the general science journal

ON THE SUMS $\sum_k \frac{1}{1/4+t_k^2} = 2+\gamma-\log 4\pi$ **AND** $\sum_\rho \rho^{-n} = \frac{2+\gamma-\log 4\pi}{2}$ **AND THEIR RELATION TO THE RIEMANN HYPOTHESIS AND THE RIEMANN-WEIL FORMULA**

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Addres: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

- *ABSTRACT: We study the sums* $\sum_k \frac{1}{1/4+t_k^2}$ *and* $\sum_\rho \rho^{-n}$ *evaluated over the zeros and the imaginary part of the zeros of the Riemann Zeta function by two methods, the first method involves the use of the Hadamard product formula for the Riemann Xi-function, the second one uses the Riemann-Weill explicit formula , which relates a sum over the imaginary part of the zeros with another sum over prime numbers , we have managed to prove that the sum* $\sum_k \frac{1}{1/4+t_k^2} = 2+\gamma-\log 4\pi$
- *Keywords: = Riemann Zeros, Explicit formula ,Riemann Hypothesis*

INTRODUCTION

For the Riemann Zeta function, $\zeta(s) = \prod_p \frac{1}{\left(1-p^{-s}\right)}$ Re(s) >1 and the Riemann Xi-function defined in terms of the Riemann Zeta in the form $\frac{1}{2}(s-1)s\Gamma\left(s/2\right)\pi^{-s/2}\zeta(s)$ we can give the Infinite Hadamard products on the Riemann Zeros [1]

$$\zeta(s)=\frac{\prod_{\rho}\left(1-\frac{s}{\rho}\right)}{(s-1)\Gamma\left(1+\frac{s}{2}\right)} \qquad \xi(s)=\xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right) \quad (1)$$

The second product on (1) can be used to evaluate the sum $\sum_{\rho}\rho^{-n}$ for n a positive integer different from 0 ,from the symmetry of the functional equation for the Riemann Zeta function if ρ is a Zero so is $1-\rho$ in this case the sum $\sum_{\rho}\rho^{-n}$ must then be understood as a sum over the Riemann zeros on the upper and lower complex plane

$$Z(n)=\sum_{\rho}\rho^{-n}=\sum_{\operatorname{Im}\rho>0}\left((\sigma+it_k)^{-n}+(1-\sigma-it_k)^{-n}\right) \quad (2)$$

In case that RH (Riemann Hypothesis) is true then $\operatorname{Re}(\sigma)=\frac{1}{2}$, so

$\sum_{\rho}\rho^{-n}=\sum_{k>0}\left(\frac{1}{4}+\gamma_k^{2}\right)^{-1}$, an straightforward method to evaluate for positive integer 'n' the sum $\sum_{\rho}\rho^{-n}$ is the following , from the infinite product (1) we can take the logarithmic derivative n-times to get the expression

$$\frac{d^n\log\xi(s)}{dx^n}=-\sum_{\rho}\frac{(n-1)!}{(\rho-s)^n}=\frac{(-1)^{n+1}}{s}+\frac{(-1)^{n+1}}{s-1}+\frac{\Psi^{(n)}(s/2)}{2^n}-\frac{\delta_n^1}{2}\log\pi+\frac{d^n\log\zeta(s)}{dx^n} \quad (3)$$

Here $\Psi^{(n)}(s)=\frac{d^n\log\Gamma(s)}{ds^n}$ is a Polygamma function, $\delta_m^n=\begin{pmatrix}1 & \text{m=n}\\ 0 & \text{m}\neq\text{n}\end{pmatrix}$ is the Kronecker's delta , and $\Gamma(n+1)=n!$ is the Gamma function. For n=1 if we expand the Digamma function near its pole at s=0 , $\Psi^{(1)}\left(\frac{s}{2}\right)\approx-\gamma-\frac{2}{s}$ with gamma the Euler-Mascheroni constant $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)$, the divergent term due to the pole at s=0 of the Digamma function cancels with the term 1/s so we get $\sum_{\rho}\frac{1}{\rho}=Z(1)=\frac{\gamma+2-\log 4\pi}{2}$

In case that RH were true , then $2\sum_{k>0}\left(\frac{1}{4}+t_k^{2}\right)^{-1}=\sum_{k}\left(\frac{1}{4}+t_k^{2}\right)^{-1}=2Z(1)=2+\gamma-\log 4\pi$, unfortunately this can not be deduced from the logarithmic derivative of the Hadamard infinite product for the Riemann Xi-function $\xi(s)=\xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right)$

RIEMANN-WEIL EXPLICIT FORMULA FOR THE SUMS $\sum_k h(t_k)$

To evaluate a sum over the imaginary part of the Riemann zeros $\sum_k h(t_k)$ we can make use of a formula conjectured by Riemann and proved by Weil , which relates a sum over the imagnary part of the zeros to another sum involving prime and prime powers

$$\sum_k h(t_k) = h\left(\frac{i}{2}\right) + h\left(\frac{i}{2}\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} dr \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)h(r) - g(0)\log\pi - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n) \quad (4)$$

The sum on the right is taken over all the imaginary part of the Riemann zeta zeros, if RH were true all the imaginary part of the zeros would be real and $t_{-k} = -t_k$ so $2\sum_{k>0} h(t_k) = \sum_k h(t_k)$, provided h(x) is an even function of ‘x’ , in general the test functions g(x) and h(x) used in (4) must satisfy the following properties

- $h(x) = h(-x)$ and $g(x) = g(-x)$ with $\frac{1}{\pi}\int_{-\infty}^{\infty} dr h(r)\cos(ur) = g(u)$
- $h(z)$ must be an analytic function on the region of complex plane with $|\operatorname{Im}(z)| < \frac{1}{2} + \delta$ for any positive and real delta
- $g(x) = O\left(e^{-mx}\right)$ as $|x| \to \infty$ $m > \frac{1}{2}$
- $h(x)$ behaves at least as $O\left(|x|^{-2}\right)$ whenever $|x| \to \infty$ to make sure that the sum $\sum_k h(t_k)$ will be convergent

Here $\Lambda(n)$ is the Mangoldt function defined like this $\Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & \text{otherwise} \end{cases}$ and ‘k’ is a positive integer, the Dirichlet generating series for this Mangoldt function is $-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$ $\operatorname{Re}(s) > 1$, a good introduction to this formula is given by Voros [8] by using the residue theorem applied to the contour integral $\frac{1}{4\pi i}\int_{C(T)} dz h(-iz) d\log G\left(\frac{1}{2}+iz\right)$ along an appropiate contour in the complex plane , with $T \to \infty$,here ‘G’ is expressed in terms of the Riemann Zeta and the Gamma function $G(x) = \pi^{-x/2}\Gamma\left(x/2\right)\zeta(x)$ and ‘h’ is a test function that satisfy certain conditions.

If we use as test functions $h(a,x) = \frac{2a}{a^2+x^2}$ and $g(a,x) = e^{-a|x|}$ with $|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$ in this case the Riemann-Weil explicit formula

$$\sum_k \frac{2a}{a^2+t_k^2} = \frac{16a}{4a^2-1} + \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{dr}{a^2+r^2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right) - \log\pi + 2\frac{\zeta'}{\zeta}\left(\frac{1}{2}+a\right) \quad (5)$$

Unfortunately inside (5) we must put $a > \frac{1}{2}$ due to a pole at $s = \pm\frac{i}{2}$ then in order to evaluate the sum $\sum_k \frac{4}{1+4t_k^2}$ we will have to use some kind of regularization including an small parameter $\varepsilon \to 0$ with $\varepsilon^2 = 0$, this regularization can be explained in the following manner

a) We replace the series $\sum_k \frac{1}{1/4+t_k^2}$ by $\sum_k \frac{1}{1/2+\varepsilon+it_k} + \sum_k \frac{1}{1/2+\varepsilon-it_k}$

b) In order to avoid the poles at $s = \pm\frac{i}{2}$ of the function $\frac{4}{4x^2+1}$ we have added an small pole near the imaginary part of the zeros $t_n^+ \to t_n^+ - i\varepsilon$ (upper complex plane) and $t_n^- \to t_n^- + i\varepsilon$ (lower complex plane)

c) We use the $\varepsilon-$dependant functions $h_\varepsilon(x) = (1+2\varepsilon)\left(\left(\frac{1}{2}+\varepsilon\right)^2 + x^2\right)^{-1}$ and $g_\varepsilon(x) = \exp\left(-|x|\left(\frac{1}{2}+\varepsilon\right)\right)$ inside the Riemann-Weil formula

In the following sections we will evaluate each term on the Riemann-Weil formula to compute the sum $\sum_k \frac{1}{1/4+t_k^2}$

- *Evaluation of the integral involving the Digamma function* $\Psi\left(\frac{1}{4}+\frac{ir}{2}\right)$*:*

In the limit $\varepsilon \to 0$ the test function $h_\varepsilon(r) \to \left(\frac{1}{4}+r^2\right)^{-1}$, then we must evaluate the following integral $\int_{-\infty}^{\infty} \frac{dr}{1/4+r^2}\Psi\left(\frac{1}{4}+\frac{ir}{2}\right)$, if we use the series representation for the Digamma function $\Psi(s) = \frac{d\log\Gamma(s)}{ds} = -\gamma - \sum_{n=1}^{\infty}\left(\frac{1}{n+s-1} - \frac{1}{n}\right)$ then

$$\int_{-\infty}^{\infty} \frac{dr}{1/4+r^2}\Psi\left(\frac{1}{4}+\frac{ir}{2}\right) = \int_0^{\infty} \frac{dr}{1/4+r^2}\left(\Psi\left(\frac{1}{4}+\frac{ir}{2}\right)+\Psi\left(\frac{1}{4}-\frac{ir}{2}\right)\right) = -\int_0^{\infty}\frac{2\gamma dr}{1/4+r^2} - \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{dr}{1/4+r^2}\left(\frac{32n-24}{(3-4n)^2+4x^2} - \frac{2}{n}\right) = -2\pi\gamma - \sum_{n=1}^{\infty}\frac{2\pi}{2n^2-n} = -2\pi\gamma - 4\pi\log 2 \quad (6)$$

Also , in the limit $\varepsilon \to 0$ $g_{\varepsilon}(r) \to \exp\left(-\frac{|r|}{2}\right)$, so the contribution of the smoot part $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dr}{1/4+r^2}\Psi\left(\frac{1}{4}+\frac{ir}{2}\right)-g(0)\log\pi$ to the Riemann-Weil explicit formula is equal to $-\gamma-\log 4\pi$

- *Evaluation of the sum* $-2\sum_{n=1}^{\infty}\Lambda(n)n^{-s-1}=2\frac{\zeta'(s)}{\zeta(s)}$:

For 's' bigger than 1 the sum $-2\sum_{n=1}^{\infty}\Lambda(n)n^{-s-1}$ is just proportional to the logarithmic derivative of the Riemann Zeta function , this logarithmic derivative of zeta $\zeta(1+s)$ can be expaned into a power series involving the Stieltjes's constants

$$-\frac{\zeta'}{\zeta}(s+1)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}=\frac{1}{s}+\sum_{j=0}^{\infty}\frac{\eta_j s^j}{j!}(-1)^j \qquad \gamma_n=\lim_{r\to\infty}\left(\sum_{k=1}^{r}\frac{\log^n(k)}{k}-\frac{\log^{n+1}r}{n+1}\right) \qquad (7)$$

The first term in the power series are $\eta_0=-\gamma_0$ (Euler constant) $\eta_1=-(2\gamma_1+\gamma_0^2)$ and $\eta_2=-(3\gamma_2+6\gamma_1\gamma_0+2\gamma_0^3)$. This formula can be deduced from the theory of the Cumulant generating function explained on [9] . In the lmit $\varepsilon \to 0$,from formula (7) the sum involving the Mangoldt function can be splitted into a finite part and a divergent part as $-2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\varepsilon}}\approx 2\gamma-\frac{2}{\varepsilon}$

- *Evaluation of the term* $h_{\varepsilon}\left(\frac{i}{2}\right)+h_{\varepsilon}\left(-\frac{i}{2}\right)$ in the limit $\varepsilon \to 0$:

If we use the Fourier transform pair $h(a,r)=\left(a^2+r^2\right)^{-1}$ and $g(a,r)=\frac{e^{-a|x|}}{2a}$ with $a=\frac{1}{2}+\varepsilon$ for positive epsilon $\varepsilon>0$, since in this case $h\left(\frac{1}{2}+\varepsilon,r\right)$ is analytic on the region of the complex plane $|\mathrm{Im}(z)|<\frac{1}{2}+\delta$ providing $\varepsilon>\delta$ for any set of positive (δ,ε) , then we can apply the analytic continuation to the parameter 'u' inside the Fourier integral to the complex plane $\int_{-\infty}^{\infty}dxg(x)e^{iux}$ $u=\pm\frac{i}{2}$ in order to get

$$h\left(\frac{1}{2}+\varepsilon,\frac{i}{2}\right)+h\left(\frac{1}{2}+\varepsilon,\frac{i}{2}\right)=2\int_{-\infty}^{\infty}dx\cosh\left(\frac{x}{2}\right)\exp\left(-\left(\frac{1}{2}+\varepsilon\right)|x|\right) \qquad (8)$$

The last integrals on the right of (8) can be evaluated using the properties of $|x|$ and making a change of variable $x \to -x$ to turn an integral over the interval $(-\infty, 0]$ into an integral over $[0, \infty)$

$$\int_{-\infty}^{0} dx e^{\frac{x+2\varepsilon-x}{2}} + \int_{0}^{\infty} dx e^{\frac{-2x+2\varepsilon}{2}} + \int_{-\infty}^{0} dx e^{\frac{2x+2\varepsilon}{2}} + \int_{0}^{\infty} dx e^{\frac{x+\varepsilon-x}{2}} = \frac{2}{\varepsilon} + \frac{2}{1+\varepsilon} \quad (9)$$

$\int_{0}^{\infty} dt e^{-st} = s^{-1}$ for positive 's' , if we put all the contributions together then we have

$$\sum_k \frac{1+2\varepsilon}{1/4+\varepsilon+t_k^2} = \frac{2}{\varepsilon} + \frac{2}{1+\varepsilon} + 2\gamma - \frac{2}{\varepsilon} - \gamma - \log 4\pi = 2 + \gamma - \log 4\pi \qquad \varepsilon^2 = 0 \quad (10)$$

The sum $\sum_k \frac{1+2\varepsilon}{1/4+\varepsilon+t_k^2} = S$ is convergent, so $\varepsilon S \to 0$ as $\varepsilon \to 0$, using the Riemann-Weil summation formula $\sum_k \frac{1}{1/4+t_k^2} = 2 + \gamma - \log 4\pi = 2Z(1)$, of course we could have simple ignored the divergent terms ε^{-1} and keep only the finite part on each term $F.p\left\{ h\left(\frac{i}{2}\right) + h\left(-\frac{i}{2}\right) \right\} = 2$, $F.p\left\{ 2\frac{\zeta'}{\zeta}(1) \right\} = 2\gamma$ to get the same result $2 + \gamma - \log 4\pi$.

The explanation to this fact would be the following , the imaginary part of the Riemann zeros are real , so from the functional equation for the Riemann zeta function $\zeta(1-s) = 2(2\pi)^{-s}\Gamma\left(\frac{s}{2}\right)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$, the imaginary part satisfy $t_k = -t_{-k}$, then $\sum_k \frac{1}{1/4+t_k^2} = 2\sum_{k>0} \frac{1}{1/4+t_k^2} = 2Z(1)$, $\sum_{k>0} \frac{1}{1/4+t_k^2} = Z(1) = \sum_\rho \frac{1}{\rho}$,we believe that this is a good indication that the Riemann Hypothesis is true and all the zeros will lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

In this paper, we have used 2 methods to evaluate the sum $\sum_\rho \frac{1}{\rho}$, the first one involves a Hadamard product over the Riemann Zeros ρ on the critical strip $0 < \mathrm{Re}(s) < 1$ and then taking the logarithmic derivative, the other method has been to evaluate a sum of the form $\sum_k \frac{1}{1/4+t_k^2}$ over ALL the imaginary part of the Riemann zeros , if we label the Riemann zeros as $\rho_n = \frac{1}{2} - it_n$ with $t_n \in C$, $\sum_k \frac{1}{1/4+t_k^2} = \sum_{\mathrm{Im}\rho>0} \left(\frac{1}{|\rho|^2} + \frac{1}{|1-\rho|^2} \right)$,this last sum is taken over ALL the Riemann zeros.

For an introduction to the Hadamard finite product $\xi(s)=\xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right)$ the best book is Apostol [1] , the Riemann-Weil explicit formula and the series $\sum_{\rho}\rho^{-n}$ for real 'n' is discussed in Voros [8] , for the integral involving the Digamma function on the critical line $\Psi\left(\frac{1}{4}+\frac{is}{2}\right)$ and the power series expansion of the derivative of zeta $\frac{d\log(s\zeta(1+s))}{dx}$, we have used the reference [7]

References

[1] Abramowitz, M. and Stegun, I. A. (Eds.). "Riemann Zeta Function and Other Sums of Reciprocal Powers." §23.2 in "*Handbook of Mathematical Functions*". New York: Dover, pp. 807-808, 1972.

[2] Apostol Tom "Introduction to Analytic Number theory" ED: Springuer-Verlag, (1976)

[3] Conrey, J. B. "*The Riemann Hypothesis*." *Not. Amer. Math. Soc.* **50**, 341-353, (2003) .http://www.ams.org/notices/200303/fea-conrey-web.pdf.

[4] Hardy G. (1949). "*Divergent Series* ". Clarendon Press, Oxford.

[5] Kammler, David (2000), *A First Course in Fourier Analysis*, Prentice Hall, ISBN 0-13-578782-3

[6] Keiper, J. B. "*Power Series Expansions of Riemann's* ξ *Function*." *Math. Comput.* **58**, 765-773, (1992)

[7] Math Stack Exchange , (Personal Communication) Forum<math.stackexchange.com/q/71230/13854>;. Avaliable oline at http://math.stackexchange.com/questions/71230/digamma-function-integral

[8] Voros A. "*Zeta function over zeros of Zeta functions*" Ed. Springer (2010) ISBN 9783642052033

[9] Weisstein, Eric W. "*Cumulant-Generating Function.*", *MathWorld*--A Wolfram Web Resource.http://mathworld.wolfram.com/Cumulant-GeneratingFunction.html