Motion in a Non-Inertial Frame of Reference vs. Motion in the Gravitomagnetical Field

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We mathematically proved that the inertial forces, which appears in a non-inertial frame of reference, such as accelerating and rotating reference frame, are equivalent to the real forces which appears, when the body moves in the gravitomagnetical field. We will remind the rotating bucket with water problem with the new proposal of the solution.

gravitation, gravitomagnetism, four-vector field of velocity, inertial forces

INTRODUCTION

Let's replace the scalar potential of a gravitational (G) field, $\varphi(\mathbf{r}, t)$, by the $V_g^2(\mathbf{r}, t)$, where $V_g^2(\mathbf{r}, t) = -\varphi(\mathbf{r}, t)$. Let's name the $V_g^2(\mathbf{r}, t)$ as *the scalar field of the square of the velocity*. Let's replace the vectorial potential of the gravitomagnetism (GM) field $\mathbf{A_g}(\mathbf{r}, t)$ by the $\mathbf{V_{gm}}(\mathbf{r}, t)$, where $\mathbf{V_{gm}}(\mathbf{r}, t) = \mathbf{A_g}(\mathbf{r}, t)$. Let's name the $\mathbf{V_{gm}}(\mathbf{r}, t)$ as *the vectorial field of the velocity* [1].

Let's replace of the G and GM four-potential $A^\mu = (\varphi/c, \mathbf{A_g})$ by the *four-vector field of the velocity* $(V_g)^\mu$, which we will define in the form

$$V_g^\mu \overset{\text{def}}{=} \left(-\frac{V_{\mathbf{g}}^{\mathbf{2}}}{c_g}, \mathbf{V_{gm}} \right)$$

where: c_g – speed of propagation of field (equal to, by General Theory of Relativity, the speed of light c). The $(V_g)^\mu$ has dimension [m/s], from here the name - *the four-vector field of the velocity* [1].

The nonrelativistic Lagrangian for the body with mass m moving in the external scalar field of the square of velocity $(V_g)^2$ in an inertial frame is[1]

$$L_0 = \frac{mv_0^2}{2} + mV_g^2 \tag{1}$$

and the corresponding equation of motion

$$m\frac{d\mathbf{v}}{dt} = m\nabla(V_g^2) \tag{2}$$

[1] In this section the suffix 0 denotes quantities pertaining to an inertial frame.

Let us now consider what the equations of motion will be in a non-inertial frame of reference. The basis of the solution of this problem is again the principle of least action, whose validity does not depend on the frame of reference chosen. Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{r}} \tag{3}$$

are likewise valid, but the Lagrangian is no longer of the form (1) and to derive it we must carry out the necessary transformation of the L_0. This transformation we will realize in two steps [2].

STEP 1. ACCELERATED TRANSLATIONAL MOTION OF THE FRAME OF REFERENCE

Let us first consider a frame of reference K' which moves with a translational velocity $\mathbf{V}(t)$ relative to the inertial frame K_0. The velocities $\mathbf{v}_0$ and $\mathbf{v'}$ of a body in the frames K_0 and K' respectively are related by

$$\mathbf{v_0} = \mathbf{v'} + \mathbf{V}(t) \tag{4}$$

Substitution of this in (1) gives the Lagrangian in the frame K':

$$L' = \frac{m\mathbf{v}'^2}{2} + m\mathbf{v'}\mathbf{V}(t) + \frac{m\mathbf{V}^2(t)}{2} + mV_g^2 \tag{5}$$

Now $\mathbf{V}^2(t)$ is a given function of time, and can be written as the total derivative with respect to t of some other function, the third term in Lagrangian function L' can therefore be omitt'ed. Next, $\mathbf{v'} = d\mathbf{r'}/dt$, where $\mathbf{r'}$ is the radius vector of the body in the frame K' [2]. Hence

$$m\mathbf{V}(t)\mathbf{v'} = m\mathbf{V}\frac{d\mathbf{r'}}{dt} = \frac{d}{dt}\left(m\mathbf{V}\mathbf{r'}\right) - m\mathbf{r'}\frac{d\mathbf{V}}{dt} \tag{6}$$

Substituting in the Lagrangian and again omitting the total time derivative, and we have finally

$$L' = \frac{m\mathbf{v}'^2}{2} - m\mathbf{a}(t)\mathbf{r'} + mV_g^2 \tag{7}$$

where $\mathbf{a}(t) = d\mathbf{V}/dt$ is *the translational acceleration* of the frame K'. The Lagrange's equation of motion derived from (7) is

$$m\frac{d\mathbf{v'}}{dt} = m\nabla(V_g^2) - m\mathbf{a}(t) \tag{8}$$

Thus an accelerated translational motion of a frame of reference is equivalent, as regards its effect on the equations of motion of a body, to the application of a uniform field of force equal to the mass of the body multiplied by the acceleration $\mathbf{a} = d\mathbf{V}/dt$, in the direction opposite to this acceleration [2].

STEP 1A. MOTION IN THE ACCELERATED $(V_g)^{\mu}$ FIELD

Let us consider the Lagrangian for the body with mass m moving in the scalar field of the square of velocity $(V_g)^2$ and in the vectorial field of velocity of the $\mathbf{V}_{\mathbf{gm}}(t)$

$$L = \frac{mv^2}{2} + m\mathbf{v}\mathbf{V}_{\mathbf{gm}}(t) + mV_g^2 \tag{9}$$

Calculations give equation of motion

$$m\frac{d\mathbf{v}}{dt} = m\nabla(V_g^2) - m\mathbf{a}_{\mathbf{gm}}(t) \tag{10}$$

where $\mathbf{a}_{\mathbf{gm}} = \partial\mathbf{V}_{\mathbf{gm}}(t)/\partial t$ is *the vectorial field of the acceleration*. Comparing two equations (8) and (10) we can see, that they are equal, if and only if, when the $\mathbf{a} = \mathbf{a}_{\mathbf{gm}}$.

STEP 2. ROTATION OF THE FRAME OF REFERENCE

Let us now bring in a further frame of reference K, whose origin coincides with that of K', but which rotates relative to K' with angular velocity $\boldsymbol{\omega}(t)$. Thus K executes both a translational and a rotational motion relative to the inertial frame K_0 [2].

The velocity **v'** of the body relative to K' is composed of its velocity **v** relative to K and the velocity $\boldsymbol{\omega}\times\mathbf{r}$ of its rotation with K: $\mathbf{v}' = \mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}$ (since the radius vectors **r** and **r'** in the frames K and K' coincide). Substituting this in the Lagrangian (7), we obtain

$$L = \frac{mv^2}{2} + m\mathbf{v}(\boldsymbol{\omega}\times\mathbf{r}) + \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 - m\mathbf{a}\cdot\mathbf{r} + mV_g^2 \tag{11}$$

This is the general form of the Lagrangian of a body in an arbitrary, not necessarily inertial, frame of reference. The equation of motion has form[2] [2].

$$m\frac{d\mathbf{v}}{dt} = m\frac{\partial V_g^2}{\partial\mathbf{r}} + m\left(\mathbf{r}\times\frac{d\boldsymbol{\omega}}{dt}\right) + 2m(\mathbf{v}\times\boldsymbol{\omega}) + m(\boldsymbol{\omega}\times(\mathbf{r}\times\boldsymbol{\omega})) \tag{12}$$

We see that the *inertia forces* due to the rotation of the frame consist of three terms:

1. the force $m\left(\mathbf{r}\times\frac{d\boldsymbol{\omega}}{dt}\right)$ is due to the non-uniformity of the rotation,
2. the force $2m(\mathbf{v}\times\boldsymbol{\omega})$ is called *the Coriolis force*,
3. the force $m(\boldsymbol{\omega}\times(\mathbf{r}\times\boldsymbol{\omega}))$ is called *the centrifugal force* [2].

STEP 2A. MOTION OF THE BODY IN THE ROTATING $(V_g)^\mu$ FIELD

Let us consider the Lagrangian function for the body with mass m moving in the scalar field of the square of velocity $(V_g)^2(\mathbf{r})$ and in the vectorial field of the $\mathbf{V}_{\mathbf{gm}}(\mathbf{r}, t)$

$$L = \frac{m\mathbf{v}^2}{2} + m\mathbf{v}\mathbf{V}_{\mathbf{gm}} + \frac{m\mathbf{V}_{\mathbf{gm}}^2}{2} + mV_g^2 \tag{13}$$

Calculations give the equation of motion [3]

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[2] We omitted the term relating to the acceleration – $m\cdot\mathbf{a}\cdot\mathbf{r}$, concentrating on the rotation only.

$$m\frac{d\mathbf{v}}{dt} = m\frac{\partial V_g^2}{\partial \mathbf{r}} + m\left(\mathbf{r}\times\frac{d\boldsymbol{\omega}_{\mathbf{gm}}}{dt}\right) + 2m\left(\mathbf{v}\times\boldsymbol{\omega}_{\mathbf{gm}}\right) + m\left(\boldsymbol{\omega}_{\mathbf{gm}}\times\left(\mathbf{r}\times\boldsymbol{\omega}_{\mathbf{gm}}\right)\right) \tag{14}$$

where: $\boldsymbol{\omega}_{\mathbf{gm}}$ *is the intensity of GM field or the vectorial field of the rotation* [1]. Comparing two equations (12) and (14) we can see, that they are equal, if and only if, when the $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathbf{gm}}$.

We mathematically proved that the inertial forces, which appears in a non-inertial frame of reference, such as accelerating and rotating reference frame, **are equivalent** to the real forces which appear, when the body moves in the GM field (the vectorial field of velocity).

Does this signify that applying the GM field (the vectorial field of velocity) we can eliminate a inertial forces? It is the very important question.

Now we will remind below the rotating bucket with water problem with the new proposal of the solution.

THE ROTATING BUCKET WITH WATER PROBLEM

Berkeley and Mach criticized the Newtonian interpretation of the experiment of the rotating bucket with water [3]. When the bucket rotates with respect to the *fixed star sphere* (FSS) (all bodies in the Universe), then the surface of water has a paraboloidal shape.

Instead of rotating the bucket, assume we could turn, by the same means, the FSS so that the relative rotation is the same.

Newton thought that if we rotated the FSS, then because the motion of the water is described with respect to the absolute space, its surface would be flat. A contrary point of view, that is, that only rotation with respect to the FSS gives curvature of the water surface in the bucket, was proposed by Berkeley and Mach. According to them, the absolute space is unobservable.

Mach pointed out that it does not matter if the Earth is rotating and the FSS is at rest, or stationary Earth is surrounded by the rotating FSS. His idea is based on an empirical fact: two measurements of the Earth’s angular velocity, **astronomical** (with respect to the FSS) and **dynamic** (by means of Foucault’s pendulum experiment), give **the same results** (in the limits of the experimental errors).

THE ROTATING LIQUID MIRROR WITH MERCURY INSTEAD THE BUCKET WITH WATER

The rotating liquid mirror (LM) [4] with mercury, instead the bucket with water, **could confirms** (**or not**) the Berkeley and Mach point of view that the rotation (only) with respect to the FSS gives curvature of the surface of the mercury. If we were to take the LM of the mercury, with the utmost care, to the Earth’s pole, we would find that the surface of the mercury assumes a paraboloidal shape, even when the LM is at the rest relative to the Earth (the Earth with LM is rotating relative to the FSS). According to the equation (12) the height of mercury h(r), in the coordinate system uniformly rotating with respect to the stationary FSS with $\boldsymbol{\omega}$ angular velocity, has the form [5]

$$h(r) = h(0) + \frac{1}{2g_E}(\omega r)^2 \tag{15a}$$

where: $g_E \approx 9{,}81\ m/s^2$ is the G acceleration, h(0) is the height of the mercury at r = 0, r is a radius of mirror. The surface of the mercury is parabolic in its dependence upon the radius of mirror.

According to the equation (14) the height of mercury h(r), in constant homogenous vectorial field of rotation $\boldsymbol{\omega}_{\mathbf{gm}}$, has the form

$$h(r) = h(0) + \frac{1}{2g_E}(\omega_{gm} r)^2 \tag{15b}$$

If we apply the motion to the mercury with the same angular velocity and in the same direction that the FSS rotates, then the surface of mercury would remain flat, because $\omega - \omega_{gm} = 0$ and $h(r) = h(0)$.

CONCLUSIONS

In this paper we mathematically proved that the inertial forces, which appears in a non-inertial frame of reference, such as accelerating and rotating reference frame, **are equivalent** to the real forces which appears, when the body moves in the GM field (the vectorial field of velocity). Does this signify that applying the GM field we can eliminate a inertial forces? It is the very important question.

The new experiment with the liquid mirror with mercury instead the bucket with water **could confirms** (**or not**) the Berkeley and Mach point of view that the rotation (only) with respect to the FSS gives curvature of the surface of the mercury.

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