Gravitational Waves versus Cosmological Perturbations: Commentary to Mukhanov's talk

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Abstract

Recently, on the conference "Quantum Theory and Gravitation" held in Zürich on June 14-24, 2011, V.F. Mukhanov has been presented talk "Massive Gravity" discussing the relationships between massive gravitational waves and Cosmological Perturbations of the Minkowski background. His crucial result was modification of the Newtonian potential of universal gravitation due to a multiplicative constant equal to $4/3$.

However, this presentation has been stirred up my negative opinion. The controversy has been caused by absence of a lot of details, what have been made the talk manifestly misleading. The lecturer did not respond to my questions satisfactory.

Mukhanov's deductions are at most half-true, and they can be easily verified by straightforward calculations. In this paper I explain shortly what is right and what is wrong in the approach propagated by Mukhanov. Particularly, I shall to show that restoration of the Newton law of universal gravitation is unambiguous.

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1 Introduction

On the most recent conference "Quantum Theory and Gravitation" held in Zürich on June 14-24, 2011, V.F. Mukhanov has presented the talk entitled "Massive Gravity" discussing the relationships between massive gravitational waves and cosmological perturbations of the Minkowski space-time. His crucial result was modification of the Newtonian potential of universal gravitation due to a multiplicative constant equal to $4/3$ in the massless limit of gravitational waves.

However, this presentation has been stirred up my negative opinion. The crucial controversy has been caused by absence of a lot of details and important references to the referred results, which have been made the talk manifestly misleading. The lecturer did not respond to my doubts and questions satisfactory.

Deductions presented by Mukhanov are, unfortunately, at most half-true. Their credibility can be easily verified by straightforward insight into detailed calculations. In this paper I explain shortly what is right and what is wrong in the approach propagated by Mukhanov. Particularly, it is shown that restoration of the Newton law of universal gravitation in this approach is not ambiguous.

2 Analysis of the Problem

Let us consider the perturbed Minkowski background

$$ds^2 = -(1+2\Phi)(dx^0)^2 + 2B_{,i}dx^i dx^0 + [(1-2\Psi)\delta_{ij} + 2s_{ij}]dx^i dx^j, \tag{1}$$

in frames the cosmological perturbation theory [1, 2]. Here

$$\Psi = -\frac{1}{6}h^k_k, \tag{2}$$

$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}h^k_k\delta_{ij}\right), \tag{3}$$

and Φ is the gravitational potential. The Einstein tensor to this case is

$$G_{00} = 3\nabla^2\Psi + \frac{1}{2}\partial_k\partial_l h^{kl}, \tag{4}$$

$$G_{0j} = \partial_0\partial_k\left(3\delta^k_j\Psi + \frac{1}{2}h^k_j\right), \tag{5}$$

$$G_{ij} = \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\Phi + \partial_0 B) - 3(\delta_{ij}\Box - \partial_i\partial_j)\Psi - \tag{6}$$

$$- \frac{1}{2}\Box h_{ij} + \partial_k\partial_{(i}h^k_{j)} - \frac{1}{2}\delta_{ij}\partial_k\partial_l h^{kl}, \tag{7}$$

where $\Box = -\partial_0^2 + \nabla^2$ is the Dealambercian. The Einstein field equations $G_{\mu\nu} = \kappa T_{\mu\nu}$ for the perturbed Minkowski metric (1) can be constructed immediately. The $0j$-component can be solved by straightforward integration. The result is

$$h^k_j = -6\delta^k_j \Psi + 2\kappa \int\int T_{0j} dx^0 dx^k. \tag{8}$$

By contraction of both sides of this solution with δ^j_k and applying the relation (2) one receives

$$\Psi = \frac{\kappa}{6} \int\int T_{0k} dx^0 dx^k, \tag{9}$$

and in this manner the solution (8) can be presented in the form

$$h^i_j = -\kappa\delta^i_j \int\int T_{0k} dx^0 dx^k + 2\kappa \int T_{0j} dx^0 dx^i. \tag{10}$$

Let us notice that now it is easy to deduce the following relations

$$\begin{aligned}
\partial_k\partial_l h^{kl} &= \kappa \int \partial^0 T_{0k} dx^0, && (11)\\
\partial_k\partial_{(i} h^k_{j)} &= \kappa \int \partial_{(i} T_{0j)} dx^0, && (12)\\
\Box h_{ij} &= \kappa\delta_{ij}\left(\int \partial^0 T_{0k} dx^k - \int \partial^k T_{0k} dx^0\right) + && (13)\\
&+ 2\kappa\left(-\int \partial^0 T_{0i} dx_j + \int \partial_j T_{0i} dx^0\right), && (14)\\
\Box\Psi &= \frac{\kappa}{6}\int \partial^k T_{0k} dx^0, && (15)\\
\partial_0^2\Psi &= \frac{\kappa}{6}\int \partial^0 T_{0k} dx^k. && (16)
\end{aligned}$$

Performing minor algebraic manipulations the 00-component can be easy presented in the form

$$\nabla^2\Psi = \frac{\kappa}{3} T_{00} - \frac{\kappa}{6}\int \partial^k T_{0k} dx^0, \tag{17}$$

while taking contraction of both sides of the $0j$-component with δ^{ij} one obtains the modified Poisson equation for the gravitational potential Φ

$$\nabla^2\Phi = -\nabla^2\partial_0 B + \frac{\kappa}{2} T + \frac{\kappa}{2} T_{00} - \frac{\kappa}{2}\int \partial^0 T_{0k} dx^k + \frac{3\kappa}{4}\int \partial^0 T_{0k} dx^0 - \frac{3\kappa}{4}\int \partial^k T_{0k} dx^0, \tag{18}$$

where $T = \delta^{ij}T_{ij} = T^k_k$.

Let us consider now the gravitational waves called also gravitons in the sense of perturbative quantum gravity. In such a situation the metric is

$$ds^2 = (\eta_{\mu\nu} + \widetilde{h}_{\mu\nu})dx^\mu dx^\nu, \tag{19}$$

where $\widetilde{h}_{\mu\nu}$ is such a perturbation that

$$|\widetilde{h}_{\mu\nu}| \ll 1. \tag{20}$$

Comparing this solution to the perturbed Minkowski background (1) one receives

$$\begin{aligned} \widetilde{h}_{00} &= -2\Phi, & (21)\\ \widetilde{h}_{0i} &= \frac{1}{2}B_{,i}, & (22)\\ \widetilde{h}_{ij} &= h_{ij}. & (23) \end{aligned}$$

Using the solution (8) and the condition (20) one receives the consistency condition

$$\det\left(-6\delta_{ij}\Psi + 2\kappa\int\int T_{0i}dx^0 dx_j\right) \ll 1 \tag{24}$$

which have not been noticed by Mukhanov.

The problem now is to take into account the energy-momentum tensor $T_{\mu\nu}$ of a graviton. Energy-momentum tensor, known from the perturbative quantum gravity i.e. the gauge field theory of spin-2 graviton, can be easy established from the Lagrangian [3]

$$\begin{aligned} L &= -\frac{1}{2}\partial_\lambda h_{\lambda\mu}\partial_\mu h_{\nu\nu} + \frac{1}{2}\partial_\lambda h_{\lambda\mu}\partial_\nu h_{\nu\mu} - \frac{1}{4}\partial_\lambda h_{\mu\nu}\partial_\lambda h_{\mu\nu} + \frac{1}{4}\partial_\lambda h_{\mu\mu}\partial_\lambda h_{\nu\nu} & (25)\\ &+ \frac{m^2}{4}\left(h_{\mu\mu}h_{\nu\nu} - h_{\mu\nu}h_{\mu\nu}\right), & (26) \end{aligned}$$

where m is mass of graviton, as the coefficient in the Taylor series

$$L = L(\eta_{\mu\nu}) + \frac{1}{2}T^{\mu\nu}h_{\mu\nu} + \ldots, \tag{27}$$

which explicitly has the form

$$T^{\mu\nu} = \frac{\delta L}{\delta h_{\mu\nu}}\left(h^{\mu\nu} = -\eta^{\mu\nu}\right), \tag{28}$$

with the result

$$T^{\mu\nu} = \frac{m^2}{2}\eta^{\mu\nu}. \tag{29}$$

In this manner in the massless limit *all components* of the energy-momentum tensor vanishes identically, while Mukhanov in his lecture remained T_{00} as nonzero component. Applying the massless limit to the obtained formulas one receives:

$$\Psi = 0, \tag{30}$$

$$h_j^k = 0, \tag{31}$$

$$\nabla^2(\Phi + \partial_0 B) = 0. \tag{32}$$

The equation (32) is the Laplace equation for the field $\Phi + \partial_0 B$, and in the case of spherically-symmetric situation and radial nature of $\Phi = \Phi(r)$ and $B = B(r,t)$ can be solved immediately as

$$\Phi + \partial_0 B = -\frac{C_1}{r} + C_0, \tag{33}$$

where C_0 and C_1 are integration constants. Now it is visible that in the massless limit the gravitational potential Φ in general must not be the Newtonian potential! Moreover, if $B = B(x^k)$ is function of the only coordinates but not time, then applying the boundary conditions $\Phi(r_0) = \Phi_0$ and $\nabla\Phi(r_0) = \Phi_0'$, where r_0 is certain fixed value of the coordinate r, one receives the solution

$$\Phi(r) = -\Phi_0'\frac{r_0^2}{r} + \Phi_0 + \Phi_0' r_0. \tag{34}$$

Now one sees that if the spherically symmetric source has the mass M then the gravitational potential is the Newtonian type if and only if the conditions are satisfied

$$\Phi_0 = \frac{GM}{r_0}, \tag{35}$$

$$\Phi_0' = -\frac{GM}{r_0^2}. \tag{36}$$

Interestingly, the force $-m\nabla\Phi$ acting on massless graviton is identically zero because of $m = 0$.

The equation (38) can be also interpreted in another way. Namely, if one wants to recover *ad hoc* the Poisson equation

$$\nabla^2\Phi = \frac{\kappa}{3}T_{00}, \tag{37}$$

which in the case $T_{00}=\rho c^2$, $\rho=\dfrac{M}{V}$, $V=\dfrac{4}{3}\pi r^3$ leads to the Newton law of universal gravitation, then it is clear that the scalar field B satisfies the equation

$$\nabla^2\partial_0 B=\frac{\kappa}{6}T_{00}+\frac{\kappa}{2}T-\frac{\kappa}{2}\int\partial^0 T_{0k}dx^k+\frac{3\kappa}{4}\int\partial^0 T_{0k}dx^0-\frac{3\kappa}{4}\int\partial^k T_{0k}dx^0. \quad (38)$$

However, in the context of gravitational waves the energy-momentum tensor does not describe the spherically symmetric source, so the Newtonian gravitation has no physical sense! In the massless limit the Einstein field equations related to such an interpretation are

$$\Psi = 0, \quad (39)$$
$$h^k_j = 0, \quad (40)$$
$$\nabla^2\Phi = 0, \quad (41)$$
$$\nabla^2\partial_0 B = 0, \quad (42)$$

and consequently the last two Laplace equations in the case of the radial character of Φ and B can be solved immediately

$$\Phi = C_0-\frac{C_1}{r}, \quad (43)$$
$$\partial_0 B = C_0'-\frac{C_1'}{r}, \quad (44)$$

and suitable boundary conditions can be applied.

Interestingly, the most general solution to the Laplace equation

$$\nabla^2\Phi=-\nabla^2\partial_0 B, \quad (45)$$

can be constructed by using of the system of equations

$$\nabla^2\Phi = f(r), \quad (46)$$
$$\nabla^2\partial_0 B = -f(r), \quad (47)$$

which leads to the solutions

$$\Phi(r) = C_2+\int_{r_0}^{r}\left(C_1+\int_{r_0}^{r'}g(r'')r''^2dr''\right)\frac{dr'}{r'^2}, \quad (48)$$
$$\partial_0 B = C_2'+\int_{r_0}^{r}\left(C_1'-\int_{r_0}^{r'}g(r'')r''^2dr''\right)\frac{dr'}{r'^2}. \quad (49)$$

3 Discussion

In General Relativity [4] the Newton law of universal gravitation is restored in a nontrivial and unique way. Namely, there is taken into account the Schwarzschild metric and the weak-field approximation. Such a procedure is fully justified by the nature of the Schwarzschild metric, which is the solution of the Einstein field equations for massive spherically-symmetric source. The weak-field approximation is the proper limiting procedure to the Newtonian gravitation.

In Mukhanov's talk presented at the conference "Quantum Theory and Gravitation", in fact there was suggested the attempt to describe gravitational waves in the framework of cosmological perturbation theory. Mukhanov has been deduced the law of universal gravitation as the massless limit of the modified Poisson equation obtained from the Einstein field equations for the perturbed Minkowski background. In his approach this law differs from the Newtonian gravitation by the constant multiplier $4/3$. This result is the mistake following from calculations and incorrect application of the massless limit.

Gravitational waves, even if massive, are appropriate for local regions of space-time. In contrast to this condition, cosmological perturbation theory is suitable for large scale behavior of space-time. From theoretical physics point of view description of gravitational waves by cosmological perturbation theory is manifestly wrong. Gravitational waves possess strict limitation to $|\widetilde{h}_{\mu\nu}| \ll 1$, which do not allow to apply them to test gravitation for large scale structure.

Applicability of cosmological perturbation theory to realistic physical problems has not been still satisfactory verified in the light of the experimental data of astrophysics and high energy physics. Cosmological perturbation theory is, in general, partially confirmed for the physics of Cosmic Microwave Background radiation, but does not explain the nature of this radiation completely. In the context of gravitational waves this theory is non-physical. By this reason cosmological perturbation theory seems to be wrong method to establish the corresponding law of universal gravitation. Mukhanov's approach is therefore also wrong from this point of view.

Lifshitz's cosmological perturbation theory is not free of problems, e.g. as we have seen the restoration of the Newton law of universal gravitation is not ambiguous. Possible solution to these problems is canonical cosmological perturbation theory recently proposed by Barbashov et al [5]. This approach is based on the old-fashioned, but still actual, version of the Hamiltonian formulation of General Relativity given by Dirac [6] and Arnowitt, Deser, and Misner [7]. Glinka and Pervushin [8] have been investigated the approach to unification of the Standard Model and General Relativity basing of this version of cosmological perturbation theory.

References

[1] E.M. Lifshitz, J. Phys. (USSR) 10, 116 (1946) (Russian version: Zh. Exp. Teor. Fiz. 16, 587 (1946)

[2] E. Bertschinger, Cosmological Dynamics, in R. Schaeffer, J. Silk, M. Spiro, and J. Zinn-Justin (Eds.), Cosmology and Large Scale Structure. Proc. Les Houches Summer School, Session LX (Elsevier Science, 1996), pp. 273-347, arXiv:astro-ph/9503125

[3] M.J.G. Veltman, Quantum Theory of Gravitation, in R. Balian and J. Zinn-Justin (Eds.), Methods in Field Theory. Proc. Les Houches, Session XXVIII (North Holland, 1976), pp. 265-328

[4] S. Weinberg, Gravitation and Cosmology. Principles and Applications of the General Theory of Relativity (Wiley, 1972)

[5] B.M. Barbashov, V.N. Pervushin, A.F. Zakharov, and V.A. Zinchuk, Phys. Lett. B 633, 458 (2006), arXiv:hep-th/0501242; Int. J. Mod. Phys. A 21, 5957 (2006), arXiv:astro-ph/0511824

[6] P.A.M. Dirac, Proc. Roy. Soc. A 246, 333 (1958); Phys. Rev. 114, 924 (1959)

[7] R. Arnowitt, S. Deser, and C.W. Misner, Phys. Rev. 117, 1595 (1960)

[8] L.A. Glinka and V.N. Pervushin, Concepts Phys. 5, 31 (2008), arXiv:0705.0655 [gr-qc]