The hyperbolic Smarandache theorem in the Poincaré upper half-plane model of hyperbolic geometry

Nilgün Sönmez and Cătălin Barbu

Abstract. In this study, we give a hyperbolic version of the Smarandache's theorem in the Poincaré upper half-plane model.

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1. Introduction

Non-Euclidean geometry in general, and hyperbolic geometry in particular, is an area of mathematics which has an interesting history and which is still being actively studied by researchers around the world. One reason for the continuing interest in hyperbolic geometry is that it touches on a number of different fields, including but not limited to Complex Analysis, Abstract Algebra and Group Theory, Number Theory, Diferential Geometry, and Low-dimensional Topology [1].

In hyperbolic geometry, a type of non-Eucliean geometry that retains the first four postulates of Euclid, the parallel postulate is replaced by the following: T*hrough a given point not on a given line there can be drawn at least two lines parallel to the given line.* The development of hyperbolic geometry is attributed to Lobachevsky, Bolyai, and Gauss, but there is certainly some evidence that Euclid had an inkling of its existence [2]. There are known many models for hyperbolic geometry, such as: Poincaré upper half- plane, Poincaré disc model, Klein model, etc. Here, in this study, we give a hyperbolic version of the Smarandache theorem in the Poincaré upper half-plane model. The Euclidean version of this well-known theorem states that if the points M_i, $i = \overline{1,n}$ are the projections of a point M on the sides A_iA_{i+1}, $i = \overline{1,n}$, where $A_{n+1} = A_1$, of the polygon $A_1A_2...A_n$, then

$$M_1A_1^2 + M_2A_2^2 + ... + M_nA_n^2 = M_1A_2^2 + M_2A_3^2 + ... + M_{n-1}A_n^2 + M_nA_1^2 \qquad (1)$$

(see [5] or [8]). The standard simple proof is based on the theorem of Pythagoras. For more details we refer to the monograph of C. Barbu [2], L. Nicolescu, V. Boskoff [7]. In fact the Smarandache's theorem is a generalization of the Carnot's theorem. We mention that O. Demirel and E. Soytürk [4] gave the hyperbolic form of Carnot's theorem in the Poincaré disc model and C. Barbu [3] gave the hyperbolic form of Smarandache's theorem in the Poincaré disc model of hyperbolic geometry. In order to introduce the Smarandache's theorem into the Poincaré upper half-plane we refer briefly some facts about the Poincaré upper half-plane.

2. Preliminaries

The nature of the x-axis is such as to make impossible any communication between the lower and the upper half-planes. We restrict our attention to the upper half-plane and refer to it as the hyperbolic plane. It is also known as the Poincaré upper half-plane. The geodesic segments of the Poincaré upper half-plane (hyperbolic plane) are either segments of Euclidean straight lines that are perpendicular to the x-axis or arc of Euclidean semicircles that are centered on the x-axis. The hyperbolic length of the Euclidean line segment joining the points $P = (a; y_1)$ and $Q = (a; y_2)$, $0 < y_1 \leq y_2$, is $\ln \frac{y_2}{y_1}$.

The hyperbolic length between the points P and Q on a Euclidean semicircle with center $C = (c; 0)$ and radius r such that the radii CP and CQ make angles α and β ($\alpha < \beta$) respectively, with the positive x-axis [9],

$$\ln \frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}.$$

As in Euclidean geometry, the polygon is one of the basic objects in hyperbolic geometry. In the Euclidean plane, a polygon is a closed convex set that is bounded by Euclidean line segments. We would like to mimic this definition as much as possible in the hyperbolic plane. A hyperbolic polygon is a closed convex set in the hyperbolic plane that can be expressed as the intersection of a locally finite collection of closed half-planes. For further details we refer to the books of J. W. Anderson [1, p.120-130], J.W.and S. Stahl [9].

Theorem 1. *Let ABC be a hyperbolic triangle with a right angle at C. If a, b, c, are the hyperbolic lengths of the sides opposite A, B, C, respectively, then*

$$\cosh c = \cosh a \cdot \cosh b. \tag{2}$$

For the proof of the theorem see [9].

3. The hyperbolic Smarandache theorem in the Poincaré upper half-plane model of hyperbolic geometry

In this section, we prove Smarandache's theorem in the Poincaré upper half-plane model of hyperbolic geometry.

Theorem 2. *Let $A_1A_2...A_n$ be a hyperbolic convex polygon in the Poincaré upper half-plane, whose vertices are the points $A_1, A_2, ..., A_n$ and whose sides (directed counterclockwise) are $a_1 = A_1A_2$, $a_2 = A_2A_3, ..., a_n = A_nA_1$. Let the points M_i, $i = \overline{1,n}$ be located on the sides $a_1, a_2, ..., a_n$ of the hyperbolic convex polygon $A_1A_2...A_n$ respectively. If the perpendiculars to the sides of the hyperbolic polygon at the points $M_1, M_2, ..., M_n$ are concurrent in a point M, then the following equalities hold*

$$\sum_{i=1}^{n} \cosh MA_i(\cosh M_iA_i - \cosh M_iA_{i+1}) = 0 \tag{3}$$

where $A_{n+1} = A_1$,

$$\frac{\cosh M_1A_1}{\cosh M_1A_2} \cdot \frac{\cosh M_2A_2}{\cosh M_2A_3} \cdot ... \cdot \frac{\cosh M_nA_n}{\cosh M_nA_1} = 1. \tag{4}$$

Proof. The geodesic segments $A_1M, A_2M, ..., A_nM, MM_1, MM_2, ..., MM_n$ split the hyperbolic polygon into $2n$ right-angled hyperbolic triangles. If we apply the Theorem 1 then

$$\cosh MA_i = \cosh M_iA_i \cdot \cosh MM_i = \cosh M_{i-1}A_i \cdot \cosh MM_{i-1}, \tag{5}$$

for all $i = \overline{1,n}$, and $M_0 = M_n$. Adding these equalities member by member, we get

$$\sum_{i=1}^{n} \cosh MA_i \cosh M_iA_i = \sum_{i=1}^{n} \cosh MA_i \cosh M_iA_{i+1},$$

and the conclusion follows. Multiplying the relations (5) member by member and making simplifications, we obtain (4). ■

Naturally, one may wonder whether the converse of the Smarandache theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 3. *Let $A_1A_2...A_n$ be a hyperbolic convex polygon in the Poincaré upper half-plane, whose vertices are the points $A_1, A_2, ..., A_n$ and whose sides (directed counterclockwise) are $a_1 = A_1A_2$, $a_2 = A_2A_3, ..., a_n = A_nA_1$. Let the points M_i, $i = \overline{1,n}$ be located on the sides $a_1, a_2, ..., a_n$ of the hyperbolic convex polygon $A_1A_2...A_n$ respectively. If the perpendiculars to the sides of the hyperbolic polygon at the points $M_1, M_2, ..., M_{n-1}$ are concurrent in a point M and the following relation holds*

$$\frac{\cosh M_1A_1}{\cosh M_1A_2} \cdot \frac{\cosh M_2A_2}{\cosh M_2A_3} \cdot ... \cdot \frac{\cosh M_nA_n}{\cosh M_nA_1} = 1, \tag{6}$$

then the point M is on the perpendicular to A_nA_1 at the point M_n.

Proof. Let M' the feet of the perpendicular from M on the side A_nA_1. Using the already proven equality (4), we obtain

$$\frac{\cosh M_1A_1}{\cosh M_1A_2} \cdot \frac{\cosh M_2A_2}{\cosh M_2A_3} \cdot ... \cdot \frac{\cosh M'A_n}{\cosh M'A_1} = 1 \tag{7}$$

By (6) and (7) we get

$$\frac{\cosh M_nA_n}{\cosh M_nA_1} = \frac{\cosh M'A_n}{\cosh M'A_1}. \tag{8}$$

We note with x, y, z the hyperbolic distances A_1M', $M'M_n$ and M_nA_n respectively. Then (8) is equivalent with

$$\cosh(x+y) \cdot \cosh(y+z) - \cosh x \cdot \cosh z = 0 \tag{9}$$

If we use the formula

$$\cosh\alpha \cdot \cosh\beta = \frac{\sinh(\alpha+\beta) - \sinh(\alpha-\beta)}{2},$$

then the relation (9) is equivalent with

$$\frac{\sinh(x+2y+z) - \sinh(x-z)}{2} - \frac{\sinh(x+z) - \sinh(x-z)}{2} = 0$$

or

$$\sinh(x+2y+z) = \sinh(x+z). \tag{10}$$

By (10) using injectivity of the function $\sinh x$ we get $y = 0$. Then the points M' and M_n are identical. ■

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AFYON KOCATEPE UNIVERSITY
FACULTY OF SCIENCE AND LITERATURES
DEPARTMENT OF MATHEMATICS
ANS CAMPUS, 03200 - AFYONKARAHISAR, TURKEY
E-mail: nceylan@aku.edu.tr

"VASILE ALECSANDRI" NATIONAL COLLEGE
DEPARTMENT OF MATHEMATICS
BACĂU, STR. VASILE ALECSANDRI NR. 37, 600011
BACĂU, ROMANIA
E-mail address: kafka mate@yahoo.com