Scope of Center of Charge in Electrostatics

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Abstract

The notion of the center of an electrostatic charge distribution is introduced. Then, it is investigated in which problems the notion may be useful. It is seen that in many problems with positive and negative charge contents (for example, image problems) the notion works nice.

I. INTRODUCTION

The concept of "center of mass" is well-known in Classical mechanics. According to Newton's 2nd law of motion, whenever linear motion is concerned, the mechanical force acting on an extended body can be assumed to act through its center of mass, the point at which the total mass of the body is thought to be concentrated.

Can we speak of a "center of charge" in Electrostatics? We all know from high school physics courses that for very special charge distributions, for example, uniform distribution of charge in a sphere or spherical shell, we can treat as if the total charge has been placed at the geometrical center of the sphere (the field point being outside the sphere). If the charge distribution depends on radial co-ordinate r, then also the conclusion remains the same. However, these are very special charge distributions. Can this notion be applied to a class of problems where geometry is not necessarily spherical or the charge density is more complicated?

To answer these questions, first we will try to see what condition needs to be satisfied to get a valid center of charge. From this condition, it will be seen that many (but not all!) image problems are amenable to this treatment. We will be able to justify it in the light of Newton's law of motion. And also, we will be able to apply the idea in problems where intuitively one could have centers of positive and negative charges.

II. CENTER OF CHARGE

We propose the 'definition' of the "Center of charge" of a charge distribution almost in the same way center of mass is defined in classical mechanics. For a continuous charge distribution of the same type, i.e. either positive or negative, center of charge $\mathbf{R}$ is given by

$$\mathbf{R} = \frac{\int \mathbf{r}'\rho(\mathbf{r}')\ d\tau\ '}{Q} \tag{1}$$

where Q represents the total charge, $\rho(\mathbf{r}')$ denotes the charge distribution over the source co-ordinate $\mathbf{r}'$. Thus, we need to determine the centers of +ve and -ve charges separately and $Q = 0$ is not allowed to occur in the denominator by construction. Notice that the equation follows directly from the definition of dipole moment of a charge distribution. Thus, we will have a center of -ve charge distribution as

$$\mathbf{R}^- = \frac{\mathbf{p}^-}{Q}$$

Notice that, the condition that the potential at a field point will indeed be that produced by a point charge placed at the center of charge demands that potential $\phi(\mathbf{r})$ to be given by

$$\phi(\mathbf{r}) = \frac{Q}{|\mathbf{r} - \mathbf{R}\ |} \tag{2}$$

If the relation is satisfied, we would get a center to which we may attribute the total charge so that potential is much easily obtained. Using (2), the total potential for a typical charge distribution would be (integrating over +ve and -ve charge densities ρ^+ and ρ^-, total charges being equal to Q^+ and Q^-)

$$\phi(\mathbf{r}) = \phi^+ + \phi^- = \int_{(+)} \frac{\rho^+(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'\ |}\ dV' + \int_{(-)} \frac{\rho^-(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'\ |}\ dV' = \frac{Q\ ^+}{|\mathbf{r} - \mathbf{R}^+\ |} + \frac{Q\ ^-}{|\mathbf{r} - \mathbf{R}^-\ |} \tag{3}$$

Notice that the potential in R.H.S. of (3) is in the form of the potential as seen in image problems. Now, we accept the challenge to find out the center of charge of a complicated charge distribution. Referring to the figure 1, consider the charge density of the grounded conducting sphere image problem (we use expression from Greiner page: 52, given in C.G.S. units)

$$\sigma(\theta) = \frac{-q}{4\pi\ a\ y} \frac{1 - \frac{a^2}{y^2}}{(1 + \frac{a^2}{y^2} - 2\frac{a}{y}\ cos\theta)^{\frac{3}{2}}} \tag{4}$$

Using (4) and noticing that here $Q = Q^- = -\frac{aq}{y}$, we have center of charge given by (translating equation (1) for surface charge density and omitting primes)

$$\mathbf{R} = \frac{\int \mathbf{r}\sigma(\theta)\ d\ S}{Q^-} \tag{5}$$

Here, dS denotes infinitesimal area element on the sphere. Expressing source co-ordinate $\mathbf{r}$ in $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ components, we see that the transverse components $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ of the integral vanish from ϕ integration-which is also clear from physical point of view, the distribution being azimuthally symmetric. The center of charge becomes (in figure, $y' = OR = \mathbf{R}$)

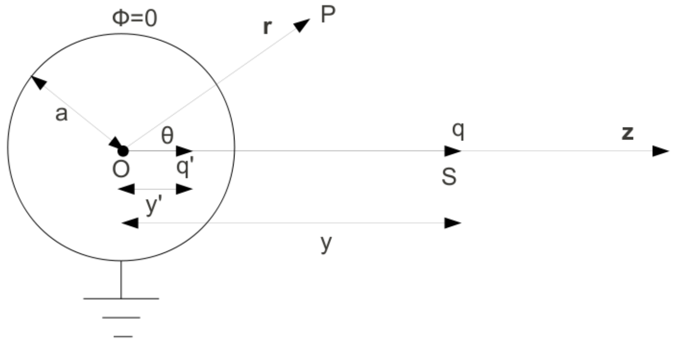

FIG. 1. Center of Charge

$$\mathbf{R} = \hat{\mathbf{z}}\ a^3 \frac{-q}{4\pi\ a\ y\ Q^-}(1-\frac{a^2}{y^2})\int \frac{cos\theta\ sin\theta\ d\theta\ d\phi}{(1+\frac{a^2}{y^2}-2\frac{a}{y}\ cos\theta)^{\frac{3}{2}}}$$

Now, we need to evaluate the integral, which is not a difficult task with the substitution:

$x =\ cos\theta$

$\alpha = \frac{a}{y}$

$z = (1+\alpha^2-2\alpha x)^{-1/2}$

The integral equals $2\pi\frac{2\alpha}{1-\alpha^2}$. After simplification, it turns out to be

$$\mathbf{R} = \frac{\mathbf{p}^-}{Q^-} = \hat{\mathbf{z}}\frac{a^2}{y} \tag{6}$$

Thus, the center of charge of the grounded conducting sphere in presence of a source charge is at the position of the image charge. And it can be shown that the electrostatic force on the sphere due to the real charge can be assumed to act through the center of charge (position of the image charge). Thus, the notion of center of charge is exactly same as that of concept of center of mass in classical mechanics. In this light, the result looks obvious; actually the justification of this method lies in establishing many intuitive but not-so-well-established results.

Now, invoking (3) and noticing that $Q^+ =\ +q$ and the center of the real charge is at $\mathbf{R}^+ = \frac{\mathbf{p}^+}{Q^+} = \mathbf{y}$; similarly, $Q^- =\ -\frac{aq}{y}$ and the center of the image charge is at $\mathbf{R}^- = \hat{\mathbf{z}}\frac{a^2}{y}$, the potential becomes

$$\phi(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{y}\ |} - \frac{aq}{y\ |\mathbf{r}-\hat{\mathbf{z}}\frac{a^2}{y}\ |} \tag{7}$$

Diving this by q, we discover that the Green's function corresponding to this image problem obtains. This means that other problems with the same Green's functions are also amenable to the same treatment. In the cases of conducting (ungrounded/insulated) sphere or sphere kept at a fixed potential, the solution is found by linear superposition of potentials. The image charge still forms at the center of charge; however, here total surface charge density includes another term (uniform charge density) corresponding to a charge sitting at the center of the sphere (field point outside the sphere, as usual). So, in these problems we talk about two centers, both of which are legitimate in their own rights, one corresponding to the position of image and another corresponding to the excess charge carried by the conducting sphere. This is an additional feature, unavailable in classical mechanics.

III. ILLUSTRATIONS AND DISCUSSIONS

What about infinite grounded conducting plane problem? The charge distribution (2 dimensional) can be integrated to confirm that the (x,y) co-ordinate of the image charge is (0,0); i.e. at the origin. The reader might wonder how to reconcile this with the 3 dimensional position of image charge. Let us remember that this infinite plane problem can be derived from the "grounded conducting sphere" (for which we have proved the center of charge is truly at the position of the image charge) problem by taking the limit radius tending to infinity. In this limit the lengths $y - a$ and $a - \frac{a^2}{y}$ becomes equal.

To see the power of the idea, let us take the example of a conducting sphere placed in a uniform field; the potential at a very large distance and surface charge density can be calculated using boundary conditions method or using the Green's function. However, given the surface charge density, one can determine the centers of +ve and -ve charges, $\mathbf{R}^+$ and $\mathbf{R}^-$ separately (taking appropriate limits on θ) and find out the expected dipole moment. This dipole (pure point dipole) produces the same potential at a very large distance (as a function of θ) as found from boundary conditions method. Of course, this would not work at nearby regions where higher moments are at work.

Again, consider the problem of conducting hemispheres at +V and -V potentials. Using the general method shown in Greiner example 2.8 (or Jackson, section 2.7) the potential may be calculated as a series solution (converging very fast). As expected, the leading term is dipolar; the other terms become less significant than 2 percent as the distance to the field point is more than 5 times the radius. Now, if we are clever enough to assume uniform charge density (the assumption is justified as it is an isolated system), the center of charge approach gives the correct the dipolar potential at large distances (given the relation between charge density and V).

Consider a problem which does not involve spherical geometry: finding the electric field on the axis of a ring charge distribution, half of which is uniformly +ve and rest half is uniformly -ve. When $z \to \infty$ is taken, dipolar nature of the field is prominent. One can evaluate the dipole moment of the pure point dipole, conceived at the origin. Now, try the center of charge approach and find $\mathbf{R}^{+} - \mathbf{R}^{-}$. Multiplying this with the magnitude of +ve charge we rediscover the dipole moment again!!!

IV. CONCLUSIONS

Of course, what we proposed is always not a labor saving tool; only when we are able to make a good guess, it reduces some labor. It shows another way of looking at these problems. What we found is, in many cases of pedagogic interest this intuitive (but not well-established) notion indeed works. The good news is that when we are trying to find the dipolar field at a very large distance, we need not calculate field and obtain the field at infinity by taking limit $r \to \infty$. From mechanics, the center of mass of many geometries are known; so, using those with dipole potential formula the exact potential may be constructed. The bad news is that this method is not too general to be applied anywhere we wish. Actually it depends on whether you get (2) or (3) satisfied. Thus, problems where the potential involves logarithmic functions (typically cylindrical charge distributions) or where higher moments are at work, this method fails.

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(ii) J.D. Jackson, "Classical Electrodynamics", (3rd Edition, Wiley, 1999, ISBN-0-471-30932-X).