On the Impossibility of Separating Clocks from Rulers

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Abstract: The Lorentz Transformations imply that time and length are in some sense interconvertible, much in contrast to our ordinary intuitions. This paper attempts to present an approach which is supposed to make it intuitively evident that time and length are in fact interconvertible and, furthermore, that this approach is compatible with two well-known phenomena predicted by SR, namely time dilation and length contraction. This is accomplished by demonstrating how a clock can be used as a ruler, and vice versa, leading to the realization that length contraction and time dilation directly imply each other in the context of the motion of the same measurement device.

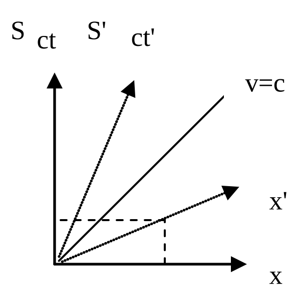
What makes both kinds of measurements using the same device really possible in the first place is the existence of a finite upper limit on motion. Because of this limit, length measurements cannot be completed without involving the passage of time, and time measurements cannot be completed without involving finite displacement. But that means any clock or ruler really measures both time and length. Hence one constraint on what is fundamentally possible in physics is our ability to build measurement devices that can be exclusively used just as rulers or just as clocks.

I. Introduction

Is it possible to interconvert time and length? The most likely immediate reaction of most of us might be that that it is not. Yet Einstein's theory of special relativity (SR), which can be regarded as our most fundamental physical theory on the relationship between time and length, seems to tell a different story. Its key mathematical feature is contained in the well-known Lorentz Transformations (LT) (1).

The LT have been unequivocally confirmed by experiment, but their implication, that relative motion leads to observations of different distances in times and space between the same events in frames that are in motion with respect to each other, seems highly counterintuitive. The non-intuitiveness of the LT's is especially evident when they are represented graphically.

Fig. 1



As fig.1 shows, a graph comparing the x and t axes of a rest Frame S against the x' and t' axes of a moving frame S' coinciding in its origin with that of the S frame clearly indicates that for any constant relative velocity smaller than c, the time and length dimensions between the two frames mix. One wonders how one is to understand that, say, a quantity of length along x' in S' corresponds to quantity of length along x *and* a quantity of time along t in S. The usual riposte to this question is to invoke the fact that this is forced upon us as a logical consequence of the postulates of SR. While this argument is logically correct, it unfortunately does not make the apparent interchangeability between time and length intuitively any more intelligible.

The purpose of this paper is to present an approach which is supposed to make it intuitively evident that time and length are in fact interconvertible and, furthermore, that this approach is compatible with two well-known phenomena predicted by SR, namely time dilation and length contraction. Perhaps such an understanding may even lead to a better intuitive understanding of the LT. In this respect, one would want to show ideally that length contraction and time dilation can be understood intuitively independent of one another. But if this were possible, then it would also seem to be possible to dispense altogether with the postulates of SR as fundamental axioms, for then one could frame the postulates as *consequences* of the LT's (which under this scenario would have been themselves intuitively derived). This seems rather like a tall order, because at least one the axioms of SR, the speed of light postulate, is itself on its face highly counterintuitive, some recent attempts at explaining it (2), including one by this author (3), notwithstanding. Thus understanding length contraction and time dilation independent of one another and without reference to the fundamental postulates of SR or an equivalent set of axioms may not be attainable. It may, however, be possible to attain a more modest goal, namely, once it is understood how time and length are interchangeable, to intuitively grasp either time dilation or length contraction when the other is already given, without direct reference to the postulates of SR.

In section II., it will be demonstrated how a clock can be used as a ruler, and that by using it in such way in a moving frame, time dilation directly forces measurements of contracted lengths when the measurement of the two is associated with the motion of the same measurement device. In the section after that, it will be demonstrated how a ruler can be used as a clock, and that by using in that way in a moving frame, length contraction directly forces the measurement of dilated time when the measurement of the two is associated with the motion of the same measurement device. Next, the role of a finite upper limit on motion as the origin of the dual usability of these devices will be highlighted, leading to the conclusion that it is impossible to build measurement devices that can be exclusively used just as rulers or just as clocks

II. The Clock as a Ruler

Consider a regular clock with a face on which the time marks are arranged in a circle and a moving arm which indicates, say, the minutes. To make it easier to see how to use this clock as a ruler, we first imagine that we can straighten the arrangement of the time marks. Thus, the time marks are now taken to be arranged in a straight line and the arm of the clock is replaced by a pointer that moves along them. Such a clock will from now on be called a *straight clock*. For simplicity, assume that the velocity of the pointer, as it moves along the time marks of the straight clock is constant. We can imagine that we will use the straight clock only for time periods shorter than 12 hours in such a way that the pointer always passes the marks sequentially. That way, we do not need to concern ourselves with issues of simultaneity at distinct locations that would otherwise arise when the pointer reaches the end on one side and needs to be matched in time against a pointer beginning at the other end. The distance between each minute mark is obviously pre-determined by the fact that it must take the pointer one minute from the instant it coincides in space with a given minute mark to the instant it coincides in space with the next minute mark. The calibration for the pointer is set in a rest frame S by a second straight clock, which we shall leave at rest at the origin in the S frame and therefore call the S-clock.

Under these conditions, it is straightforward to see that a clock can also serve as a ruler: The distance swept out by the pointer can be measured in terms of the minute marks, and the speed of the pointer, since its velocity is constant, provides a simple conversion factor from time to length: If the speed of the pointer is given by v, then by going from the zero mark to the mark which indicates time t, the distance swept out is simply vt. So, just by making what is really supposed to be a *time measurement* (reading off the time mark on a clock) one can determine a distance, which in this case is that of the pointer from the origin.

It does not appear to be essential to use straight clocks as we did. A regular clock with time marks arranged in a circle could have been used in the above scenario to measure distances (it would be an excellent device for measuring angular distances associated with radii equal to the clock's arm). Similarly, a clock for which the time marks are, say, audio signals could have been used, so long as we could quantitatively identify the motion inherent in the process of time measurement. That's because motion at a finite speed (at most the speed of light) guarantees that a time measurement can be converted into a distance measurement. For those clocks, the analysis might be much more cumbersome than above, but not impossible (at least in principle) because the essential ingredient, motion, can never be completely eliminated from a time measurement when it does not occur infinitely fast. Put more generally, a measurement of time is really a measurement of change in a standardized manner (operationally defined through some naturally occurring periodicity), and change implies motion at finite speed.

Now let us suppose that a straight clock is at rest at the origin in a frame S' which is moving at a constant velocity relative to S in the standard configuration (from left to right, in the direction of positive x), and call this the S'-clock. Further suppose that the time marks of both straight clocks are arranged *perpendicularly* to the direction of relative motion (say, along the y-axis) and the distances between them are the same in both frames.

By arranging them in this way, we can exclude any direct length contraction effects, since we want to see how time dilation by itself affects the function of our moving clock as a ruler (*we take time dilation as a given*). For clarity, further suppose that as the origin of S' coincides with that of S, the pointers of both clocks coincide with each other at the origin (i.e. $x(t_0) = x(0) = x'(t'_0) = x'(0) = 0$), as in fig. 2a:

Fig.2a

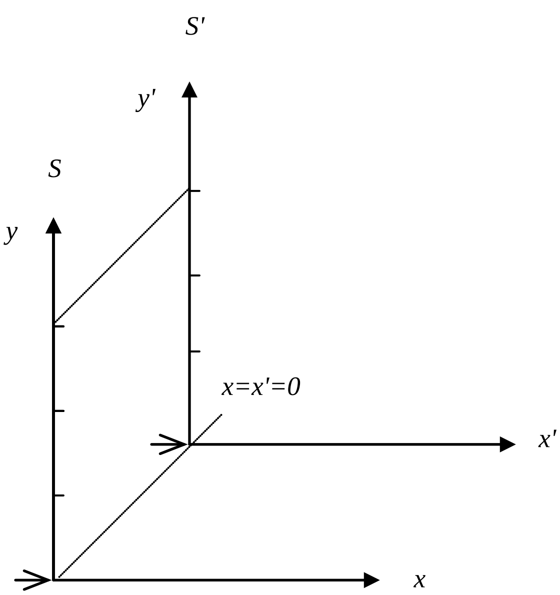
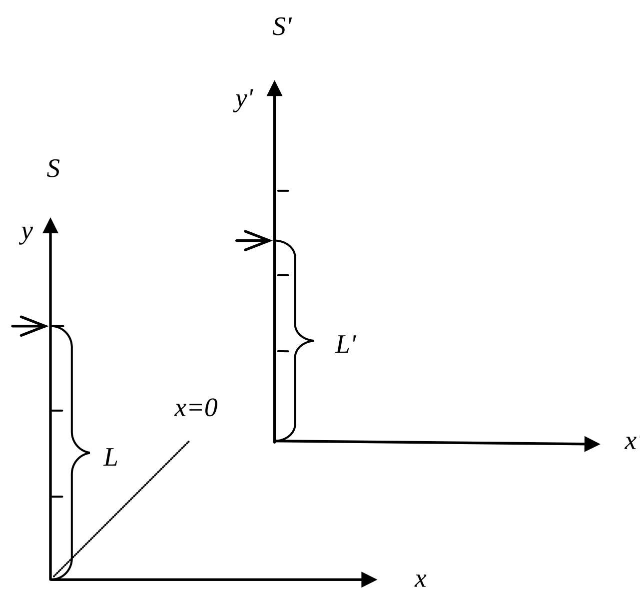


Fig. 2b



If $t_1 - t_0 = t_1 - 0 \equiv t$ is the time interval measured by the S-clock and $t'_1 - t'_0 = t'_1 - 0 \equiv t'$ is the time interval measured by the S'-clock, then the consequence of time dilation in the moving frame is that for any time interval measured by the S-clock in the S frame, the time interval measured by the S'-clock will in the S frame be observed to be shorter, since time is observed to pass more slowly in S'. The factor by which it passes more slowly is the well-known $\gamma -$ factor, but this means is that the length L' swept out by the pointer of the S'-clock will be shorter than the length L swept out by the pointer of the S-clock by the same factor (see fig 2b.).

Here, then, is a direct demonstration that the observation of slowing time by itself leads to the observation of a corresponding shortening of distances *when the two measured quantities are measured by the same device in motion*. To put it more provocatively, we have *interchanged* a quantity of time with a quantity of length by using a clock as a ruler, and by using it in motion we have converted a time dilation phenomenon into a length contraction phenomenon. We will now attempt to demonstrate the reverse, namely how to interchange a quantity of time with a quantity of length, by using a ruler as a clock.

III. The Ruler as a Clock

Consider a regular measuring rod with distance marks of equal length intervals. We now wish to use this ruler as a clock. To do so, consider first how we would use it to measure distances. To keep it simple, let us suppose that our procedure is as follows: we put the ruler parallel to the distance to be measured in such a way that its zero mark coincides with one end of the distance to be measured, and then measure the distance by determining where the other end of the distance coincides with a distance mark on the ruler. Note that for a single observer to make this kind of a length measurement, he must be in motion, because the measurement requires him to be (eventually) at two distinct locations: the zero mark and the location at which the other end of the distance coincides with the rod. We tend to neglect this aspect mainly because the velocity of this motion is totally unimportant as long as the observer's position eventually coincides with the locations of the two length marks. In contrast, for a time measurement the velocity of the motion is vitally important. Consider, for instance, that if our observer were to move at a constant velocity along his ruler, he would become functionally indistinguishable from one of the pointers of the straight clocks mentioned above. The magnitude of his velocity would then serve as a conversion factor from length to time: If he moved at a constant velocity v, we would know that when his location coincides with a mark on the ruler at x, an amount of time equal to x/v has passed since the instant he was coincident with the ruler's zero mark. In this case, then, what is supposed to be a *length measurement* (reading off the distance mark on a ruler) allows the determination of a time interval. So in the most straightforward case, namely that involving constant velocity along a non-zero length or time interval, the operational procedure for measuring distance and time is exactly the same!

Even if we replace our single observer by two observers, one located at each end of the distance to be measured, we still cannot avoid motion. That's because, ultimately, the measurement of a distance consists of a comparison between two observations at distinct spatial locations which must be brought together in a single entity. This last step, which is usually ignored, is necessary to complete the measurement. Given a finite universal speed limit, this entity can never be located at both ends of the distance at once, but that is precisely the condition that is required to completely eliminate motion from the measurement. With two observers, the motion of a single observer as above is replaced by the motion associated with the transmission of the information about the data to the entity that makes the distance determination, and this motion can only go as fast as c. Thus, even if two observers at distinct locations a distance L apart carry out a 'simultaneous' length measurement, the *completion* of the measurement still requires at minimum an amount of time equal to $L/2c$ (when the entity in which the two data are brought together is located exactly between the two observers and the information is transmitted by them at the speed of light).

Using two observers or more complicated arrangements to make a length measurement is therefore analogous to using clocks that are more complicated than the straight clocks mentioned above in that both further obscure the essential role of motion subject to a finite upper limit.

If we now wish to compare two rulers in relative motion to each other in their function as clocks, we take length contraction as a given, and must exclude the effect of time dilation on our experiment. Suppose then, that there is a ruler at rest with respect to the S'-frame aligned *parallel* to the direction of relative motion and call it the S'-ruler. Suppose further that in its rest frame, the S'-ruler measures out a distance L' and that its zero mark coincides with the origin of the S'-frame. Suppose we have another ruler at rest in the S-frame that measures out the distance L, such that if the two rulers were at rest relative to each other,

$L = L'$. Let us call the second ruler the S-ruler and align it also parallel to the direction of relative motion in such way that its zero mark coincides with the origin of the S-frame and its distance marker at L is to the right of the origin of the S-frame (i.e. it is located at $x = L$). Now consider the situation in which the S'-frame is in constant motion relative to the S-frame which, as usual, is from left to right in the direction of positive x. As Fig. 3a shows, we consider as our initial situation the instant the origin of the S-frame coincides with a distance at in S' at which the L' mark of the ruler is located. Now, as the S' frame continues to move to the right, we have a situation in which the two rulers can clearly function as clocks for their own frames: The zero mark of the S-ruler functions as a pointer indicating the passage of time measured by the S'-ruler in the S' frame, and the L' mark of the S'-ruler, as it moves by, functions as a pointer indicating the passage of time in the S frame measured by the S-ruler. Notice that by virtue of this set-up, time dilation plays no role in this situation.

Fig. 3a

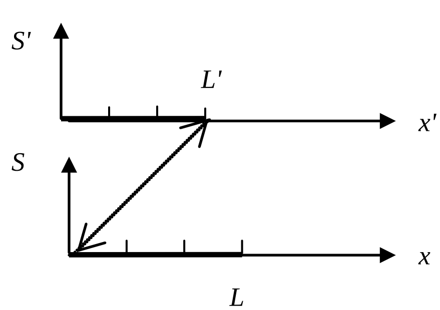
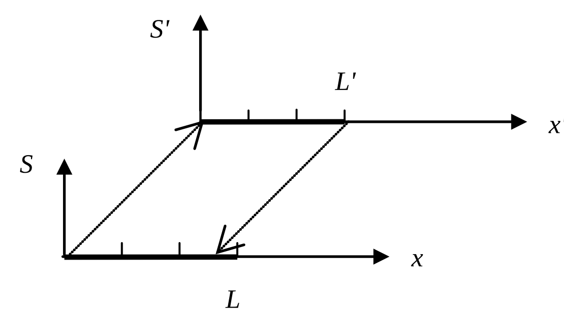


Fig. 3b



Given length contraction, we know that the S'-ruler is contracted relative to the S-ruler in the S-frame by a factor equal to γ. Therefore, when the zero mark of the S'-ruler finally coincides with the zero mark of the S-ruler, as shown in fig. 3b, the L' mark has not yet reached the L mark. But this means that the time interval $t' = L'/v$ measured out by using the S'-ruler as a clock must be shorter by the same factor than the time interval $t = L/v$ measured by using the S-ruler as a clock, which means that the S'-ruler was observed to measure time more slowly than the S-ruler.

Here, then, is a direct demonstration that merely the observation of shortening of distances leads to the observation of a corresponding slowing of time when the two are measured in the context of the motion of the same measurement device. To put it more provocatively again, we have *interchanged* a quantity of length with a quantity of time by using a ruler in motion as a clock and converted a length contraction phenomenon into a time dilation phenomenon.

IV. The Importance of an Upper Limit on Motion

In one sense, the physical findings presented here are already implied by the LT's and even their graphic representation. However, it may not have been realized that time dilation and length contraction *directly* imply each other, i.e. that if one of the two is given, no direct invocation of the postulates of SR is necessary to predict the other. The realization of this may make it possible to understand in a less abstract way how time and length really are interchangeable.

The easiest way to visualize this interchangeability is by imagining the relationship between the straight clock and the ruler in which the motion associated with a distance measurement is constant. In that situation, it is most obvious how the operational procedure for measuring time and length intervals is exactly the same. What differs is the emphasis on the quantities that are measured. For a length measurement, it does not matter how much time it takes to get from one mark to the other, as long as one eventually gets there, as that is required to complete the length measurement. On the other hand, it is very important that the distance between the marks be correctly calibrated. For a time measurement, the emphasis is exactly reversed: Now it does not matter how far away the marks are from each other (as long as the ratio of the distances between time marks to their face value is less than the speed of light), but it is very important that the time it takes to get from one mark to the next be correctly calibrated.

What makes both kinds of measurements using the same device really possible in the first place is motion. The importance of motion is due to the existence of a finite upper speed limit, the speed of light. If no finite limit existed, that is, if it were possible to transmit information at an infinite speed, then this would eliminate the requirement for motion: An observer could then receive information about two distinct locations the very moment they are made, which for the purposes of a length measurement is equivalent to an observer who is able to exist at two distinct locations at once. But an entity which exists at two distinct locations at once undergoes, in effect, *no* motion at all to get from one location to the other. Hence, it would in principle become possible to use a ruler ‘just as a ruler’ without its also functioning, however

subtly, as a clock, because the length measurement could then be completed in zero time. Similarly, if an observer could be at two (or more) time markers on a clock in the same instant, it would then become possible in principle to use a clock ‘just as a clock’ without its also subtly functioning as a ruler: The *same* observer could simply stand by at each time mark and mark off the time at each location, and because he is located at two distinct time marks simultaneously, he undergoes in effect no displacement.
Both of these consequences are operational manifestations of the fact that length and time have decoupled, just as we should expect in a Galilean universe, i.e. one in which $c = \infty$ and $t = t'$.

V. Conclusion

The Lorentz transformations ultimately tell us that in our actual universe, no clock is just a clock and no ruler is just a ruler, but each is both. However, the LT also abstract out the fact that any clocks and rulers we could use to measure distance and time intervals are made out of matter. But all matter is fundamentally in motion: it is made up of atoms which continually are in motion that is subject to a finite upper limit. One can imagine that as atoms move relative to one another, they function both as distance and time markers, thereby casting onto themselves and onto other atoms the roles of tiny rulers and clocks, which is then aggregated in the macroscopic picture of matter into something which functions both as a ruler and a clock. In that case, the findings from this paper apply: If there is a finite upper limit to motion, then length measurements cannot be completed without involving the passage of time, and time measurements cannot be completed without involving finite displacement. But this means that any length measurement also subtly involves a measurement of time, and any time measurement subtly involves a measurement of length. The LT's tell us that just because we ignore one function in deference to the other, that does not mean the former does not physically manifest itself, which is why they seem so counterintuitive to us. In the end this means that one constraint on what is fundamentally possible in physics is our ability to build measurement devices that can be exclusively used just as rulers or just as clocks.

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