähler magnetic confinement: electro-weak confinement holds however true also now. The above argument generalizes as such to the pairs formed by wormhole throats at the ends of string like object. One can of course imagine also more complex hybrids of these basic options but the general idea remains the same.

Note that the argument involves in an essential manner non-locality , which is indeed the defining property of the Yangian algebra and also the fact that physical particles are bound states. The massivation of the physical particles brings in the IR cutoff.

4.3 Number theoretical constraints on the virtual momenta

One can consider also further assumptions motivated by the recent view about the generalized eigenvalues of Chern-Simons Dirac operator having interpretation as pseudo-momentum. The details of this view need not of course be final.

1. Assume that the pseudo-momentum assigned to fermion lines by the modified Dirac equation [16] corresponds to the actual four-momentum associated with the line of the generalized Feynman diagram. That this assumption makes sense is far from obvious but conforms with the assumption that incoming particles are build out of massless fermions. It also implies that the propagators are massless propagators as required by twistorialization and Yangian generalization of super-conformal invariance.

2. Since (pseudo)-mass squared is number theoretically quantized as the length of a hyper-complex prime in preferred plane M^2 of momentum space fermionic propagators are massless propagators with virtual masses restricted on discrete mass shells. Lorentz invariance suggests that M^2 cannot be common to all particles but corresponds to preferred reference frame for the virtual particle having interpretation as plane spanned by the quantization axes of energy and spin.

3. Hyper-complex primeness means also the quantization of momentum components so that one has hyper-complex primes of form $\pm((p+1)/2, \pm(p-1)/1)$ corresponding to mass squared $M^2 = p$ and hypercomplex primes $\pm(p, 0)$ with mass squared $M2 = p^2$. Space-like fermionic momenta are not needed since for opposite signes of energy wormhole throats can have space-like net momenta. If they are allowed/needed they could correspond to space-like hyper-complex primes $\pm((p-1)/2, \pm(p+1)/1)$ and $\pm(0, p)$ so that one would obtains also discretization of space-like mass shells also. The number theoretical mass squared is proportional to p, whereas p-adic mass squared is proportional to $1/p$. For p-adic mass calculations canonical identification $\sum x_n p^n$ maps p-adic mass squared to its real counterpart. The simplest mapping consistent with this would be $(p_0, p_1) \to (p_0, p_1)/p$. This could be assumed from the beginning in real context and would mean that the mass squared scale is proportional to $1/p$.

4. Lorentz invariance requires that the preferred coordinate system in which this holds must be analogous to the rest system of the virtual fermion and thus depends on the virtual particle. In accordance with the general vision discussed in [16] Lorentz invariance could correspond to a discrete algebraic subgroup of Lorentz group spanned by transformation matrices expressible in terms of roots of unity. This would give a discrete version of mass shell and the preferred coordinate system would have a precise meaning also in the real context. Unless one allows algebraic extension of p-adic numbers p-adic mass shell reduces to the set of above number-theoretic momenta. For algebraic extensions of p-adic numbers the same algebraic mass shell is obtained as in real correspondence and is essential for the number theoretic universality. The interpretation for the algebraic discretization would be in terms of a finite measurement resolution. In real context this would mean discretization inducing a decomposition of the

mass shell to cells. In the p-adic context each discrete point would be replaced with a p-adic continuum. As far as loop integrals are considered, this vision means that they make sense in both real and p-adic context and reduce to summations in p-adic context. This picture is discussed in detail in [16].

5. Concerning p-adicization the beautiful aspect of residue integral is that it makes sense also in p-adic context provided one can circumvent the problems related to the identification of p-adic counterpart of π requiring infinite-dimensional transcendental extension coming in powers of π. Together with the discretization of both real and virtual four-momenta this would allow to define also p-adic variants of the scattering amplitudes.

4.4 Could recursion formula allow interpretation in terms of zero energy ontology?

One can ask whether the recursion formula could allow interpretation in terms of zero energy states assigned to causal diamonds (CDs) containing CDs containing... In this framework loops should be assigned with sub-CDs. Of course, one can make only questions and a word of caution is in order: Grassmannian diagrams are not generalized Feynman diagrams and it might not make sense to assign concrete interpretations in terms of zero energy ontology to them. One can also ask whether the generalized Feynman diagrams are actually more like twistor diagrams.

1. The interpretation of leading order singularities forming basic building blocks of the twistor approach in zero ontology is the basic question. Suppose that the massivation of virtual fermions and their super partners allows only ladder diagrams in which the intermediate states contain on mass shell massless states. Should one allow this kind of ladder diagrams? Can one identify them in terms of leading order singularities? Could one construct the generalized Feynman diagrams from Yangian invariant tree diagrams associated with the hierarchy of sub-CDs and using BCFW bridges and entangled pairs of massless states having interpretation as box diagrams with on mass shell momenta at microscopic level? Could it make sense to say that scattering amplitudes are represented by tree diagrams inside CDs in various scales and that the fermionic momenta associated with throats and emerging from sub-CDs are always massless?

2. In this framework the interpretation of BCFW bridge would be in terms of a scattering in which the four on mass shell massless fermions are exchanged between four sub-CDs.

3. Could the addition of 2-particle zero energy state having interpretation in terms of the cutting of line carrying loop momentum correspond to an addition of sub-CD such that the 2-particle zero energy state has its positive and negative energy part on its past and future boundaries? Could this mean that one cuts a propagator line by adding CD and leaves only the portion of the line within CD. Could the reverse operation mean to the addition of zero energy "thermally entangled" states in shorter time and length scales and assignable as a zero energy state to a sub-CD. Could one interpret the Cutkosky rule for propagator line in terms of this cutting or its reversal. Why only pairs are needed in the recursion formula? Why not more general states? Does the recursion formula imply that they are included? Does this relate to the fact that these zero energy states have interpretation as single particle states in the positive energy ontology and that the basic building block of Feynman diagrams is single particle state? Could one regard the unitarity as an identity which states that the discontinuity of T-matrix characterizing zero energy state over cut is expressible in terms of $TT^\dagger$ and T matrix is the relevant quantity?

4. If I have understood correctly the genuine l-loop term results from $l-1$-loop term by the addition of the zero energy pair and integration over GL(2) as a representative of loop integral reducing $n+2$ to n and calculating the added loop at the same time. The integrations over the two momentum twistor variables associated with a line in twistor space defining off mass shell four-momentum and integration over the lines represent the integration over loop momentum. The reduction to $GL(2)$ integration should result from the delta functions relating the additional momenta to $GL(2)$ variables (note that GL(2) performs linear transformations in the space spanned by the twistors Z_A and Z_B and means integral over the positions of Z_A an Z_B). The resulting object is formally Yangian invariant but IR divergences along some contours of integration breaks Yangian symmetry.

The question is what happens in TGD framework. The previous arguments suggests that the reduction of the the loop momentum integral to integrals over discrete mass shells and possibly to a sum over their discrete subsets does not spoil the reduction to contour integrals for loop integrals in the example considered in [34]. Furthermore, the replacement of mass continuum with a discrete set of mass shells should eliminate IR divergences and might allow to preserve Yangian symmetry. One can however wonder whether the loop corrections with on mass shell massless fermions are needed. If so, one would have at most finite number of loop diagrams with on mass shell fermionic momenta and one of the TGD inspired dreams already forgotten would be realized.

5 Questions

There are many questions to be asked. There would be in-numerable questions upwelling from my very incomplete understanding of the technical issues. In the following I restrict only to the questions which relate to the relationship of TGD approach to Witten's twistor string approach [26] and M-theory like frameworks.

5.1 $M^4 \times CP_2$ from twistor approach

The first question which comes to mind relates to the origin of the Grassmannians. Do they have some deeper interpretation in TGD context. In twistor string theory Grassmannians relate to the moduli spaces of holomorphic surfaces defined by string world sheets in twistor space. Could partonic 2-surfaces have analogous intepretation and could one assign Grassmannians to thei moduli spaces. If so, one could have rather direct connection with topological QFT defining twistor strings [26] and the almost topological QFT defining TGD. There are some hints to this direction which could be of course seen as figments of a too wild imagination.

1. The geometry of CD brings strongly in mind Penrose diagram for the conformally compactified Minkowski space [24], which indeed becomes CD when its points are replaced with spheres. This would suggest the information theoretic idea about interaction between observer and externals as a map in which M^4 is mapped to its conformal compactification represented by CD. Compactification means that the light-like points at the light-like boundaries of CD are identified and the physical counterpart for this in TGD framework is conformal invariance along light-rays along the boundaries of CD. The world of conscious observer for which CD is identified as a geometric correlate would be conformally compactified M^4 (plus CP_2 or course).

2. Since the points of the conformally compactified M^4 correspond to twistor pairs [30], which are unique only apart from opposite complex scalings, it would be natural to assign twistor space to CD and represent its points as pairs of twistors. This suggest an interpretation for the basic formulas of Grassmannian approach involving integration over twistors. The incoming and outgoing massless particles could be assigned at point-like limit light-like points at the lower and upper boundaries of CD and the lifting of the points of the light-cone boundary at partonic surfaces would give rise to the description in terms of ordinary twistors. The assumption that massless collinear fermions at partonic 2-surfaces are the basic building blocks of physical particles at partonic 2-surfaces defined as many particles states involving several partonic 2-surfaces would lead naturally to momentum twistor description in which massless momenta and described by twistors and virtual momenta in terms of twistor pairs. It is important to notice that in TGD framework string like objects would emerge from these massless fermions.

3. Partonic 2-surfaces are located at the upper and lower light-like boundaries of the causal diamond (CD) and carry energies of opposite sign in zero energy ontology. Quite generally, one can assign to the point of the conformally compactified Minkowski space a twistor pair using the standard description. The pair of twistors is determined apart from $Gl(2)$ rotation. At the lightcone boundary M^4 points are are light-like so that the two spinors of the two twistors differ from each other only by a complex scaling and single twistor is enough to characterize the space-time poin this degenerate situation. The components of the twistor are related by the well known twistor equation $\mu^{a'} = -ix^{aa'}\lambda_a$. One can therefore lift each point of the partonic 2-surface to single twistor determined apart from opposite complex scalings of μ and λ so that the lift

of the point would be 2-sphere. In the general case one must lift the point of CD to a twistor pair. The degeneracy of the points is given by $Gl(2)$ and each point corresponds to a 2-sphere in projective twistor space.

4. The new observation is that one can understand also CP_2 factor in twistor framework. The basic observation about which I learned in [30] (giving also a nice desription of basics of twistor geometry) is that a pair (X,Y) of twistors defines a point of CD on one hand and complex 2-planes of the dual twistor space -which is nothing but CP_2- by the equations

$$X_\alpha W^\alpha = 0 \ , \quad Y_\alpha W^\alpha = 0 \ .$$

The intersection of these planes is the complex line $CP_1 = S^2$. The action of $G(2)$ on the twistor pair affects the pair of surfaces CP_2 determined by these equations since it transforms the equations to their linear combination but not the the point of conformal CD resulting as projection of the sphere. Therefore twistor pair defines both a point of M^4 and assigns with it pair of CP_2:s represented as holomorphic surfaces of the projective dual twistor space. Hence the union over twistor pairs defines $M^4 \times CP_2$ via this assignment if it is possible to choose "the other" CP_2 in a unique manner for all points of M^4. The situation is similar to the assignment of a twistor to a point in the Grassmannian diagrams forming closed polygons with light-like edges. In this case one assigns to the the "region momenta" associated with the edge the twistor at the either end of the edge. One possible interpretation is that the two CP_2:s correspond to the opposite ends of the CD. My humble hunch is that this observation might be something very deep.

Recall that the assigment of CP_2 to M^4 point works also in another direction. M^8-H duality associates with so called hyper-quaternionic 4-surface of M^8 allowing preferred hyper-complex plane at each point 4-surfaces of $M^4 \times CP_2$. The basic observation behind this duality is that the hyperquaternionic planes (copies of M^4) with preferred choices of hyper-complex plane M^2 are parameterized by points of CP_2. One can therefore assign to a point of CP_2 a copy of M^4. Maybe these both assignments indeed belong to the core of quantum TGD. There is also an interesting analogy with Uncertainty Principle: complete localization in M^4 implies maximal uncertainty of the point in CP_2 and vice versa.

5.2 Does twistor string theory generalize to TGD?

With this background the key speculative questions seem to be the following ones.

1. Could one relate twistor-string theory to TGD framework? Partonic 2-surfaces at the boundaries of CD are lifted to 4-D sphere bundles in twistor space. Could they serve as a 4-D counterpart for Witten's holomorphic twistor strings assigned to point like particles? Could these surfaces be actually lifts of the holomorphic curves of twistor space replaced with the product $CP_3 \times CP_2$ to 4-D sphere bundles? If I have understood correctly, the Grassmannians $G(n,k)$ can be assigned to the moduli spaces of these holomorphic curves characterized by the degree of the polyomial expressible in terms of genus, number of negative helicity gluons, and the number of loops for twistor diagram.

   Could one interpret $G(n,k)$ as a moduli space for the δCD projections of n partonic 2-surfaces to which k negative helicity gluons and $n-k$ positive helicity gluons are assigned (or something more complex when one considers more general particle states)? Could quantum numbers be mapped to integer valued algebraic invariants? IF so, there would be a correlation between the geometry of the partonic 2-surface and quantum numbers in accordance with quantum classical correspondence.

2. Could one understand light-like orbits of partonic 2-surfaces and space-time surfaces in terms of twistors? To each point of the 2-surface one can assign a 2-sphere in twistor space CP_3 and CP_2 in its dual. These CP_2s can be identified. One should be able to assign to each sphere S^2 at least one point of corresponding CP_2s associated with its points in the dual twistor space and identified as single CP_2 union of CP_2:s in the dual twistor space a point of CP_2 or even several

of them. One should be also able to continue this correspondence so that it applies to the light-like orbit of the partonic 2-surface and to the space-time surface defining a preferred extremal of Kähler action. For space-time sheets representable as graph of a map $M^4 \rightarrow CP_2$ locally one should select from a CP_2 assigned with a particular point of the space-time sheet a unique point of corresponding CP_2 in a manner consistent with field equations. For surfaces with lower dimensional M^4 projection one must assign a continuum of points of CP_2 to a given point of M^4. What kind equations-could allow to realize this assigment? Holomorphy is strongly favored also by the number theoretic considerations since in this case one has hopes of performing integrals using residue calculus.

(a) Could two holomorphic equations in $CP_3 \times CP_2$ defining 6-D surfaces as sphere bundles over $M^4 \times CP_2$ characterize the preferred extremals of Kähler action? Could partonic 2-surfaces be obtained by posing an additional holomorphic equation reducing twistors to null twistors and thus projecting to the boundaries of CD? A philosophical justification for this conjecture comes from effective 2-dimensionality stating that partonic 2-surfaces plus their 4-D tangent space data code for physics. That the dynamics would reduce to holomorphy would be an extremely beautiful result. Of course this is only an additional item in the list of general conjectures about the classical dynamics for the preferred extremals of Kähler action.

(b) One could also work in $CP_3 \times CP_3$. The first CP_3 would represent twistors endowed with a metric conformally equivalent to that of $M^{2,4}$ and having the covering of $SU(2,2)$ of $SO(2,4)$ as isometries. The second CP_3 defining its dual would have a metric consistent with the Calabi-Yau structure (having holonomy group $SU(3)$). Also the induced metric for canonically imbeddied CP_2s should be the standard metric of CP_2 having $SU(3)$ as its isometries. In this situation the linear equations assigning to M^4 points twistor pairs and $CP_2 \subset CP_3$ as a complex plane would hold always true. Besides this two holomorphic equations coding for the dynamics would be needed.

(c) The issues related to the induced metric are important. The conformal equivalence class of M^4 metric emerges from the 5-D light-cone of $M^{2,4}$ under projective identification. The choice of a proper projective gauge would select M^4 metric locally. Twistors inherit the conformal metric with signature $(2,4)$ form the metric of 4+4 component spinors with metric having $(4,4)$ signature. One should be able to assign a conformal equivalence class of Minkowski metric with the orbits of pairs of twistors modulo $GL(2)$. The metric of conformally compactified M^4 would be obtained from this metric by dropping from the line element the contribution to the S^2 fiber associated with M^4 point.

(d) Witten related [26] the degree d of the algebraic curve describing twistor string, its genus g, the number k of negative helicity gluons, and the number l of loops by the following formula

$$d = k - 1 + l \ , \ \ g \leq l \ . \tag{5.1}$$

One should generalize the definition of the genus so that it applies to 6-D surfaces. For projective complex varieties of complex dimension n this definition indeed makes sense. Algebraic genus [20] is expressible in terms of the dimensions of the spaces of closed holomorphic forms known as Hodge numbers $h^{p,q}$ as

$$g = \sum (-1)^{n-k} h^{k,0} \ . \tag{5.2}$$

The first guess is that the formula of Witten generalizes by replacing genus with its algebraic counterpart. This requires that the allowed holomorphic surfaces are projective curves of twistori space, that is described in terms of homogenous polynomials of the 4+4 projective coordinates of $CP_3 \times CP_3$.

5.3 What is the relationship of TGD to M-theory and F-theory?

There are also questions relating to the possible relationship to M-theory and F-theory.

1. Calabi-Yau-manifolds [21, 22] are central for the compactification in super string theory and emerge from the condition that the super-symmetry breaks down to $\mathcal{N}=1$ SUSY. The dual twistor space CP_3 with Euclidian signature of metric is a Calabi-Yau manifold [26]. Could one have in some sense two Calabi-Yaus! Twistorial CP_3 can be interpreted as a four-fold covering and conformal compactification of $M^{2,4}$. I do not know whether Calabi-Yau property has a generalization to the situation when Euclidian metric is replaced with a conformal equivalence class of flat metrics with Minkowskian signature and thus having a vanishing Ricci tensor. As far as differential forms (no dependence on metric) are considered there should be no problems. Whether the replacement of the maximal holonomy group $SU(3)$ with its non-compact version $SU(1,2)$ makes sense is not clear to me.

2. The lift of the CD to projective twistor space would replace $CD\times CP_2$ with 10-dimensional space which inspires the familiar questions about connection between TGD and M-theory. If Calabi-Yau with a Minkowskian signature of metric makes sense then the Calabi-Yau of the standard M-theory would be replaced with its Minkowskian counterpart! Could it really be that M-theory like theory based on $CP_3\times CP_2$ reduces to TGD in $CD\times CP_2$ if an additional symmetry mapping 2-spheres of CP_3 to points of CD is assumed? Could the formulation based on 12-D $CP_3\times CP_3$ correspond to F-theory which also has two time-like dimensions. Of course, the additional conditions defined by the maps to M^4 and CP_2 would remove the second time-like dimension which is very difficult to justify on purely physical grounds.

3. One can actually challenge the assumption that the first CP_3 should have a conformal metric with signature $(2,4)$. Metric appears nowhere in the definition holomorphic functions and once the projections to M^4 and CP_2 are known, the metric of the space-time surface is obtained from the metric of $M^4\times CP_2$. The previous argument for the necessity of the presence of the information about metric in the second order differential equation however suggests that the metric is needed.

4. The beginner might ask whether the 6-D 2-sphere bundles representing space-time sheets could have interpretation as Calabi-Yau manifolds. In fact, the Calabi-Yau manifolds defined as complete intersections in $CP_3\times CP_3$ discovered by Tian and Yau are defined by three polynomials [22]. Two of them have degree 3 and depend on the coordinates of single CP_3 only whereas the third is bilinear in the coordinates of the CP_3:s. Obviously the number of these manifolds is quite too small (taking into account scaling the space defined by the coefficients is 6-dimensional). All these manifolds are deformation equivalent. These manifolds have Euler characteristic $\chi=\pm 18$ and a non-trivial fundamental group. By divididing this manifold by Z_3 one obtains $\chi=\pm 6$, which guarantees that the number of fermion generations is three in heterotic string theory. This manifold was the first one proposed to give rise to three generations and $\mathcal{N}=1$ SUSY.

5.4 What could the field equations be in twistorial formulation?

The fascinating question is whether one can identify the equations determining the 3-D complex surfaces of $CP_3\times CP_3$ in turn determining the space-time surfaces.

The first thing is to clarify in detail how space-time $M^4\times CP_2$ results from $CP_3\times CP_3$. Each point $CP_3\times CP_3$ define a line in third CP_3 having interpretation as a point of conformally compactified M^4 obtained by sphere bundle projection. Each point of either CP_3 in turn defines CP_2 in in fourth CP_3 as a 2-plane. Therefore one has $(CP_3\times CP_3)\times(CP_3\times CP_3)$ but one can reduce the consideration to $CP_3\times CP_3$ fixing $M^4\times CP_2$. In the generic situtiion 6-D surface in 12-D $CP_3\times CP_3$ defines 4-D surface in the dual $CP_3\times CP_3$ and its sphere bundle projection defines a 4-D surface in $M^4\times CP_2$.

1. The vanishing of three holomorphic functions f^i would characterize 3-D holomorphic surfaces of 6-D $CP_3\times CP_3$. These are determined by three real functions of three real arguments just like a holomorphic function of single variable is dictated by its values on a one-dimensional curve of complex plane. This conforms with the idea that initial data are given at 3-D surface. Note that either the first or second CP_3 can determine the CP_2 image of the holomorphic 3-surface

unless one assumes that the holomorphic functions are symmetric under the exchange of the coordinates of the two CP_3s. If symmetry is not assumed one has some kind of duality.

2. Effective 2-dimensionality means that 2-D partonic surfaces plus 4-D tangent space data are enough. This suggests that the 2 holomorphic functions determining the dynamics satisfy some second order differential equation with respect to their three complex arguments: the value of the function and its derivative would correspond to the initial values of the imbedding space coordinates and their normal derivatives at partonic 2-surface. Since the effective 2-dimensionality brings in dependence on the induced metric of the space-time surface, this equation should contain information about the induced metric.

3. The no-where vanishing holomorphic 3-form Ω, which can be regarded as a "complex square root" of volume form characterizes 6-D Calabi-Yau manifold [21, 22], indeed contains this information albeit in a rather implicit manner but in spirit with TGD as almost topological QFT philosophy. Both CP_3:s are characterized by this kind of 3-form if Calabi-Yau with $(2,4)$ signature makes sense.

4. The simplest second order- and one might hope holomorphic- differential equation that one can imagine with these ingredients is of the form

$$\Omega_1^{i_1 j_1 k_1}\Omega_2^{i_2 j_2 k_2}\partial_{i_1 i_2}f^1\partial_{j_1 j_2}f^2\partial_{k_1 k_2}f^3 \quad = \quad 0 \ , \quad \partial_{ij}\equiv\partial_i\partial_j \ . \tag{5.3}$$

   Since Ω_i is by its antisymmetry equal to $\Omega_i^{123}\epsilon^{ijk}$, one can divide Ω^{123}:s away from the equation so that one indeed obtains holomorphic solutions. Note also that one can replace ordinary derivatives in the equation with covariant derivatives without any effect so that the equations are general coordinate invariant.

   One can consider more complex equations obtained by taking instead of (f^1, f^2, f^3) arbitrary combinations (f^i, f^j, f^k) which results uniquely if one assumes antisymmetrization in the labels $(1,2,3)$. In the sequel only this equation is considered.

5. The metric disappears completely from the equations and skeptic could argue that this is inconsistent with the fact that it appears in the equations defining the weak form of electric-magnetic duality as a Lagrange multiplier term in Chern-Simons action. Optimist would respond that the representation of the 6-surfaces as intersections of three hyper-surfaces is different from the representation as imbedding maps $X^4 \rightarrow H$ used in the usual formulation so that the argument does not bite, and continue by saying that the metric emerges in any case when one endows space-time with the induced metric given by projection to M^4.

6. These equations allow infinite families of obvious solutions. For instance, when some f^i depends on the coordinates of either CP_3 only, the equations are identically satisfied. As a special case one obtains solutions for which $f^1 = Z\cdot W$ and $(f^2, f^3) = (f^2(Z), f^3(W))$ This family contains also the Calabi-Yau manifold found by Yau and Tian, whose factor space was proposed as the first candidate for a compactication consistent with three fermion families.

7. One might hope that an infinite non-obvious solution family could be obtained from the ansatz expressible as products of exponential functions of Z and W. Exponentials are not consistent with the assumption that the functions f_i are homogenous polynomials of finite degree in projective coordinates so that the following argument is only for the purpose for learning something about the basic character of the equations.

$$\begin{aligned}
&f^1 = E_{a_1,a_2,a_3}(Z)E_{\hat{a}_1,\hat{a}_2,\hat{a}_3}(W) \ , \qquad f^2 = E_{b_1,b_2,b_3}(Z)E_{\hat{b}_1,\hat{b}_2,\hat{b}_3}(W) \ , \\
&f^3 = E_{c_1,c_2,c_3}(Z)E_{\hat{c}_1,\hat{c}_2,\hat{c}_3}(W) \ , \\
&E_{a,b,c}(Z) = exp(az_1)exp(bz_2)exp(cz_3) \ .
\end{aligned} \tag{5.4}$$

The parameters a, b, c, and $\hat{a}, \hat{b}, \hat{c}$ can be arbitrary real numbers in real context. By the basic properties of exponential functions the field equations are algebraic. The conditions reduce to the vanishing of the products of determinants $det(a, b, c)$ and $det(\hat{a}, \hat{b}, \hat{c})$ so that the vanishing of either determinant is enough. Therefore the dependence can be arbitry either in Z coordiantes or in W coordinates. Linear superposition holds for the modes for which determinant vanishes which means that the vectors (a, b, c) or $(\hat{a}, \hat{b}, \hat{c})$ are in the same plane.

Unfortunately, the vanishing conditions reduce to the conditions $f^i(W) = 0$ for case a) and to $f^i(Z) = 0$ for case b) so that the conditions are equivalent with those obtained by putting the "wave vector" to zero and the solutions reduce to obvious ones. The lesson is that the equations do not commute with the multiplication of the functions f^i with nowhere vanishing functions of W and Z. The equation selects a particular representation of the surfaces and one might argue that this should not be the case unless the hyper-surfaces defined by f^i contain some physically relevant information. One could consider the possibility that the vanishing conditions are replaced with conditions $f^i = c_i$ with $f^i(0) = 0$ in which case the information would be coded by a family of space-time surfaces obtained by varying c_i.

One might criticize the above equations since they are formulated directly in the product $CP_3 \times CP_3$ of projective twistor by choosing a specific projective gauge by putting$z^4 = 1, w^4 = 1$. The manifestly projectively invariant formulation for the equations is in full twistor space so that 12-D space would be replaced with 16-D space. In this case one would have 4-D complex permutation symbol giving for these spaces Calabi-Yau structure with flat metric. The product of functions $f = z^4 = constant$ and $g = w^4 = constant$ would define the fourth function $f_4 = fg$ fixing the projective gauge

$$\epsilon^{i_1 j_1 k_1 l_1} \epsilon^{i_2 j_2 k_2 l_2} \partial_{i_1 i_2} f^1 \partial_{j_1 j_2} f^2 \partial_{k_1 k_2} f^3 \partial_{l_1 l_2} f^4 = 0 \ , \ \partial_{ij} \equiv \partial_i \partial_j \ . \quad (5.5)$$

The functions f^i are homogenous polynomials of their twistor arguments to guarantee projective invariance. These equations are projectively invariant and reduce to the above form which means also loss of homogenous polynomial property. The undesirable feature is the loss of manifest projective invariance by the fixing of the projective gauge.

A more attractive ansatz is based on the idea that one must have one equation for each f^i to minimize the non-determism of the equations obvious from the fact that there is single equation in 3-D lattice for three dynamical variables. The quartets (f^1, f^2, f^3, f^i), $i = 1, 2, 3$ would define a possible minimally non-linear generalization of the equation

$$\epsilon^{i_1 j_1 k_1 l_1} \epsilon^{i_2 j_2 k_2 l_2} \partial_{i_1 i_2} f^1 \partial_{j_1 j_2} f^2 \partial_{k_1 k_2} f^m \partial_{l_1 l_2} f^4 = 0 \ , \ \partial_{ij} \equiv \partial_i \partial_j \ , \ m = 1, 2, 3 \ . \quad (5.6)$$

Note that the functions are homogenous polynomials of their arguments and analogous to spherical harmonics suggesting that they can allow a nice interpretation in terms of quantum classical correspondence.

The minimal non-linearity of the equations also conforms with the non-linearity of the field equations associated with Kähler action. Note that also in this case one can solve the equations by diagonalizing the dynamical coefficient matrix associated with the quadratic term and by identifying the eigen-vectors of zero eigen values. One could consider also more complicated strongly non-linear ansätze such as (f^i, f^i, f^i, f^i), $i = 1, 2, 3$, but these do not seem plausible.

5.4.1 The explicit form of the equations using Taylor series expansion for multi-linear case

In this section the equations associated with (f_1, f_2, f_3) ansatz are discussed in order to obtain a perspective about the general structure of the equations. This experience can be applied directly to the (f^1, f^2, f^3, f^i) ansatz.

The explicit form of the equations is obtained as infinite number of conditions relating the coefficients of the Taylor series of f^1 and f^2. The treatment of the two variants for the equations is essentially identical and in the following only the manifestly projectively invariant form will be considered.

1. One can express the Taylor series as

$$\begin{aligned} f^1(Z,W) &= \sum_{m,n} C_{m,n} M_m(Z) M_n(W) \ , \\ f^2(Z,W) &= \sum_{m,n} D_{m,n} M_m(Z) M_n(W) \ , \\ f^3(Z,W) &= \sum_{m,n} E_{m,n} M_m(Z) M_n(W) \ , \\ M_{m\equiv(m_1,m_2,m_3)}(Z) &= z_1^{m_1} z_2^{m_2} z_3^{m_3} \ . \end{aligned} \tag{5.7}$$

2. The application of derivatives to the functions reduces to a simple algebraic operation

$$\partial_{ij}(M_m(Z)M_n(W)) = m_i n_j M_{m_1-e_i}(Z) M_{n-e_j}(W) \ . \tag{5.8}$$

   Here e_i denotes i:th unit vector.

3. Using the product rule $M_m M_n = M_{m+n}$ one obtains

$$\begin{aligned} & \partial_{ij}(M_m(Z)M_n(W))\partial_{rs}(M_k(Z)M_l(W)) \\ = \ & m_i n_j k_r l_s \times M_{m-e_i}(Z) M_{n-e_j}(W) \times M_{k-e_r}(Z) M_{k-e_s}(W) \\ = \ & m_i n_j k_r l_s \times M_{m+k-e_i-e_r}(Z) \times M_{n+l-e_j-e_l}(W) \ . \end{aligned} \tag{5.9}$$

4. The equations reduce to the trilinear form

$$\begin{aligned} & \sum_{m,n,k,l,r,s} C_{m,n} D_{k,l} E_{r,s} (m,k,r)(n,l,s) M_{m+k+r-E}(Z) M_{n+l+s-E}(W) = 0 \ , \\ E &= e_1+e_2+e_3 \ , \quad (a,b,c) = \epsilon^{ijk} a_i b_j c_c \ . \end{aligned} \tag{5.10}$$

   Here (a,b,c) denotes the determinant defined by the three index vectors involved. By introducing the summation indices

$$(M = m+k+r-E, k, r) \ , \quad (N = n+l+s-E, l, s)$$

   one obtains an infinite number of conditions, one for each pair (M,N). The condition for a given pair (M,N) reads as

$$\sum_{k,l,r,s} C_{M-k-r+E,N-l-s+E} D_{k,l} E_{r,s} \times (M-k-r+E,k,r)(N-l-s+E,l,s) = 0 \ . \tag{5.11}$$

   These equations can be regarded as linear equations by regarding any matrix selected from $\{C,D,E\}$ as a vector of linear space. The existence solutions requires that the determinant associated with the tensor product of other two matrices vanishes. This matrix is dynamical. Same applies to the tensor product of any of the matrices.

5. Hyper-determinant [23] is the generalization of the notion of determinant whose vanishing tells that multilinear equations have solutions. Now the vanishing of the hyper-determinant defined for the tensor product of the three-fold tensor power of the vector space defined by the coefficients of the Taylor expansion should provide the appropriate manner to characterize the conditions

for the existence of the solutions. As already seen, solutions indeed exist so that the hyper-determinant must vanish. The elements of the hyper matrix are now products of determinants for the exponents of the monomials involved. The non-locality of the Kähler function as a functional of the partonic surface leads to the argument that the field equations of TGD for vanishing n:th variations of Kähler action are multilinear and that a vanishing of a generalized hyper-determinant characterizes this [16].

6. Since the differential operators are homogenous polynomials of partial derivatives, the total degrees of $M_m(Z)$ and $M_m(W)$ defined as a sum $D = \sum m_i$ is reduced by one unit by the action of both operators ∂_{ij}. For given value of M and N only the products

$$M_m(Z)M_n(W)M_k(Z)M_r(W)M_s(Z)M_l(W)$$

for which the vector valued degrees $D_1 = m + k + r$ and $D_2 = n + l + s$ have the same value are coupled. Since the degree is reduced by the operators appearing in the equation, polynomial solutions for which f^i contain monomials labelled by vectors m_i, n_i, r_i for which the components vary in a finite range $(0, n_{max})$ look like a natural solution ansatz. All the degrees $D_i \leq D_{i,max}$ appear in the solution ansatz so that quite a large number of conditions is obtained.

What is nice is that the equation can be interpreted as a difference equation in 3-D lattice with "time direction" defined by the direction of the diagonal.

1. The counterparts of time=constant slices are the planes $n_1 + n_2 + n_3 = n$ defining outer surfaces of simplices having E as a normal vector. The difference equation does not seem to say nothing about the behavior in the transversal directions. M and N vary in the simplex planes satisfying $\sum M_i = T_1$, $\sum N_i = T_2$. It seems natural to choose $T_1 = T_2 = T$ so that Z and W dynamics corresponds to the same "time". The number of points in the $T = constant$ simplex plane increases with T which is analogous to cosmic expansion.

2. The "time evolution" with respect to T can be solved iteratively by increasing the value of $\sum M_i = N_i = T$ by one unit at each step. Suppose that the values of coefficients are known and satisfy the conditions for (m, k, r) and (n, l, s) up to the maximum value T for the sum of the components of each of these six vectors. The region of known coefficients -"past"- obviously corresponds to the interior of the simplex bounded by the plane $\sum M_i = \sum N_i = T$ having E as a normal. Let (m_{min}, n_{min}), (k_{min}, l_{min}) and (r_{min}, s_{min}) correspond to the smallest values of 3-indices for which the coefficiens are non-vanishing- this could be called the moment of "Big Bang". The simplest but not necessary assumption is that these indices correspond zero vectors $(0, 0, 0)$ analogous to the tip of light-cone.

3. For given values of M and N corresponding to same value of "cosmic time" T one can separate from the formula the terms which correspond to the un-knownn coefficients as the sum $C_{M+E,N+E}D_{0,0}E_{0,0} + D_{M+E,N+E}D_{0,0}C_{0,0} + E_{M+E,N+E}C_{0,0}D_{0,0}$. The remaining terms are by assumption already known. One can fix the normalization by choosing $C_{0,0} = D_{0,0} = E_{0,0} = 1$. With these assumptions the equation reduces at each point of the outer boundary of the simplex to the form

$$C_{M+E,N+E} + D_{M+E,N+E} + E_{M+E,N+E} = X$$

where X is something already known and contain only data about points in the plane $m+k+r = M$ and $n + r + s = N$. Note that these planes have one "time like direction" unlike the simplex plane so that one could speak about a discrete analog of string world sheet in 3+3+3-D lattice space defined by a 2-plane with one time-like direction.

4. For each point of the simplex plane one has equation of the above form. The equation is non-deterministic since only constrain only the sum $C_{M+E,N+E} + D_{M+E,N+E} + E_{M+E,N+E}$ at each point of the simplex plane to a plane in the complex 3-D space defined by them. Hence the number of solutions is very large. The condition that the solutions reduce to polynomials poses conditions on the coefficients since the quantities X associated with the plane $T = T_{max}$ must

vanish for each point of the simplex plane in this case. In fact, projective invariance means that the functions involved are homogenous functions in projective coordinates and thus polynomials and therefore reduce to polynomials of finite degree in 3-D treatment. This obviously gives additional condition to the equations.

5.4.2 The minimally non-linear option

The simple equation just discussed should be taken with a caution since the non-determinism seems to be too large if one takes seriously the analogy with clasical dynamics. By the vacuum degeneracy also the time evolution associated with Kähler action breaks determinism in the standard sense of the word. The non-determinism is however not so strong and removed completely in local sense for non-vacuum exrtremals. One could also try to see the non-determinism as the analog for non-deterministic time evolution by quantum jumps.

One can however consider the already mentioned possibility of increasing the number of equations so that one would have three equations corresponding to the three unknown functions f^i so that the determinism associated with each step would be reduced. The equations in question would be of the same general form but with (f^1, f^2, f^3) replaced with some some other combination.

1. In the genuinely projective situation where one can consider the (f^1, f^2, f^3, f^i), $i = 1, 2, 3$ as a unique generalization of the equation. This would make the equations quadratic in f_i and reduce the non-determinism at given step of the time evolution. The new element is that now only monomials $M_m(z)$ associated with the f^i with same degree of homegenuity defined by $d = \sum m_i$ are consistent with projective invariance. Therefore the solutions are characterized by six integers $(d_{i,1}, d_{i,2})$ having interpretation as analogs of conformal weights since they correspond to eigenvalues of scaling operators. That homogenous polynomials are in question gives hopes that a generalization of Witten's approach might make sense. The indices m vary at the outer surfaces of the six 3-simplices defined by $(d_{i,1}, d_{i,2})$ and looking like tedrahedrons in 3-D space. The functions f^i are highly analogous to the homogenous functions appearing in group representations and quantum classical correspondence could be realized through the representation of the space-time surfaces in this manner.

2. The 3-determinants (a, b, c) appearing in the equations would be replaced by 4-determinants and the equations would have the same general form. One has

$$\begin{gathered} \sum_{k,l,r,s,t,u} C_{M-k-r-t+E,N-l-s-u+E} D_{k,l} E_{r,s} C_{t,u} \times \\ \times (M-k-r-t+E, k, r, t)(N-l-s-u+E, l, s, u) = 0 \ , \\ E = e_1 + e_2 + e_3 + e_4 \ , \ \ (a, b, c, d) = \epsilon^{ijkl} a_i b_j c_k d_l \ . \end{gathered} \tag{5.12}$$

   and its variants in which D and E appear quadratically. The values of M and N are restricted to the tedrahedrons $\sum M_i = \sum d_{k,1} + d_{1,i}$ and $\sum N_i = \sum d_{i,2} + d_{i,2}$, $i = 1, 2, 3$. Therefore the dynamics in the index space is 3-dimensional. Since the index space is in a well-defined sense dual to CP_3 as is also the CP_3 in which the solutions are represented as counterparts of 3-surfaces, one could say that the 3-dimensionality of the dynamics corresponds to the dynamics of Chern-Simons action at space-like at the ends of CD and at light-like 3-surfaces.

3. The view based on 4-D time evolution is not useful since the solutions are restricted to time=constant plane in 4-D sense. The elimination of one of the projective coordinates would lead however to the analog of the above describe time evolution. In four-D context a more appropriate form of the equations is

$$\sum_{m,n,k,l,r,s} C_{m,n} D_{k,l} E_{r,s} C_{t,u} (m, k, r, t)(n, l, s, u) M_{m+k+r-E}(Z) M_{n+l+s-E}(W) = 0 \tag{5.13}$$

with similar equations for f^2 and f^3. If one assumes that the CP_2 image of the holomorphic 3-surface is unique (it can correspond to either CP_3) the homogenous polynomials f^i must be symmetric under the exchange of Z and W so that the matrices C, D, and E are symmetric. This is equivalent to a replacement of the product of determinants with a sum of 16 products of determinants obtained by permuting the indices of each index pair (m, n), (k, l),(r, s) and (t, u).

4. The number N_{cond} of conditions is given by the product $N_{cond} = N(d_M)N(d_N)$ of numbers of points in the two tedrahedrons defined by the total conformal weights

$$\sum M_r = d_M = \sum_k d_{k,1} + d_{i,1} \text{ and } \sum N_r = d_N = \sum_k d_{k,2} + d_{i,2} \ , \ \ i = 1, 2, 3.$$

The number N_{coeff} of coefficients is

$$N_{coeff} = \sum_k n(d_{k,1}) + \sum_k n(d_{k,2}) \ ,$$

where $n(d_{k,i})$ is the number points associated with the tedrahedron with conformal weight $d_{k,i}$. Since one has $n(d) \propto d^3$, N_{cond} scales as

$$N_{cond} \propto d_M^3 d_N^3 = (\sum_k d_{k,1} + d_{1,i})^3 \times (\sum_k d_{k,2} + d_{i,2})^3$$

whereas the number N_{coeff} of coefficients scales as

$$N_{coeff} \propto \sum_k (d_{k,1}^3 + d_{k,2}^3) \ .$$

N_{cond} is clearly much larger than N_{coeff} so the solutions are analogous to partial waves and that the reduction of the rank for the matrices involved is an essential aspect of being a solution. The reduction of the rank for the coefficient matrices should reduce the effective number of coefficients so that solutions can be found. An interesting question is whether the coefficients are rationals with a suitable normalization allowed by independent conformal scalings. An analogy for the dynamics is quantum entanglement for 3+3 systems respecting the conservation of conformal weights and quantum classical correspondence taken to extreme suggests something like this.

5. One can interpret these equations as linear equations for the coefficients of the either linear term or as quadratic equations for the non-linear term. Also in the case of quadratic term one can apply general linear methods to identify the vanishing eigen values of the matrix of the quadratic form involved and to find the zero modes as solutions. The rank of the dynamically determined multiplier matrix must be non-maximal for the solutions to exist. One can imagine that the rank changes at critical surfaces in the space of Taylor coefficients meaning a multifurcation in the space determined by the coefficients of the polynomials. Also the degree of the polynomial can change at the critical point.

Solutions for which either determinant vanishes for all terms present in the solution exist. This is is achieved if either the index vectors (m, l, r, t) or (n, l, s, u) in their respective parallel 3-planes are also in a 3-plane going through the origin. These solutions might seen as the analogs of vacuum extremals of Chern-Simons action for which the CP_2 projection is at most 2-D Lagrangian manifold.

Quantum classical correspondence requires that the space-time surface carries also information about the momenta of partons. This information is quasicontinuous. Also information about zero modes should have representation in terms of the coefficients of the polynomials. Is this really possible if only products of polynomials of fixed conformal weights with strong restrictions on coefficients can be used? The counterpart for the vacuum degeneracy of Kähler action might resolve the problem. The analog for the construction of space-time surfaces as deformations of vacuum extremals would be starting from a trivial solution and adding to the building blocks of f^i some terms of same degree for which the wave vectors are not in the intersection of a 3-plane and simplex planes. The still existing "vacuum part" of the solution could carry the needed information.

6. One can take "obvious solutions" characterized by different common 3-planes for the "wave vectors" characterizing the 8 monomials $M_a(Z)$ and $M_b(W)$, $a \in \{m,k,r,t\}$ and $b \in \{n,l,s,u\}$. The coefficient matrices C,D,E,F are completely free. For the sum of these solutions the equations contain interaction terms for which at least two "wave vectors" belong to different 3-planes so that the corresponding 4-determinants are non-vanishing. The coefficients are not anymore free. Could the "obvious solutions" have interpretation in terms of different space-time sheets interacting via wormhole contacts? Or can one equate "obvious" with "vacuum" so that interaction between different vacuum space-time sheets via wormhole contact with 3-D CP_2 projection would deform vacuum extremals to non-vacuum extremals? Quantum classical correspondence inspires the question whether the products for functions f_i associated with an obvious solution associated with a particular plane correspond to a tensor products for quantum states associated with a particular partonic 2-surface or space-time sheet.

7. Effective 2-dimensionality realized in terms of the extremals of Chern-Simons actions with Lagrange multiplier term coming from the weak form of electric magnetic duality should also have a concrete counterpart if one takes the analogy with the extremals of Kähler action seriously. The equations can be transformed to 3-D ones by the elimination of the fourth coordinate but the interpretation in terms of discrete time evolution seems to be impossible since all points are coupled. The total conformal weights of the monomials vary in the range $[0, d_{1,i}]$ and $[0, d_{2,i}]$ so that the non-vanishing coefficients are in the interior of 3-simplex. The information about the fourth coordinate is preserved being visible via the four-determinants.

8. It should be possible to relate the hierarchy with respect to conformal weights would to the geometrization of loop integrals if a generalization of twistor strings is in question. One could hope that there exists a hierarchy of solutions with levels characterized by the rank of the matrices appearing in the linear representation. There is a temptation to associate this hierarchy with the hierarchy of deformations of vacuum extremals of Kähler action forming also a hierarchy. If this is the case the obvious solutions would correspond to vacuum exremals. At each step when the rank of the matrices involved decreases the solution becomes nearer to vacuum extremal and there should exist vanishing second variation of Kähler action. This structural similarity gives hopes that the proposed ansatz might work. Also the fact that a generalization of the Penrose's twistorial description for the solutions of Maxwell's equations to the situation when Maxwell field is induced from the Kähler form of CP_2 raises hopes. One must however remember that the consistency with other proposed solution ansatze and with what is believed to be known about the preferred extremals is an enormously powerful constraint and a mathematical miracle would be required.

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