On a strengthened Hardy-Hilbert's type inequality

Guangsheng Chen*

Department of Computer Engineering, Guangxi Modern Vocational Technology College,

Hechi,Guangxi, 547000, P.R. China

Abstract: In this paper, by using the Euler-Maclaurin expansion for the zeta function and estimating the weight function effectively, we derive a strengthenment of a Hardy-Hilbert's type inequality proved by W.Y. Zhong. As applications, some particular results are considered.

Keywords: Hardy-Hilbert type inequality; weight coefficient; Hölder inequality

MSC: 26D15

1 Introduction

If $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, a_n, $b_n\ge 0$, $0<\sum\limits_{n=1}^{\infty}a_n^p<\infty$ and $0<\sum\limits_{n=1}^{\infty}b_n^q<\infty$, then[1] have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{m+n}<\frac{\pi}{\sin(\pi/p)}[\sum_{n=1}^{\infty}a_n^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}b_n^q]^{\frac{1}{q}}, \tag{1.1}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m,n\}}<pq[\sum_{n=1}^{\infty}a_n^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}b_n^q]^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ and pq are the best possible. Inequality (1.1) is well known as Hardy- Hilbert's inequality and (1.2) is named a Hardy- Hilbert's type inequality. Both of them are important in analysis and its applications [2]. In the recent years, a lot of results with generalizations of this type of inequality were obtained (see [3]). Under the same conditions as (1.1) and (1.2), there are some Hardy-Hilbert's type inequalities similar to (1.1) and (1.2), which also had been studied and generalized by some mathematicians.

Recently, by introducing a parameter Yang [4] gave a generalization of inequality (1.2) with the best constant factor as follows:

If $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-\min\{p,q\}<\lambda\le 2$, a_n, $b_n\ge 0$, such that $0<\sum\limits_{n=1}^{\infty}n^{1-\lambda}a_n^p<\infty$ and $0<\sum\limits_{n=1}^{\infty}n^{1-\lambda}b_n^q<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^{\lambda},n^{\lambda}\}}<k_{\lambda}(p)\left\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{1.3}$$

*E-mail address: cgswavelets@126.com(Chen)

where the constant factor $k_\lambda(p) = \frac{\lambda pq}{(p+\lambda-2)(q+\lambda-2)}$ is the best possible.

Very recently, by introducing a parameter and two pairs of conjugate exponents, Zhong [5] gave a generalization of inequality (1.3) with the best constant factor as follows:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \leq \min\{r, s\}$, a_n, $b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{r})-1} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < k_\lambda(r) \left\{\sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{r})-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q\right\}^{\frac{1}{q}}, \tag{1.4}$$

where the constant factor $k_\lambda(r) = \frac{rs}{\lambda}$ is the best possible.

In this paper, by introducing a parameter and estimating the weight coefficient, we obtain a strengthenment of (1.4). As applications, some particular results are considered.

2 Some preliminary results

First, we need the following formula of the Riemann-ζ function (see [6]):

$$\begin{aligned}\zeta(\rho) = &\sum_{n=1}^{m} \frac{1}{n^\rho} - \frac{m^{1-\rho}}{1-\rho} - \frac{1}{2m^\rho} - \sum_{n=1}^{l-1} \frac{B_{2n}}{2n} \begin{pmatrix} -\rho \\ 2n-1 \end{pmatrix} \frac{1}{m^{\rho+2n-1}} \\ &- \frac{B_{2l}}{2l} \begin{pmatrix} -\rho \\ 2l-1 \end{pmatrix} \frac{\varepsilon}{m^{\rho+2l-1}},\end{aligned} \tag{2.1}$$

where $\rho > 0$, $\rho \neq 1$, m, $l \geq 1$, $m, l \in \mathrm{N}$, $0 < \varepsilon = \varepsilon(\rho, l, m) < 1$. The numbers $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30, \ldots$ are Bernoulli numbers. In particular, $\zeta(\rho) = \sum_{n=1}^{\infty} \frac{1}{n^\rho}$ $(\rho > 1)$.

Since $\zeta(0) = -1/2$, then the formula of the Riemann-ζ function (2.1) is also true for $\rho = 0$.

Lemma 2.1 *If $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \leq \min\{r, s\}$, define the weight coefficients $\omega(m, \lambda, s)$ and $\omega(n, \lambda, r)$ as*

$$\omega(m, \lambda, s) = \sum_{n=1}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}}, \tag{2.2}$$

$$\omega(n, \lambda, r) = \sum_{m=1}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{1-\frac{\lambda}{r}}, \tag{2.3}$$

then we have

$$\omega(m, \lambda, s) < m^{1-\lambda}[k_\lambda - \frac{s}{3\lambda m^{\frac{\lambda}{s}}}], \tag{2.4}$$

and

$$\omega(n, \lambda, r) < n^{1-\lambda}[k_\lambda - \frac{r}{3\lambda m^{\frac{\lambda}{r}}}], \tag{2.5}$$

Where $k_\lambda = \frac{rs}{\lambda}$.

Proof. For $0 < \lambda \le \min\{r, s\}$, taking $\rho = 1 - \frac{\lambda}{s} \ge 0$, $l = 1$ in (2.1), we get

$$\zeta(1 - \tfrac{\lambda}{s}) = \sum_{n=1}^{m} \frac{1}{n^{1-\frac{\lambda}{s}}} - \frac{sm^{\frac{\lambda}{s}}}{\lambda} - \frac{1}{2m^{1-\frac{\lambda}{s}}} + \frac{1-\frac{\lambda}{s}}{12pm^{2-\frac{\lambda}{s}}}\varepsilon_1, \tag{2.6}$$

Where $0 < \varepsilon_1 < 1$.

Taking $\rho = 1 + \frac{\lambda}{r}$, we obtain

$$\zeta(1 + \tfrac{\lambda}{r}) = \sum_{n=1}^{m-1} \frac{1}{n^{1+\frac{\lambda}{r}}} + \frac{rm^{-\frac{\lambda}{r}}}{\lambda} + \frac{1}{2m^{1+\frac{\lambda}{r}}} + \frac{1+\frac{\lambda}{r}}{12m^{2+\frac{\lambda}{r}}}\varepsilon_2, \tag{2.7}$$

Where $0 < \varepsilon_2 < 1$.

Thus we get

$$\begin{aligned}
\omega(m, \lambda, s) &= \sum_{n=1}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} \\
&= \sum_{n=1}^{m} \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} - \frac{1}{m^\lambda} + \sum_{n=m}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} \\
&= \sum_{n=1}^{m} \frac{1}{m^\lambda} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} - \frac{1}{m^\lambda} + \sum_{n=m}^{\infty} \frac{1}{n^\lambda} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} \\
&= \frac{1}{m^{\lambda+\frac{\lambda}{s}-1}} \sum_{n=1}^{m} \frac{1}{n^{1-\frac{\lambda}{s}}} - \frac{1}{m^\lambda} + m^{1-\frac{\lambda}{s}} \sum_{n=m}^{\infty} \frac{1}{n^{1+\frac{\lambda}{r}}}.
\end{aligned}$$

By (2.6) and (2.7), we have

$$\begin{aligned}
\omega(m, \lambda, s) &< \frac{1}{m^{\lambda+\frac{\lambda}{s}-1}} [\zeta(1 - \tfrac{\lambda}{s}) + \tfrac{sm^{\frac{\lambda}{s}}}{\lambda} + \tfrac{1}{2m^{1-\frac{\lambda}{s}}}] - \tfrac{1}{m^\lambda} \\
&+ m^{1-\frac{\lambda}{s}} [\frac{rm^{-\frac{\lambda}{r}}}{\lambda} + \frac{1}{2m^{1+\frac{\lambda}{r}}} + \frac{1+\frac{\lambda}{r}}{12m^{2+\frac{\lambda}{r}}}] \\
&= \frac{1}{m^{\lambda+\frac{\lambda}{s}-1}} \zeta(1 - \tfrac{\lambda}{s}) + \tfrac{sm^{1-\lambda}}{\lambda} + \tfrac{1}{2m^\lambda} - \tfrac{1}{m^\lambda} + \tfrac{rm^{1-\lambda}}{\lambda} + \tfrac{1}{2m^\lambda} + \tfrac{1+\frac{\lambda}{r}}{12m^{1+\lambda}} \\
&= \frac{1}{m^{\lambda+\frac{\lambda}{s}-1}} \zeta(1 - \tfrac{\lambda}{s}) + \tfrac{rsm^{1-\lambda}}{\lambda} + \tfrac{1+\frac{\lambda}{r}}{12m^{1+\lambda}} \\
&= m^{1-\lambda} \left\{ \frac{rs}{\lambda} - \frac{1}{m^{\frac{\lambda}{s}}} [-\zeta(1 - \tfrac{\lambda}{s}) - \tfrac{1+\frac{\lambda}{r}}{12m^{2-\frac{\lambda}{s}}}] \right\}.
\end{aligned}$$

In (2.6), taking $m = 1$, by $0 < \lambda \le \min\{r, s\}$, we obtain

$$\begin{aligned}
\zeta(1 - \tfrac{\lambda}{s}) &= 1 - \tfrac{s}{\lambda} - \tfrac{1}{2} + \tfrac{(1-\frac{\lambda}{s})\varepsilon_1}{12} \\
&< \frac{1}{2} - \frac{s}{\lambda} + \frac{1-\frac{\lambda}{s}}{12} = \frac{6\lambda - 12s - \lambda(1 - \frac{\lambda}{s})}{12\lambda} \\
&< \frac{6\lambda - 12s - (\lambda - s)}{12\lambda} = \frac{5\lambda - 11s}{12\lambda} = -\frac{11s - 5\lambda}{12\lambda} < 0.
\end{aligned}$$

Therefore for $m \geq 1$, $m \in \mathrm{N}$, $0 < \lambda \leq \min\{r, s\}$,we have

$$
\begin{aligned}
&-\zeta(1-\tfrac{\lambda}{s}) - \frac{1+\frac{\lambda}{r}}{12m^{2-\frac{\lambda}{s}}} > \tfrac{11s-5\lambda}{12\lambda} - \frac{1+\frac{\lambda}{r}}{12} \\
&= \frac{11s-5\lambda-\lambda(1+\frac{\lambda}{r})}{12\lambda} \geq \frac{11s-5\lambda-2\lambda}{12\lambda} = \frac{4s+7(s-\lambda)}{12\lambda} \geq \frac{4s}{12\lambda} = \frac{s}{3\lambda}.
\end{aligned}
$$

Applying the last result and the inequality for $\omega(m, \lambda, p)$ above, we obtain (2.4). Similarly, we can prove (2.5).The lemma is proved.

3 Main results

Theorem 3.1 *If* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \leq \min\{r, s\}$, $a_n \geq 0$, $b_n \geq 0$, *such that* $0 < \sum\limits_{n=1}^{\infty} n^{p(1-\frac{\lambda}{r})-1} a_n^p < \infty$ *and* $0 < \sum\limits_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q < \infty$, *then*

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} &< \left\{\sum_{n=1}^{\infty} [k_\lambda - \frac{s}{3\lambda n^{\frac{\lambda}{s}}}] n^{p(1-\frac{\lambda}{r})-1} a_n^p \right\}^{\frac{1}{p}} \\
&\times \left\{\sum_{n=1}^{\infty} [k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}] n^{q(1-\frac{\lambda}{s})-1} b_n^q \right\}^{\frac{1}{q}},
\end{aligned} \tag{3.1}
$$

$$
\sum_{n=1}^{\infty} \frac{n^{\frac{p\lambda}{s}-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}} [\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^p < \sum_{n=1}^{\infty} [k_\lambda - \frac{s}{3\lambda n^{\frac{\lambda}{s}}}] n^{p(1-\frac{\lambda}{r})-1} a_n^p, \tag{3.2}
$$

Where $k_\lambda = \frac{rs}{\lambda} > 0$. *Inequality (3.1) is equivalent to (3.2). In particular, we have the following equivalent inequalities:*

$$
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < k_\lambda \left\{\sum_{n=1}^{\infty} [1 - \frac{s}{k_\lambda 3\lambda n^{\frac{\lambda}{s}}}] n^{p(1-\frac{\lambda}{r})-1} a_n^p \right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q \right\}^{\frac{1}{q}} \tag{3.3}
$$

$$
\sum_{n=1}^{\infty} n^{\frac{p\lambda}{s}-1} [\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^p < k_\lambda^p \sum_{n=1}^{\infty} [1 - \frac{s}{3k_\lambda \lambda n^{\frac{\lambda}{s}}}] n^{p(1-\frac{\lambda}{r})-1} a_n^p. \tag{3.4}
$$

Proof. By Hölder inequality(see[7]), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} \frac{n^{(\lambda/s-1)/p}}{m^{(\lambda/r-1)/q}} \frac{m^{(\lambda/r-1)/q}}{n^{(\lambda/s-1)/p}} \\
&\leq \{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m^p m^{p(1-\lambda/r)+\lambda-2}}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} \}^{\frac{1}{p}} \{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{b_n^q n^{q(1-\lambda/s)+\lambda-2}}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{1-\frac{\lambda}{r}} \}^{\frac{1}{q}} \\
&= \{\sum_{m=1}^{\infty} \omega(m, \lambda, s) m^{p(1-\lambda/r)+\lambda-2} a_m^p \}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} \omega(n, \lambda, r) n^{q(1-\lambda/s)+\lambda-2} b_n^q \}^{\frac{1}{q}}.
\end{aligned}
$$

Hence, by (2.4), (2.5), inequality (3.1) holds.

Setting b_n as

$$b_n = \frac{n^{p\lambda/s-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}}[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^{p-1},$$

By (3.1), we have

$$\sum_{n=1}^{\infty}[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]n^{q(1-\frac{\lambda}{s})-1}b_n^q = \sum_{n=1}^{\infty} \frac{n^{p\lambda/s-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}}[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^{p}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} \leq \left\{\sum_{n=1}^{\infty}[k_\lambda - \frac{s}{3\lambda n^{\frac{\lambda}{s}}}]n^{p(1-\frac{\lambda}{r})-1}a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty}[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]n^{q(1-\frac{\lambda}{s})-1}b_n^q\right\}^{\frac{1}{q}}, \tag{3.5}$$

Hence we obtain

$$0 < \sum_{n=1}^{\infty} \frac{n^{\frac{p\lambda}{s}-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}}[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^{p} < \sum_{n=1}^{\infty}[k_\lambda - \frac{s}{3\lambda n^{\frac{\lambda}{s}}}]n^{p(1-\frac{\lambda}{r})-1}a_n^p < \infty, \tag{3.6}$$

By (3.1), both (3.5) and (3.6) take the form of strict inequality, and we have (3.2).

On the other hand, suppose that (3.2) is valid, by Hölder inequality, we find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}}$$
$$= \sum_{n=1}^{\infty} \frac{n^{[q(\lambda/s-1)+1]/q}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{\frac{1}{q}}}[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}][k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{\frac{1}{q}} n^{[q(1-\lambda/s)-1]/q} b_n$$
$$\leq \{\sum_{n=1}^{\infty} \frac{n^{p\lambda/s-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}}[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^{p}\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]n^{q(1-\lambda/s)-1}b_n^q\}^{\frac{1}{q}},$$

Then by (3.2), we have (3.1). Hence (3.2) and (3.1) are equivalent. The proof of Theorem 3.1 is completed.

Since $0 < \lambda \leq \min\{r, s\}$, by Theorem 3.1, we have

Corollary 3.2 *If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \leq \min\{r, s\}$, $a_n \geq 0$, $b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{r})-1}a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1}b_n^q < \infty$, then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \left\{\sum_{n=1}^{\infty}[k_\lambda - \frac{1}{3n^{\frac{\lambda}{s}}}]n^{p(1-\frac{\lambda}{r})-1}a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty}[k_\lambda - \frac{1}{3n^{\frac{\lambda}{r}}}]n^{q(1-\frac{\lambda}{s})-1}b_n^q\right\}^{\frac{1}{q}}, \tag{3.7}$$

$$\sum_{n=1}^{\infty}\frac{n^{\frac{p\lambda}{s}-1}}{[k_\lambda-\frac{1}{3n^{\frac{\lambda}{r}}}]^{p-1}}[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}]^p<\sum_{n=1}^{\infty}[k_\lambda-\frac{1}{3n^{\frac{\lambda}{s}}}]n^{p(1-\frac{\lambda}{r})-1}a_n^p, \tag{3.8}$$

Where $k_\lambda=\frac{rs}{\lambda}>0$. *Inequality (3.7) is equivalent to (3.8).*

For $r=s=2$, by (3.1) and (3.2), we have

Corollary 3.3 *If* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda\le 2$, $a_n\ge 0$, $b_n\ge 0$, *such that* $0<\sum_{n=1}^{\infty}n^{p(1-\frac{\lambda}{2})-1}a_n^p<\infty$ *and* $0<\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q<\infty$, *then*

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}&<\left\{\sum_{n=1}^{\infty}[k_\lambda-\frac{2}{3\lambda n^{\frac{\lambda}{2}}}]n^{p(1-\frac{\lambda}{2})-1}a_n^p\right\}^{\frac{1}{p}}\\&\times\left\{\sum_{n=1}^{\infty}[k_\lambda-\frac{2}{3\lambda n^{\frac{\lambda}{2}}}]n^{q(1-\frac{\lambda}{2})-1}b_n^q\right\}^{\frac{1}{q}},\end{aligned}\tag{3.9}$$

$$\sum_{n=1}^{\infty}\frac{n^{\frac{p\lambda}{2}-1}}{[k_\lambda-\frac{2}{3\lambda n^{\frac{\lambda}{2}}}]^{p-1}}[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}]^p<\sum_{n=1}^{\infty}[k_\lambda-\frac{2}{3\lambda n^{\frac{\lambda}{2}}}]n^{p(1-\frac{\lambda}{2})-1}a_n^p, \tag{3.10}$$

Where $k_\lambda=\frac{4}{\lambda}>0$. *Inequality (3.9) is equivalent to (3.10). In particular, we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}<k_\lambda\left\{\sum_{n=1}^{\infty}[1-\frac{2}{k_\lambda 3\lambda n^{\frac{\lambda}{2}}}]n^{p(1-\frac{\lambda}{2})-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right\}^{\frac{1}{q}} \tag{3.11}$$

$$\sum_{n=1}^{\infty}n^{\frac{p\lambda}{2}-1}[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}]^p<k_\lambda^p\sum_{n=1}^{\infty}[1-\frac{2}{3k_\lambda\lambda n^{\frac{\lambda}{2}}}]n^{p(1-\frac{\lambda}{2})-1}a_n^p. \tag{3.12}$$

For $r=q$, $s=p$, by (3.1) and (3.2), we have

Corollary 3.4 *If* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda\le\min\{p,q\}$, $a_n\ge 0$, $b_n\ge 0$, *such that* $0<\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}a_n^p<\infty$ *and* $0<\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}b_n^q<\infty$, *then*

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}&<\left\{\sum_{n=1}^{\infty}[k_\lambda-\frac{p}{3\lambda n^{\frac{\lambda}{p}}}]n^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\\&\times\left\{\sum_{n=1}^{\infty}[k_\lambda-\frac{q}{3\lambda n^{\frac{\lambda}{q}}}]n^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}},\end{aligned}\tag{3.13}$$

$$\sum_{n=1}^{\infty}\frac{n^{\lambda-1}}{[k_\lambda-\frac{q}{3\lambda n^{\frac{\lambda}{q}}}]^{p-1}}[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}]^p<\sum_{n=1}^{\infty}[k_\lambda-\frac{p}{3\lambda n^{\frac{\lambda}{p}}}]n^{(p-1)(1-\lambda)}a_n^p, \tag{3.14}$$

Where $k_\lambda = \frac{pq}{\lambda} > 0$. *Inequality (3.13) is equivalent to (3.14). In particular, we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < k_\lambda \left\{\sum_{n=1}^{\infty}[1-\frac{p}{k_\lambda 3\lambda n^{\frac{\lambda}{p}}}]n^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}}, \tag{3.15}$$

$$\sum_{n=1}^{\infty}n^{\lambda-1}[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^p < k_\lambda^p\sum_{n=1}^{\infty}[1-\frac{p}{3k_\lambda\lambda n^{\frac{\lambda}{p}}}]n^{(p-1)(1-\lambda)}a_n^p. \tag{3.16}$$

For $r = p$, $s = q$, by (3.1) and (3.2), we have

Corollary 3.5 *If* $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < \lambda \le \min\{p, q\}$, $a_n \ge 0$, $b_n \ge 0$, *such that* $0 < \sum\limits_{n=1}^{\infty} n^{p-\lambda-1}a_n^p < \infty$ *and* $0 < \sum\limits_{n=1}^{\infty} n^{q-\lambda-1}b_n^q < \infty$, *then*

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} &< \left\{\sum_{n=1}^{\infty}[k_\lambda-\frac{q}{3\lambda n^{\frac{\lambda}{q}}}]n^{p-\lambda-1}a_n^p\right\}^{\frac{1}{p}} \\ &\times\left\{\sum_{n=1}^{\infty}[k_\lambda-\frac{p}{3\lambda n^{\frac{\lambda}{p}}}]n^{q-\lambda-1}b_n^q\right\}^{\frac{1}{q}},\end{aligned} \tag{3.17}$$

$$\sum_{n=1}^{\infty}\frac{n^{(p-1)\lambda-1}}{[k_\lambda-\frac{p}{3\lambda n^{\frac{\lambda}{p}}}]^{p-1}}[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^p < \sum_{n=1}^{\infty}[k_\lambda-\frac{q}{3\lambda n^{\frac{\lambda}{q}}}]n^{p-\lambda-1}a_n^p, \tag{3.18}$$

Where $k_\lambda = \frac{pq}{\lambda} > 0$. *Inequality (3.17) is equivalent to (3.18). In particular, we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < k_\lambda \left\{\sum_{n=1}^{\infty}[1-\frac{q}{k_\lambda 3\lambda n^{\frac{\lambda}{q}}}]n^{p-\lambda-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q-\lambda-1}b_n^q\right\}^{\frac{1}{q}}, \tag{3.19}$$

$$\sum_{n=1}^{\infty}n^{(p-1)\lambda-1}[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda, n^\lambda\}}]^p < k_\lambda^p\sum_{n=1}^{\infty}[1-\frac{q}{3k_\lambda\lambda n^{\frac{\lambda}{q}}}]n^{p-\lambda-1}a_n^p. \tag{3.20}$$

Taking $\lambda = 1$, by (3.1) and (3.2), we have

Corollary 3.6 *If* $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$,$r > 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $a_n \ge 0$, $b_n \ge 0$, *such that* $0 < \sum\limits_{n=1}^{\infty} n^{\frac{p}{s}}-1a_n^p < \infty$ *and* $0 < \sum\limits_{n=1}^{\infty} n^{\frac{q}{r}}-1b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m, n\}} < \left\{\sum_{n=1}^{\infty}[rs-\frac{s}{3n^{\frac{1}{s}}}]n^{\frac{p}{s}}-1a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}[rs-\frac{r}{3n^{\frac{1}{r}}}]n^{\frac{q}{r}}-1b_n^q\right\}^{\frac{1}{q}}, \tag{3.21}$$

$$\sum_{n=1}^{\infty} \frac{n^{\frac{p}{s}-1}}{[rs - \frac{r}{3n^{\frac{1}{r}}}]^{p-1}} [\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\}}]^p < \sum_{n=1}^{\infty} [rs - \frac{s}{3n^{\frac{1}{s}}}] n^{\frac{p}{s}-1} a_n^p. \tag{3.22}$$

In particular, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < rs \left\{ \sum_{n=1}^{\infty} [1 - \frac{1}{3rn^{\frac{1}{s}}}] n^{\frac{p}{s}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} b_n^q \right\}^{\frac{1}{q}}, \tag{3.23}$$

$$\sum_{n=1}^{\infty} n^{\frac{p}{s}-1} [\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\}}]^p < (rs)^p \sum_{n=1}^{\infty} [1 - \frac{1}{3rn^{\frac{1}{s}}}] n^{\frac{p}{s}-1} a_n^p. \tag{3.24}$$

Taking $p = q = r = s = 2$, in (3.23) and (3.24), we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < 4 \left\{ \sum_{n=1}^{\infty} [1 - \frac{1}{6\sqrt{n}}] a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} [1 - \frac{1}{6\sqrt{n}}] b_n^2 \right\}^{\frac{1}{2}}, \tag{3.25}$$

$$\sum_{n=1}^{\infty} [\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\}}]^2 < 16 \sum_{n=1}^{\infty} [1 - \frac{1}{6\sqrt{n}}] a_n^2. \tag{3.26}$$

References

[1] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge Univ. Press, 1952.

[2] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, Inequalities Involving Functions and Their Integrals and Derivatives, Kluwer Academic Publishers, Boston, 1991.

[3] B.C.Yang, T.M. Rassias, On the way of weight coefficient and research for the Hilbert-type inequalities, Math. Inequal. Appl. 6 (4) (2003)625–658.

[4] B.C.Yang, On a generalization of the Hilbert's type inequality and its applications, Chinese Journal of Engineering Mathematics 21 (4) (2004)821-824.

[5] W.Y Zhong,A best extension of Hilbert-type inequality with some parameters, Chinese Journal of Guangdong Education Institute 27 (5) (2007)14-18.

[6] B. C.Yang, The Norm of Operator and Hilbert-type Inequality, Science Press, Beijing, 2008.

[7] J. Kuang. Applied Inequalities, Shandong Science Press, Jinan, 2003.